Casimir Cascades in Two-Dimensional Turbulence

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Summary. The Kraichnan–Leith–Batchelor theory of two-dimensional turbulence is based on the fact that the nonlinear terms of the two-dimensional Navier–Stokes equation conserve both energy and enstrophy. In an infinite domain and in the limit of infinite Reynolds number, the net energy and enstrophy transfers out of a low-wavenumber forcing region must consequently be independent of wavenumber. The resulting dual cascade of energy to larger scales and enstrophy to smaller scales is readily observed in numerical simulations of two-dimensional turbulence in a finite domain.

While it is well known that the nonlinearity also conserves the global integral of any arbitrary C^1 function of the scalar vorticity field, the direction of transfer of these quantities in wavenumber space remains unclear. Numerical investigations of this problem are hampered by the fact that pseudospectral simulations, which necessarily truncate the wavenumber domain, do not conserve these higher-order Casimir invariants.

A fundamental question is whether these invariants also play an underlying role in the turbulent cascade, in addition to the *rugged* quadratic (energy and enstrophy) invariants, which do survive spectral truncation. Polyakov's minimal conformal field theory model [1] has suggested that the higher-order Casimir invariants cascade to large scales, while Eyink [2] suggests that they might instead cascade to small scales.

In this work we develop estimates for the degree of nonconservation of the Casimir invariants and demonstrate, using sufficiently well-resolved simulations, that the fourth power of the vorticity cascades to small scales.

1 Two-Dimensional Turbulence

We begin with the 2D incompressible Navier–Stokes equation for the *vorticity* $\omega \doteq \widehat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\times}\boldsymbol{u}$:

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\omega = -\nu\nabla^2\omega + f, \tag{1}$$

where the constant ν is the kinematic viscosity and f is an external stirring force. In the inviscid unforced limit $\nu = f = 0$, both the *energy* $E \doteq \frac{1}{2}\int u^2\, d\boldsymbol{x}$ and *enstrophy* $Z \doteq \frac{1}{2}\int \omega^2\, d\boldsymbol{x}$ are conserved.

However, as is well known, inviscid unforced 2D turbulence has uncountably many other *Casimir* invariants: any continuously differentiable function g of the (scalar) vorticity is conserved by the nonlinearity:

$$\begin{aligned}\frac{d}{dt}\int g(\omega)\,d\boldsymbol{x} &= \int g'(\omega)\frac{\partial\omega}{\partial t}\,d\boldsymbol{x} = -\int g'(\omega)\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega\,d\boldsymbol{x}\\ &= -\int \boldsymbol{u}\cdot\boldsymbol{\nabla}g(\omega)\,d\boldsymbol{x} = \int g(\omega)\boldsymbol{\nabla}\cdot\boldsymbol{u}\,d\boldsymbol{x} = 0.\end{aligned}$$

Do these invariants also play a fundamental role in the turbulent dynamics, in addition to the quadratic (energy and enstrophy) invariants? In particular, do they exhibit *cascades*? In the theoretical literature, this remains an open question: Polyakov [1] has predicted that the higher-order Casimir invariants cascade to large scales, while Eyink [2] suggests that they might cascade to small scales. What is certain is that only the quadratic invariants are *rugged*, meaning that their conservation, being a consequence of detailed triadic balance, survives high-wavenumber truncation. To see this, let us express (1) in Fourier space:

$$\frac{\partial\omega_{\boldsymbol{k}}}{\partial t} + \nu k^2\omega_{\boldsymbol{k}} = \sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\omega^*_{\boldsymbol{p}}\,\omega^*_{\boldsymbol{q}} + f_{\boldsymbol{k}}, \tag{2}$$

where $\epsilon_{\boldsymbol{kpq}} \doteq (\widehat{\boldsymbol{z}}\cdot\boldsymbol{p}\times\boldsymbol{q})\,\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$ is antisymmetric under interchange of any two indices. When $\nu = f_{\boldsymbol{k}} = 0$, the enstrophy is readily seen to be conserved:

$$\frac{d}{dt}\sum_{\boldsymbol{k}}|\omega_{\boldsymbol{k}}|^2 = \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\omega^*_{\boldsymbol{k}}\omega^*_{\boldsymbol{p}}\,\omega^*_{\boldsymbol{q}} = 0.$$

In the absence of high-wavenumber truncation, the invariance of $Z_3 \doteq \int \omega^3\,d\boldsymbol{x}$ also arises from a product of antisymmetric and symmetric tensors:

$$0 = \sum_{\boldsymbol{k},\boldsymbol{r},\boldsymbol{s}}\left[\sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\omega^*_{\boldsymbol{p}}\,\omega^*_{\boldsymbol{q}}\omega^*_{\boldsymbol{r}}\omega^*_{\boldsymbol{s}} + 2 \text{ other similar terms}\right].$$

However, the absence of an explicit $\omega_{\boldsymbol{k}}$ in the first term means that setting $\omega_{\boldsymbol{\ell}} = 0$ for $\ell > K$ breaks the symmetry in the summations. Nevertheless, since the missing terms involve $\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{q}}$ for p and q higher than the truncation wavenumber K, one might expect that a very well-resolved simulation would lead to almost exact invariance of Z_3. Indeed, we will see that this is the case.

In terms of the nonlinearity $S_{\boldsymbol{k}} \doteq \sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\omega^*_{\boldsymbol{p}}\,\omega^*_{\boldsymbol{q}}$, the enstrophy spectrum $Z(k)$ is seen to satisfy a balance equation of the form

$$\frac{\partial}{\partial t}Z(k) + 2\nu k^2 Z(k) = 2T(k) + G(k),$$

where $T(k)$ and $G(k)$ are the angular averages of $\mathrm{Re}\,\langle S_{\boldsymbol{k}}\omega^*_{\boldsymbol{k}}\rangle$ and $\mathrm{Re}\,\langle f_{\boldsymbol{k}}\omega^*_{\boldsymbol{k}}\rangle$, respectively. It is convenient to define the *nonlinear enstrophy transfer function* $\Pi(k)$, which measures the cumulative nonlinear transfer of enstrophy into $[k,\infty)$:

$$\Pi(k) = 2\int_k^\infty T(p)\,dp.$$

On integrating from k to ∞, we find

$$\frac{d}{dt}\int_k^\infty Z(p)\,dp = \Pi(k) - \epsilon(k),$$

where $\epsilon(k) \doteq 2\nu \int_k^\infty p^2 Z(p)\,dp - \int_k^\infty G(p)\,dp$ is the total enstrophy transfer, *via* dissipation and forcing, *out* of wavenumbers higher than k. A positive (negative) value for $\Pi(k)$ represents a flow of enstrophy to wavenumbers higher (lower) than k. When $\nu = f_{\boldsymbol{k}} = 0$ enstrophy conservation implies that

$$0 = \frac{d}{dt}\int_0^\infty Z(p)\,dp = 2\int_0^\infty T(p)\,dp,$$

so that

$$\Pi(k) = 2\int_k^\infty T(p)\,dp = -2\int_0^k T(p)\,dp. \tag{3}$$

We note that $\Pi(0) = \Pi(\infty) = 0$. Moreover, in a steady state, $\Pi(k) = \epsilon(k)$; this provides an excellent numerical diagnostic for validating a steady state.

The cumulative nonlinear enstrophy transfer Π_3 for the globally integrated invariant $Z_3 = \int \omega^3\,d\boldsymbol{x}$ can be defined similarly and measured numerically. However, we found no systematic cascade: Z_3 appears to slosh back and forth between the large and small scales. In hindsight, this should be expected since ω^3 is not a sign-definite quantity.

Of much more interest is the determination from a pseudospectral code of the cascade direction of a sign-definite quantity like the fourth-order Casimir invariant $Z_4 \doteq \int \omega^4\,d\boldsymbol{x}$. If we Fourier decompose $Z_4 = N^3 \sum_{\boldsymbol{j}} \omega^4(x_{\boldsymbol{j}})$ in terms of N spatial collocation points $x_{\boldsymbol{j}}$, we find

$$Z_4 = \sum_{\boldsymbol{k},\boldsymbol{p}} \omega_{\boldsymbol{k}}\,\omega_{\boldsymbol{p}}\,\omega_{\boldsymbol{q}}\,\omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}}.$$

In terms of the nonlinear source term $S_{\boldsymbol{k}}$, the evolution of Z_4 follows

$$\begin{aligned}
\frac{d}{dt}Z_4 &= \sum_{\boldsymbol{k}} \left[S_{\boldsymbol{k}} \sum_{\boldsymbol{p},\boldsymbol{q}} \omega_{\boldsymbol{p}}\,\omega_{\boldsymbol{q}}\,\omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} + 3\omega_{\boldsymbol{k}} \sum_{\boldsymbol{p},\boldsymbol{q}} S_{\boldsymbol{p}}\,\omega_{\boldsymbol{q}}\,\omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} \right] \\
&= N^2 \sum_{\boldsymbol{k}} \left[S_{\boldsymbol{k}} \sum_{\boldsymbol{j}} \omega^3(x_{\boldsymbol{j}}) e^{2\pi i \boldsymbol{j}\cdot\boldsymbol{k}/N} + 3\omega_{\boldsymbol{k}} \sum_{\boldsymbol{j}} S(x_{\boldsymbol{j}})\omega^2(x_{\boldsymbol{j}}) e^{2\pi i \boldsymbol{j}\cdot\boldsymbol{k}/N} \right] \\
&\doteq \sum_{\boldsymbol{k}} T_4(k).
\end{aligned} \tag{4}$$

To determine the cascade direction of Z_4, we considered a double-periodic pseudospectral simulation forced at wavenumber 2, with the dissipation

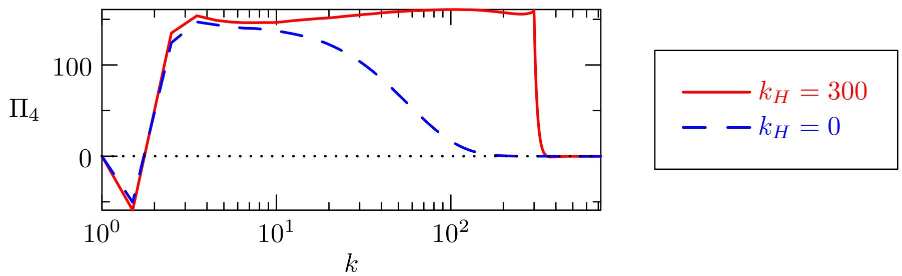

Fig. 1. Downscale nonlinear transfer Π_4 of Z_4 averaged over $t \in [200, 450]$.

term νk^2 replaced by $\nu k^2 H(k - k_H)$, where H is the Heaviside step function. A positive cutoff k_H mimics a pristine inertial range, à la Kolmogorov. In Fig. 1, we see that the time-averaged nonlinear transfer Π_4 of Z_4 exhibits the clear signature of a downward cascade (positive Π_4 in the enstrophy inertial range) at small scales. As a check that sufficient numerical resolution has been used to resolve the contribution of the nonlinear terms to the evolution of Z_4, we note that $\Pi(0) = \Pi(\infty) = 0$, as desired. An important point to emphasize in computing Z_4 is that (4) requires the computation of a double convolution, in terms of the Fourier transform of the cubic quantity ω^3. Correctly dealiasing therefore requires a 2/4 zero padding rule (instead of the usual 2/3 rule for a quadratic convolution). This means that even though a 2048×2048 pseudospectral simulation was used, the maximum physical wavenumber retained in each direction was 512.

We also point out an important distinction between nonlinear enstrophy *transfer* and *flux*. The mean rate of enstrophy transfer to $[k, \infty)$ is given by (3). In a steady state, $\Pi(k)$ will thus trivially be constant throughout an inertial range. In contrast, the enstrophy flux through a wavenumber k, as considered by Kolmogorov, is the amount of enstrophy transferred to small scales *via* triad interactions involving mode k. Independence of the flux on k is highly nontrivial, based on the conjectured self-similarity of the inertial range.

Even though higher-order Casimir invariants do not survive wavenumber truncation, it is appears possible, with sufficiently well-resolved simulations, to check whether they cascade to large or small scales. In this work, we computed the transfer function of the globally integrated ω^4 inviscid invariant and provided strong numerical evidence supporting Eyink's conjecture that in the enstrophy inertial range there is a direct cascade of (positive-definite) high-order invariants to small scales.

References

1. A.M. Polyakov, PUPT–1369 (1992).
2. Gregory L. Eyink, Physica D 91 (1996) 97–142.