FIELD THEORY MODEL FOR TWO-DIMENSIONAL TURBULENCE: VORTICITY-BASED APPROACH

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Renormalization group analysis is applied to the two-dimensional Navier–Stokes vorticity equation driven by a Gaussian random stirring. The energy-range spectrum $C_K\varepsilon^{2/3}k^{-5/3}$ obtained in the one-loop approximation coincides with earlier double epsilon expansion results, with $C_K = 3.634$. This result is in good agreement with the value $C_K = 3.35$ obtained by direct numerical simulation of the two-dimensional turbulent energy cascade using the pseudospectral method.

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Statistical hydrodynamics constitutes one of the most promising applications of quantum field theory methods to the physics of classical systems. The statistical description of *two-dimensional* hydrodynamic turbulence is in turn one of the most challenging problems in hydrodynamics. In addition to having intrinsic mathematical beauty, the renormalization group (RG) model of turbulence in three dimensions has provided realistic values of the Kolmogorov constant in the range 1.4 to 1.7 [1, 2, 3]. Along with the energy, which is conserved by the nonlinear terms of the incompressible Navier–Stokes equation in any dimension; there exists in two dimensions an additional positive-semidefinite invariant, the mean-squared vorticity. This *enstrophy* conservation law forbids universal scaling in the whole range of scales from energy injection to energy dissipation, making the dynamical RG technique [4, 2] hardly applicable in two dimensions [5, 6].

The goal of applying the RG technique to hydrodynamic turbulence is to decrease the number of modes required to describe the system, retaining at the same time all basic symmetries and properties of the system [4]. The same idea underlies the spectral reduction method, where the effect of discarded modes is taken into account by enhancing the nonlinear interaction coefficients [7]. This is accomplished by coarse-graining the vorticity equation in Fourier space in

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such a way that the averaged equations automatically respect the energy and enstrophy invariants. For the two-dimensional enstrophy cascade, spectral reduction has been shown to provide an efficient numerical approximation to the turbulent statistics. In the RG approach, the effect of the deleted small scale modes is taken into account by integrating over these modes in the presence of a random stirring force added to the right-hand side of the Navier–Stokes equation.

Technically, two-dimensional turbulence RG calculations are more complicated than three-dimensional ones because of extra one-loop divergences that require a counterterm having no counterpart in the original action [8]. A significant breakthrough in this problem was achieved when the double parametric renormalization group [6] was applied to the field theory formalism of two-dimensional turbulence to cancel the extra one-loop divergence [9]. However, it has been difficult to estimate by other theoretical means if the obtained value of $C_K = 3.634$ is relevant and to confirm it with appropriate numerical simulations.

In this paper we apply the double parametric expansion RG method to essentially the same vorticity equation in Fourier space that is used for pseudospectral simulations of two-dimensional turbulence. Our analytical calculations match the result $C_K = 3.634$ obtained by Honkonen [9] and M. Hnatich *et al.* [3]. The numerical simulation performed with the pseudospectral code gives $C_K \approx 3.35$. The RG method for two-dimensional turbulence is based on a two-term force correlator that models the phase transition between laminar flow and developed turbulence: the first term matches the (asymmetric) turbulent regime, while the second term matches the symmetric (with respect to (3)) laminar regime.

We briefly review the field theory approach to stochastic hydrodynamics [4, 8], starting from the two-dimensional Navier–Stokes equation $\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \nu\Delta\boldsymbol{v} - \boldsymbol{\nabla}P + \boldsymbol{f}$, where the velocity field $\boldsymbol{v}$ is incompressible ($\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$), ν is the viscosity, P is the pressure, and $\boldsymbol{f}$ denotes an external stirring force. The curl of this equation describes the evolution of the vorticity vector $\boldsymbol{\zeta}(t,\boldsymbol{x}) = \boldsymbol{\nabla}\times\boldsymbol{v}(t,\boldsymbol{x}) = \zeta\hat{\boldsymbol{z}}$, where $\hat{\boldsymbol{z}}$ is the normal to the plane of flow [10],

$$\partial_t\boldsymbol{\zeta} - \nu\Delta\boldsymbol{\zeta} = -\boldsymbol{\nabla}\times\left[\Delta^{-1}(\boldsymbol{\nabla}\times\boldsymbol{\zeta})\times\boldsymbol{\zeta}\right] + \xi\hat{\boldsymbol{z}}; \tag{1}$$

the incompressible velocity field being the inverse Laplacian of the curl of the vorticity field: $\boldsymbol{v}(t,\boldsymbol{x}) = -\Delta^{-1}\boldsymbol{\nabla}\times\boldsymbol{\zeta}(t,\boldsymbol{x})$. The random stirring force $\xi\hat{\boldsymbol{z}} = \boldsymbol{\nabla}\times\boldsymbol{f}$ is supposed to be Gaussian:

$$\langle\xi(x)\xi(x')\rangle = D(x-x'), \quad P[\xi] = \exp\left(-\frac{1}{2}\int \xi(x)D^{-1}(x-x')\xi(x')\,dx\,dx'\right). \tag{2}$$

Here we use the $1+d$ notation $x \equiv (x_0, \boldsymbol{x})$, where $x_0 = t$ and $d = 2$ is the dimension.

Consider a stochastic process $u(t,\boldsymbol{x})$, governed by the stochastic integro-differential equation $F[u] = \xi - \hat{\eta}$, where F is a nonlinear integro-differential operator, $\hat{\eta}(x)$ is a regular force, and $\xi(x)$ is a Gaussian random force satisfying (2). The field theory approach is based on the characteristic functional

$$Z[\eta,\hat{\eta}] = \left\langle \exp\left(i\int dx\,\eta(x)u(x)\right)\right\rangle, \quad \langle u^n\rangle = \left.\frac{\delta^n}{i^n\delta\eta^n}\right|_{\eta=0}.$$

By prescribing to each trajectory $u = u(t,\boldsymbol{x})$ the weight $P[u,\xi] = \delta(F[u]-\xi+\hat{\eta})P[\xi]$ and representing the δ-function as a functional integral over an auxiliary field $\hat{u}$, one finds that $Z[\eta,\hat{\eta}] = \int \mathcal{D}u\mathcal{D}\hat{u}\,\exp\left(iS[\hat{u},u] + i\eta u + i\hat{\eta}\hat{u}\right)$, where the action $S[\hat{u},u]$ can be expressed in terms of the random force correlation function D as $S[\hat{u},u] = \hat{u}F[u] + i\hat{u}D\hat{u}/2$.

In the absence of a random force, the vorticity equation (1) is invariant under the Lie group G of scale transformations:

$$t' = e^{2\gamma}t, \quad \boldsymbol{x}' = e^{\gamma}\boldsymbol{x}, \quad \zeta' = e^{-2\gamma}\zeta, \tag{3}$$

where γ is the infinitesimal transformation parameter. The corresponding Lie algebra generators are $\hat{\Gamma}_t = 2t\partial_t$, $\hat{\Gamma}_{\boldsymbol{x}} = \boldsymbol{x}\partial_{\boldsymbol{x}}$, $\hat{\Gamma}_\zeta = -2\zeta\partial_\zeta$. In terms of the characteristic functional, invariance under a Lie group implies [11]:

$$\hat{\Gamma}\left[\frac{\delta}{i\delta\eta}, i\eta, x, \partial_x\right] Z[\eta,\hat{\eta}] = 0 \tag{4}$$

for all generators $\hat{\Gamma}[u, \partial_u, x, \partial_x]$ of the symmetry group of the equation $F[u] = 0$.

Following [11], we can easily find the constraints on the correlators that underly the symmetry (3) even in the case of a random force ($D \neq 0$). Since $\hat{\zeta}$ is just an auxiliary field with no dynamical constraints imposed on it, we can fix the transformation law of $\hat{\zeta}$ with respect to scale transformations to satisfy the invariance of $\int \hat{\zeta}(t,\boldsymbol{x})F[\zeta(t,\boldsymbol{x})]dt\, d^d\boldsymbol{x}$ identically with respect to (3). This means that $\hat{\zeta}' = e^{\gamma D_{\hat{\zeta}}}\hat{\zeta}$, where $D_{\hat{\zeta}} = 2 - d$.

Now the invariance of the action $S[\zeta, \hat{\zeta}]$ with respect to scale transformations is completely determined by the transformation properties of $\hat{\zeta}D\hat{\zeta}$ with respect to (3). To keep it invariant, the identity $\int \hat{\zeta}_1' D' \hat{\zeta}_2' d^d\boldsymbol{x}'_1 dt_1' d^d\boldsymbol{x}'_2 dt_2' = \int \hat{\zeta}_1 D \hat{\zeta}_2 d^d\boldsymbol{x}_1 dt_1 d^d\boldsymbol{x}_2 dt_2$ should hold. Simple power counting gives $D' = e^{\gamma D_D}D$, with $D_D = -8$, or $D = 0$. The second possibility corresponds to the free equation, i.e. decaying turbulence. The first case corresponds to a special power-type stirring, say

$$D(\boldsymbol{x},t) \sim \delta(t)|\boldsymbol{x}|^{-6} \quad \text{or} \quad D(\boldsymbol{x},t) \sim |t|^{\alpha}|\boldsymbol{x}|^{\beta} \text{ with } 2\alpha + \beta = -8. \tag{5}$$

Thus, for a white-noise random forcing, a correlator that is invariant under (3) must have a Fourier transform proportional to k^4.

Now, assuming the invariance of the characteristic functional with respect to scale transformations (3), let us derive the scaling equations for the correlation and response functions. Here we follow the method already used to derive the governing statistical equations for the Navier–Stokes velocity field in a similar setting [11]. The correlation and response functions are

$$C(12) = -\frac{\delta^2 \ln Z[\hat{\eta},\eta]}{\delta\eta(x_1)\delta\eta(x_2)}, \quad G(12) = i\frac{\delta^2 \ln Z[\hat{\eta},\eta]}{i\delta\eta(x_1)i\delta\hat{\eta}(x_2)}. \tag{6}$$

Since the action should be identically invariant under (3), we have to account only for the variation of the "generating part" $\int d\boldsymbol{x}\, dt\,(\eta\zeta + \hat{\eta}\hat{\zeta})$ in the exponent of the generating functional, so that the invariance of the characteristic functional under the group of scale transformations $\langle \delta_G \int d\boldsymbol{x}\, dt\,(\eta\zeta + \hat{\eta}\hat{\zeta})\rangle = 0$, where δ_G means variation with respect to (3), will hold.

After straightforward calculations, similar to those presented in [11], we obtain the symmetries of the correlation and Green function, which hold when the scaling symmetry (3) is observed, i.e. for the class (5) of random force correlators,

$$[\boldsymbol{x}\partial_{\boldsymbol{x}} + 2t\partial_t + 4]\, C(t,\boldsymbol{x}) = 0, \quad [\boldsymbol{k}\partial_{\boldsymbol{k}} + 2\omega\partial_\omega - 4]\, C(\omega,\boldsymbol{k}) = 0, \tag{7}$$

$$[\boldsymbol{x}\partial_{\boldsymbol{x}} + 2t\partial_t + d]\, G(t,\boldsymbol{x}) = 0, \quad [\boldsymbol{k}\partial_{\boldsymbol{k}} + 2\omega\partial_\omega - d]\, G(\omega,\boldsymbol{k}) = 0. \tag{8}$$

The vorticity equation (1) gives rise to the field theory

$$Z[J]=\int e^{iS[\Phi]+iJ\cdot\Phi}D\Phi,\quad S[\Phi]=i\frac{1}{2}\hat{\zeta}D\hat{\zeta}+\hat{\zeta}\big[\partial_t\zeta-\nu\Delta\zeta-U[\zeta]\big]. \tag{9}$$

The notation $\Phi=(\zeta,\hat{\zeta})$, $J=(\eta,\hat{\eta})$ is used for book-keeping, where the nonlinear interaction in two dimensions is given by the nonlocal vertex

$$U[\zeta]=\int U(k|p,q)\zeta(p)\zeta(q)\,dp\,dq,\quad U(k|p,q)=\frac{\delta(k-p-q)}{2(2\pi)^3}(\boldsymbol{p}\times\boldsymbol{q})_z\left(\frac{1}{|\boldsymbol{q}|^2}-\frac{1}{|\boldsymbol{p}|^2}\right).$$

The force correlation operator D is chosen in accord with the double parametric expansion: $\langle\xi(k_1)\xi(k_2)\rangle=(2\pi)^{d+1}\delta(k_1+k_2)D(\boldsymbol{k}_1)$ and $D(\boldsymbol{k})=\nu^3(g_1\mu^{2\epsilon}\boldsymbol{k}^{4-2\epsilon}+g_2\boldsymbol{k}^4)$, where μ is the renormalization mass [6, 9, 3].

The coupling constants g_1 and g_2 can be made dimensionless by rescaling both vorticity and time (1): $t'=t\nu_0\Lambda^2$, $\zeta'=\zeta\sqrt{\nu_0/D_0}$, $\eta'=\eta/(\Lambda^2\sqrt{\nu_0D_0})$, where $D_0=g_{1,0}\nu_0^3$ is the unrenormalized strength of the vorticity forcing and Λ is the ultraviolet momentum cutoff.

The renormalized viscosity is given by a one-loop contribution to the Green function $\langle\hat{\zeta}\zeta\rangle$,

$$G_2(k)=4G_0^2(k)\int\left[\frac{(\boldsymbol{q}\times\boldsymbol{k})_z}{2}\left(\frac{1}{|\boldsymbol{q}^2|}-\frac{1}{|\boldsymbol{k}-\boldsymbol{q}|^2}\right)\right]D(q)|G_0(q)|^2 \left[\frac{(-\boldsymbol{q}\times\boldsymbol{k})_z}{2}\left(\frac{1}{|\boldsymbol{q}|^2}-\frac{1}{|\boldsymbol{k}|^2}\right)\right]G_0(k-q)\frac{d^{d+1}q}{(2\pi)^{d+1}},$$

where $G_0(q)=(-iq_0+\nu_0\,|\boldsymbol{q}|^2)^{-1}$.

In the long-wave limit $k\to0$ this results in a turbulent dressing of the viscosity,

$$\tilde{\nu}=\nu_0\left[1+g_{1,0}\frac{1}{32\pi}\frac{1}{2\epsilon}+g_{2,0}\frac{1}{32\pi}\ln\frac{\Lambda}{m}\right], \tag{10}$$

(for $\epsilon>0$), where m and Λ are the lower (infrared) and the upper (ultraviolet) limits of integration and $g_{1,0}$ and $g_{2,0}$ are the "bare" constants of the vorticity force correlation. The positive turbulent dressing of the viscosity (10) can be absorbed into the renormalization of the viscosity $\nu_0=\nu Z_\nu$,

$$Z_\nu=1-A\left[g_1\frac{1}{2\epsilon}+g_2\ln\frac{\Lambda}{m}\right],\quad A=\frac{1}{32\pi}. \tag{11}$$

Similarly, the one-loop divergent contribution to the correlation function $\langle\zeta\zeta\rangle$ is

$$D_2(k)=2\nu^6\int_m^\Lambda\frac{qdq}{(2\pi)^2}d\theta\frac{1}{4}q^2k^2\sin^2\theta\left[\frac{-k^2+2kq\cos\theta}{q^2(k^2+q^2-2kq\cos\theta)}\right]^2 \frac{(g_1q^{4-2\epsilon}+g_2q^4)(g_1|\boldsymbol{k}-\boldsymbol{q}|^{4-2\epsilon}+g_2|\boldsymbol{k}-\boldsymbol{q}|^4)}{2\nu^3(2q^2+k^2-2kq\cos\theta)q^2(k^2+q^2-2kq\cos\theta)}.$$

In the limit $k\to0$ the one-loop contribution to the correlator is

$$D_2(k)=\frac{\nu^3k^2}{32\pi}\left(g_{1,0}^2\frac{1}{4\epsilon}+g_{1,0}g_{2,0}\frac{1}{\epsilon}+g_{2,0}^2\ln\frac{\Lambda}{m}\right), \tag{12}$$

which can be absorbed into the renormalization of the coupling constant g_2,

$$Z_2 = 1 - A\left(\frac{g_1^2}{g_2}\frac{1}{4\epsilon} + g_1\frac{1}{\epsilon} + g_2 \ln\frac{\Lambda}{m}\right). \tag{13}$$

The renormalization of g_1 is related to Z_ν in the usual way: $g_{1,0} = \mu^{2\epsilon} g_1 Z_1$, $g_{2,0} = g_2 Z_1 Z_2$, $Z_1 = Z_\nu^{-3}$. The corresponding anomalous dimensions $\gamma_i = \mu\partial_\mu \ln Z_i|_{g_{1,0},g_{2,0},\nu_0=\mathrm{const}}$ and the functions $\beta_i = \mu\partial_\mu g_i|_{g_{1,0},g_{2,0},\nu_0=\mathrm{const}}$ are evaluated as a series in the coupling constants g_1 and g_2. Retaining up to second-order terms in g_1 and g_2, we find $\beta_1 = g_1(-2\epsilon + 3Ag_1 + 3Ag_2)$ and $\beta_2 = A(-g_1^2 + g_1 g_2 + 2g_2^2)$. This yields the nontrivial infrared fixed point (g_1^*, g_2^*), where [6]

$$g_1^* = \frac{4\epsilon}{9A}, \quad g_2^* = \frac{2\epsilon}{9A}. \tag{14}$$

Our results (11) and (13), obtained in the one-loop approximation, exactly match the results obtained in two dimensions for the forced stream function [9] and velocity [3] equations. Among a variety of controversial theoretical and numerical results for two-dimensional turbulence obtained by different means, it seems most natural to compare the value of the Kolmogorov constant obtained by a two parameter renormalization group (Z_ν, Z_2) with a quasistationary numerical result obtained by pseudospectral simulation of the same vorticity equation used to derive the values Z_ν and Z_2. Referring the reader to [9, 3] for details of the derivation of the energy spectrum determined by the fixed point (14), we just state the final results.

The mean energy dissipation rate ε is related to the field theory model parameters: $\varepsilon = \nu_0^3 g_{1,0} k_d^{4-2\epsilon}/(16\pi)$, where k_d is the dissipation wavenumber and $\epsilon = 2$.

We now evaluate the constant C_K in the Kolmogorov law for the energy cascade, $E(\kappa) = C_K \varepsilon^{2/3}\kappa^{-5/3}$, where $\kappa = |\boldsymbol{k}|$ and the two-dimensional energy spectrum $E(\kappa)$ is defined by

$$E(\kappa) = \frac{1}{2}\mathrm{Tr}\int 2\pi\kappa\, d\kappa \int \frac{d\omega}{2\pi}\frac{\langle v(k)v(-k)\rangle}{(2\pi)^2} = \pi\int \kappa\, d\kappa \int \frac{d\omega}{2\pi}\frac{\langle \zeta(k)\zeta(-k)\rangle}{(2\pi\kappa)^2}.$$

This yields [9, 3]

$$C_K = \left(\frac{1}{2\pi}\right)^{1/3}\frac{g_1^* + g_2^*}{g_1^{*2/3}} = (24\epsilon)^{1/3}, \tag{15}$$

so that $C_K \approx 3.634$ for $\epsilon = 2$ and $d = 2$.

Euclidean field theory (9) describing two-dimensional turbulence is similar to the Landau-Ginzburg phase transition theory. In the absence of a random force or a symmetry preserving force ($\epsilon = 0$), two-dimensional turbulence in an unbounded domain is invariant under the Lie group of scale transformations (3), which formally corresponds to the vorticity correlator $\langle\zeta(k)\zeta(-k)\rangle \sim \boldsymbol{k}^4$. The Kolmogorov regime requires that the mean energy dissipation rate ε is exactly compensated by energy injection. This state ($\epsilon = 2$) corresponds to the breaking of the original Lie group (3) symmetry down to a new state of the system determined by the anomalous dimensions $\gamma_i = \gamma_i(\epsilon, g_1, g_2)$. For a given value of ϵ, the existence of the stable fixed point $\beta_i(\epsilon, g_{1,\epsilon}^*, g_{2,\epsilon}^*) = 0$ implies, in the language of phase transitions, the existence of a stable phase. The trivial fixed point $g_1 = g_2 = 0$ corresponds to the symmetric phase (laminar flow); the infrared stable fixed point (14) at $\epsilon = 2$ corresponds to the asymmetric phase, the Kolmogorov

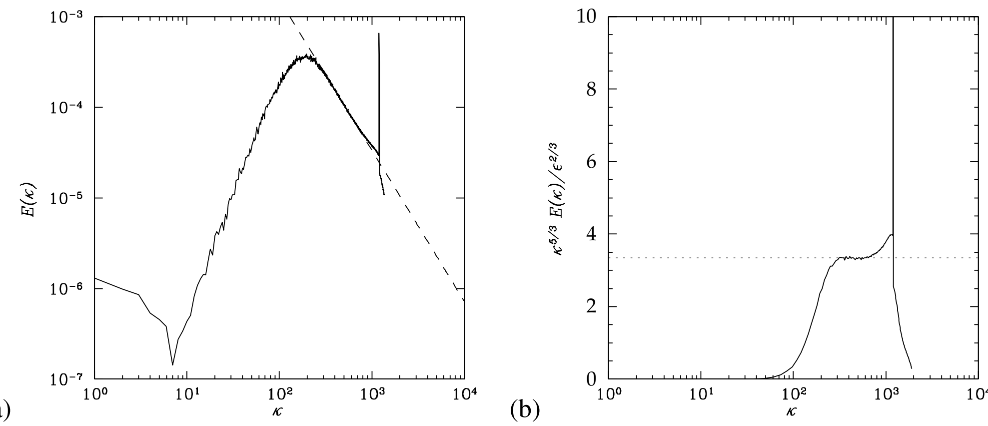

Fig. 1. (a) Transient spectral energy density $E(\kappa)$ obtained from a 2731×2731 dealiased pseudospectral simulation. The dashed line indicates the fitted Kolmogorov spectrum $3.35\varepsilon^{2/3}\kappa^{-5/3}$. (b) Variation of the estimated value of Kolmogorov constant for the inverse energy cascade with wavenumber.

regime. The diffeomorphism between phases is controlled by a single parameter ϵ that determines the symmetry of the forcing. If other fields are incorporated (e.g. passive scalar advection) the phase diagram of the combined system becomes multidimensional [3].

In Fig. 1, we depict the transient energy spectrum and estimated Kolmogorov constant $C_K = 3.35$ obtained with a dealiased pseudospectral simulation of the two-dimensional energy cascade driven by a white-noise random forcing restricted to $\kappa \in [1198, 1202]$, taking $\varepsilon = 1$ and $\nu = 6.4 \times 10^{-5}$. To minimize the required resolution, we replaced the viscous term $\nu\kappa^2\zeta$ by $\nu\kappa^2 H(\kappa - 1202)\zeta$, where $H(\kappa)$ denotes the Heaviside unit step function. An adaptive fifth–order Cash–Karp–Fehlberg Runge–Kutta integrator was used to advance (1) and its second moment.

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