On the Dual Cascade in Two-Dimensional Turbulence

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Abstract

We study the dual cascade scenario for two-dimensional turbulence driven by a spectrally localized forcing applied over a finite wavenumber range $[k_{\min}, k_{\max}]$ (with $k_{\min} > 0$) such that the respective energy and enstrophy injection rates ϵ and η satisfy $k_{\min}^2\epsilon \leq \eta \leq k_{\max}^2\epsilon$. The classical Kraichnan–Leith–Batchelor paradigm, based on the simultaneous conservation of energy and enstrophy and the scale-selectivity of the molecular viscosity, requires that the domain be unbounded in both directions. For two-dimensional turbulence either in a doubly periodic domain or in an unbounded channel with a periodic boundary condition in the across-channel direction, a direct enstrophy cascade is not possible. In the usual case where the forcing wavenumber is no greater than the geometric mean of the integral and dissipation wavenumbers, constant spectral slopes must satisfy $\beta > 5$ and $\alpha + \beta \geq 8$, where $-\alpha$ $(-\beta)$ is the asymptotic slope of the range of wavenumbers lower (higher) than the forcing wavenumber. The influence of a large-scale dissipation on the realizability of a dual cascade is analyzed. We discuss the consequences for numerical simulations attempting to mimic the classical unbounded picture in a bounded domain.

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1 Introduction

Since Kraichnan [13,14], Batchelor [2], and Leith [15] (referred to as KLB) adapted Kolmogorov's theory of self-similarity in three-dimensional turbulence to two-dimensional (2D) fluids, the conventional wisdom for decades has been that 2D turbulence simultaneously exhibits a direct cascade of enstrophy to large wavenumbers, up to a dissipation wavenumber k_ν, and an inverse cascade of energy to small wavenumbers, down to wavenumber $k = 0$. In the limit of small viscosity, the inverse cascade is thought to proceed indefinitely in time to ever-larger scales, transferring virtually all of the energy input to wavenumber zero. The direct cascade is thought to come into balance with viscosity, transferring virtually all of the enstrophy input to k_ν, where it will then be dissipated. In the long-time limit, a quasi-steady state is reached, in which two inertial ranges are established. According to the KLB theory, the energy range, which is only quasi-steady, scales as $k^{-5/3}$. The enstrophy range, which is in absolute equilibrium, should scale as k^{-3}.

The idea of a dual cascade was first suggested by Fjørtoft [10], who examined nonlinear transfer by individual interacting wavenumber triads. A later study by Merilees and Warn [23] provides more quantitative detail. They showed that

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roughly 70% (60%) of triads containing a given intermediate wavenumber k predominantly exchange energy (enstrophy) with lower (higher) wavenumbers. These analyses of nonlinear transfer, although carried out at the triad level, provide only a necessary basis for the KLB theory. The theory requires all triads to work collectively such that virtually all of the energy (enstrophy) input gets transferred to the large (small) scales; this is neither suggested nor implied by [10,23]. Nevertheless, a dual cascade from a spectrally localized initial spectrum is consistent with [10,23] and has been well confirmed by numerical simulations with various resolutions (e.g. Borue [4,5]; Frisch and Sulem [12]; Lilly [17]; Smith and Yakhot [34]). In particular, the inverse energy cascade is observed in the laboratory experiments of Dubos *et al.* [8] and Paret and Tabeling [26,27].[1] What has not been established beyond doubt is the realization of the inertial spectral scaling k^{-3}. In fact, there exist other theories that propose very different spectral slopes (Moffatt [24]; Saffman [31]; Sulem and Frisch [37]).

Numerical simulations, aiming to verify the KLB picture, face a number of formidable tasks. First, the simulations are performed for fluids in a rectangular box instead of an infinite domain. This turns out to be a serious shortcoming, as the equilibrium dynamics of a fluid in a doubly periodic domain differs considerably from that of an unbounded fluid in the KLB picture (see Constantin, Foias, and Manley [7]; Tran and Shepherd [39]). In particular, the k^{-3} range and the direct enstrophy cascade have been shown to be unreal-

[1] The inverse energy cascade and the $k^{-5/3}$ range are also seen to be robust in many numerical simulations, at least until the energy reaches the lower spectral boundary; however, the evidence for a direct enstrophy cascade is inconclusive (see the discussion in Paret and Tabeling [27] and references therein).

izable in a bounded domain. The main reason seems to be that in the bounded case, where an absolute equilibrium will be reached, there is no analogue of a persistent upscale flow of energy that eventually evades viscous dissipation altogether. To mimic the inverse cascade, one needs to introduce a dissipation that removes energy at the large scales (cf. [36]). Linear Ekman drag (proportional to k^0 and restricted to a few low wavenumbers) and inverse viscosity ($\nu_\mu k^{2\mu}$, with $\mu < 0$) have often been used for that purpose. Unfortunately, the inverse energy cascade, subject to this large-scale dissipation, carries with it a significant fraction of the enstrophy (in contrast to the asymptotic KLB inverse cascade, which carries no enstrophy). Moreover, there is no guarantee that the large-scale dissipation will absorb virtually all of the energy input. This is crucial in the KLB picture, as any fraction of the energy input that gets reflected would ultimately be trapped in the inertial ranges. The trapped energy, being in a virtually inviscid region, would then considerably change the dynamics and the slopes of the inertial ranges. Second, testing the theory requires the achievement of high Reynolds numbers, but current computers are only able to resolve relatively low Reynolds numbers. To overcome these resolution limitations, researchers often resort to introducing a hyperviscosity $\nu_\mu k^{2\mu}$, where the degree of viscosity μ is greater than one. This numerical device helps to compress the dissipation range, allowing simulations to be performed at relatively low resolutions. It is hoped that the effect of this modification on the inertial-range dynamics is negligible. However, a significant dissipation of energy at the small scales is inevitable in all numerical schemes.

In this study, we establish a theoretical basis highlighting the dynamical differences between bounded and unbounded 2D turbulence. The consequences for numerical simulations that aim to verify the KLB theory will be addressed.

In particular, we argue that instead of the familiar inertial-range spectral scalings $k^{-5/3}$ and k^{-3} conjectured by KLB, one would expect steeper scalings, such as k^{-3} and k^{-5}, respectively, in these ranges for bounded systems in equilibrium. In fact, it is shown in [39] that the enstrophy range must be steeper than k^{-5}; the physical implication is that no direct enstrophy cascade is possible. This result is easily generalized, using the Poincaré inequality, to turbulence in semi-unbounded 2D fluids, i.e. fluids confined to unbounded channels with a periodic boundary condition in the across-channel direction. Moreover, we investigate the effectiveness of using a large-scale dissipation to obtain a dual cascade in a bounded fluid. This information should be particularly useful for numerical simulations.

In Section 2, we summarize the KLB theory and contrast it to the dynamics of a bounded fluid. We also show that the theory does not apply to a fluid in an infinite channel with a periodic boundary condition in the across-channel direction. In Section 3, we review a result in [39] related to the unrealizability of a direct enstrophy cascade and discuss the fundamental differences between the dynamics of bounded fluids in equilibrium and that of unbounded fluids (or bounded fluids in transient phase) in the KLB picture. In Section 4, we derive a new constraint on the spectral slope of the energy range for bounded fluids in equilibrium and a condition for a persistent inverse energy cascade for unbounded fluids. In Section 5, we point out some effects of including a large-scale dissipation and the implications for numerical simulations attempting to verify the KLB picture. We conclude with some remarks in the final section. Further estimates on spectral slopes are given in the appendices.

2 KLB picture for unbounded 2D fluids

The evolution of the ensemble-averaged energy spectrum $E(k)$, which represents the energy density associated with the wavenumber k, is governed by (see Frisch [11] and Kraichnan [13])

$$\frac{d}{dt}E(k) = T(k) - 2\nu k^2 E(k) + F(k). \tag{1}$$

Here $T(k)$ and $F(k)$ are, respectively, the ensemble-averaged energy transfer and energy input rate and ν is the kinematic viscosity coefficient. Since waves of the same scale do not nonlinearly interact, $T(k)$ is linear in the modal component corresponding to wavenumber k. Moreover, $T(k)$ satisfies, by virtue of energy and enstrophy conservation,

$$\int_0^\infty T(k)\,dk = \int_0^\infty k^2 T(k)\,dk = 0. \tag{2}$$

For a fluid in a doubly periodic domain, which for convenience we call a bounded fluid, the integral in (2) and elsewhere in this section should be replaced by a discrete sum over all wavenumbers. This constraint on $T(k)$ imposes certain restrictions on its distribution and is thought to give rise to the dual cascade, believed to be a distinct feature of 2D turbulence.

We multiply (1) by k^2 and integrate both the original and resulting equations over all wavenumbers, noting from (2) that the nonlinear terms drop out, to obtain evolution equations for the total energy density $E = \int_0^\infty E(k)\,dk$ and enstrophy density $Z = \int_0^\infty k^2 E(k)\,dk$,

$$\frac{d}{dt}E = -2\nu Z + \epsilon, \tag{3}$$

$$\frac{d}{dt}Z = -2\nu P + \eta, \tag{4}$$

in terms of the energy and enstrophy injection rates $\epsilon = \int_0^\infty F(k)\,dk$ and $\eta = \int_0^\infty k^2 F(k)\,dk$, respectively, and the palinstrophy density $P = \int_0^\infty k^4 E(k)\,dk$. We assume that the forcing is spectrally localized to a wavenumber interval $[k_{\min}, k_{\max}]$ in the sense that

$$0 \le k_{\min}^2 \epsilon \le \eta \le k_{\max}^2 \epsilon. \tag{5}$$

This hypothesis is employed in [39]. This is a classical (although not exclusive) scenario for the KLB theory (cf. Kraichnan [13], p. 1421b; Pouquet et al. [28], p. 314; and Lesieur [16], p. 291), and is furthermore a common setup in numerical simulations of forced 2D turbulence (cf. Lilly [18]; Basdevant et al. [1]; Shepherd [33]). This assumption seems plausible for time-independent or white-noise forcing over $[k_{\min}, k_{\max}]$, given the assemble-averaged nature of ϵ and η. In particular, a monoscale forcing at a wavenumber s satisfies (5) with $k_{\min} = k_{\max} = s$ for each individual realization. Another example of such a forcing over $[k_{\min}, k_{\max}]$ is described in [39]; in each realization it yields time-independent energy and enstrophy injection rates ϵ and η such that $\eta = s^2\epsilon$, where s^2 is the mean of k^2 over $[k_{\min}, k_{\max}]$. A similar forcing was used by Shepherd [33] in a study of 2D turbulence in a large-scale zonal jet on the so-called beta-plane. In these examples, the characteristic forcing wavenumber s is constant in time.

The dual cascade scenario can be best appreciated if one examines the evolution equation

$$s^2 \frac{d}{dt}E - \frac{d}{dt}Z = 2\nu(P - s^2 Z), \tag{6}$$

obtained from (3) and (4). Here s is the forcing wavenumber $\sqrt{\eta/\epsilon}$ (for $\epsilon > 0$), which according to hypothesis (5), must lie in the interval $[k_{\min}, k_{\max}]$. If $\epsilon = 0$, then $\eta = 0$ and we take s to be any wavenumber in $[k_{\min}, k_{\max}]$. A direct enstrophy cascade requires that the characteristic enstrophy dissipation wavenumber $\sqrt{P/Z}$ be much larger than the forcing wavenumber s; hence, the right-hand side of (6) must be positive. The positiveness of $P - s^2 Z$ implies the positiveness of the left-hand side of (6) as well. If, in accord with the quasi-steady KLB theory, the total enstrophy reaches a steady state, we deduce that dE/dt must be positive. Equation (6) reflects the fact that the total energy must increase without limit, due to the inverse energy cascade toward wavenumber zero. (In a bounded domain, Tran and Shepherd [39] showed that $P = s^2 Z$ and concluded from this that no direct cascade is possible.)

Equation (6) satisfies $P/Z \gg s^2$ not only for the KLB scaling $k^{-5/3}$ and k^{-3} but also for a rich variety of spectra. This condition only requires an energy spectrum shallower than k^{-5} for a sufficiently wide range of wavenumbers $k > s$, provided that the energy spectrum for $k \ll s$ is shallower than k^{-3}. (In Section 4, it is shown that the quantity $P - s^2 Z$ can be positive even if the energy spectrum for $k > s$ is steeper than k^{-5}. This allows for the possibility of an inverse energy cascade in the absence of a direct enstrophy cascade.) However, the KLB theory insists on the specific scalings $k^{-5/3}$ and k^{-3} (with a logarithmic correction proposed by Kraichnan [13,14] and further investigated by Bowman [6]), respectively, for the energy and enstrophy inertial ranges. The $k^{-5/3}$ scaling is analogous to the Kolmogorov spectrum for 3D turbulence; the k^{-3} scaling implies that successive octaves in the enstrophy range contain equal amounts of enstrophy, so that the enstrophy grows logarithmically with dissipation wavenumber k_ν. There exists a number of predictions for the numerical value of the enstrophy inertial range slope $-\beta$ in the literature. Saffman [31] proposes $\beta = 4$, while Moffat [24] favours a slightly smaller value: $\beta = 11/3$. Sulem and Frisch [37] instead propose the upper bound $\beta \leq 11/3$.

In the long-time limit, the KLB inverse energy cascade (or any inverse energy cascade with a spectral scaling shallower than k^{-3} near $k = 0$) carries no enstrophy with it. Therefore, the enstrophy necessarily approaches an absolute equilibrium in a quasi-steady state. As a consequence, (6) reduces to

$$s^2 \frac{d}{dt} E = 2\nu(P - s^2 Z). \tag{7}$$

This equation indicates a simple and interesting fact about the KLB theory for unbounded 2D turbulence: in a quasi-steady state (for which $dZ/dt = 0$), the strength of an inverse energy cascade (the rate of the energy growth), if realizable, is primarily determined by the rate $2\nu P$ of enstrophy dissipation.

Besides the simultaneous conservation of energy and enstrophy, other essential features of 2D turbulence that underly the KLB theory are the scale-selectivity of the molecular viscosity and the unboundedness of the domain (in both directions). Together, they give rise to an infinite reservoir of energy in the vicinity of $k = 0$ that allows for the possibility of the KLB inverse energy cascade (which contains no enstrophy if the spectrum near $k = 0$ is shallower than k^{-3}). The theory breaks down when either the scale-selectivity of the dissipation or the unboundedness of the domain is absent. We demonstrate the former case and the semi-bounded case below. The case of a fluid in a doubly periodic domain is studied in [39] and will be reviewed in the next section.

Consider (1) with the viscous dissipation term νk^2 replaced by a constant

$\sigma > 0$; this scale-neutral frictional dissipation is often called Ekman drag in geophysical contexts. Equations (3) and (4) become

$$\frac{d}{dt}E = -2\sigma E + \epsilon, \tag{8}$$

$$\frac{d}{dt}Z = -2\sigma Z + \eta, \tag{9}$$

which, for bounded injection rates ϵ and η, implies that both the energy and enstrophy are bounded. This simple fact precludes the KLB type of inverse energy cascade. (As argued below, an inverse energy cascade dissipated by the friction at the large scales is not possible either.) Moreover, upon applying (5), we find

$$\frac{d}{dt}(k_{\min}^2 E - Z) \leq -2\sigma(k_{\min}^2 E - Z), \tag{10}$$

$$\frac{d}{dt}(k_{\max}^2 E - Z) \geq -2\sigma(k_{\max}^2 E - Z). \tag{11}$$

Hence, in the limit $t \to \infty$ the following holds

$$k_{\min}^2 E - Z \leq 0 \leq k_{\max}^2 E - Z, \tag{12}$$

or equivalently,

$$k_{\min}^2 \leq \frac{Z}{E} \leq k_{\max}^2. \tag{13}$$

Equation (13) implies that the redistribution of energy and enstrophy obeys exactly the same constraint as that imposed on the energy and enstrophy injection rates. Now, the boundedness of energy (enstrophy) prohibits an infinitely wide range of wavenumbers $k < k_{\min}$ ($k > k_{\max}$) in which the energy spectrum can scale as k^{-1} (k^{-3}), as this would imply a logarithmic divergence of the energy (enstrophy) as $k \to 0$ ($k \to \infty$). In fact, an energy spectrum k^{-1} (k^{-3}) or steeper (shallower) on the large (small) scales is inconsistent with (13).

Spectra consistent with (13) require that the energy and enstrophy be primarily dissipated near the region of forcing; hence no inverse (direct) energy (enstrophy) cascade is possible.

Besides the boundedness of the energy density in the present case, the spectral distribution of energy and enstrophy obeying (13) is profoundly different from that of the classical picture. The energy range is seen to be much shallower than $k^{-5/3}$ for a friction that acts uniformly on all scales. What seems curious is that the slopes of the enstrophy range in both cases do not differ by much. In fact, one could argue that a k^{-3} enstrophy-range spectrum with a logarithmic correction is consistent with the constraint (13), provided that the spectrum in the energy range scales as k^{-1} (with a similar correction). Nevertheless, we see that the scale-selectivity of the molecular viscosity plays an important role in the dual cascade picture. Perhaps, instead of studying the 2D Navier–Stokes dual cascade, one could replace the molecular dissipation νk^2 in (1) by another scale-selective dissipation $\nu_\mu k^{2\mu}$ (with $\mu > 0$) and examine the realizability of a dual cascade in this hypothetical unbounded system.

Finally, it is interesting to note that the KLB picture is not realizable for a 2D Navier–Stokes fluid confined to an unbounded channel with a periodic boundary condition in the across-channel direction ($0 \le y \le L$) and vanishing velocity (and derivatives thereof) at $x = \pm\infty$. This system is also furnished with the zero-mean flow condition

$$\int_0^L \boldsymbol{u}(x,y,t)\,dy = \mathbf{0}, \tag{14}$$

where $\boldsymbol{u}(x,y,t)$ is the fluid velocity. To see why the dynamics of this system are incompatible with the KLB picture, consider the Poincaré inequality for this domain: there exists a constant $\lambda > 0$ such that

$$\lambda \int_0^\infty k^\mu E(k)\, dk \leq \int_0^\infty k^{2+\mu} E(k)\, dk, \tag{15}$$

where $\mu \geq 0$. The optimal value of the constant λ depends on L. Now, the energy equation (3) and enstrophy equation (4) can be bounded *via* (15) as follows

$$\frac{d}{dt} E \leq -2\nu\lambda E + \epsilon, \tag{16}$$
$$\frac{d}{dt} Z \leq -2\nu\lambda Z + \eta. \tag{17}$$

These inequalities render the boundedness of both energy and enstrophy. The boundedness of the energy rules out the persistent inverse cascade needed for the positiveness of $P - s^2 Z$; rather, an absolute equilibrium is more plausible for this system. This result is quite physically reasonable: if one visualizes the inverse cascade as a result of the coalescence of same-sign vortices to form ever-larger ones, the system will tend toward equilibrium as the radii of the vortices approach the channel width L. This dynamical behaviour is observed by Rutgers [30] in an experiment of turbulence in a long channel, where, as equilibrium is approached, the vortices grow until the channel width is reached.

Now, if an equilibrium is achieved, (6) necessarily reduces to

$$P - s^2 Z = 0. \tag{18}$$

This equation, which also applies to the case of bounded fluids in equilibrium considered in the next section, implies that the dissipation of enstrophy mainly occurs in the forcing region and admits much steeper spectra than the KLB spectrum.

Remark 1. Some theoretical studies of the 2D Navier–Stokes equations in unbounded domains make an *a priori* assumption that the Poincaré inequality holds (e.g. Rosa [29]; see also Temam [38], p. 307). Such studies automatically exclude the possibility of the KLB dynamics.

3 Dynamics of bounded fluids in equilibrium

In bounded systems, the energy is also in absolute equilibrium for arbitrary (but positive) viscosity coefficient ν, so that (18) holds in equilibrium. This implies that $P/Z = s^2$, ruling out the existence of an enstrophy inertial range, as argued in [39]. Moreover, the energy spectrum for $k > k_{\max}$ is steeper than k^{-5}. This constraint clearly indicates a dramatic departure from the KLB theory. It is consistent with numerous numerical results, in which large-scale vortices, known as coherent structures, are observed (see for example Borue [5]; McWilliams [21,22]; Santangelo, Benzi, and Legras [32]; Smith and Yakhot [34,35]); these are often blamed for causing spectra steeper than those predicted by KLB. Although the mechanism behind these structures is not fully understood, we argue that it is the steepness of the spectrum (steeper than k^{-5}) that allows coherent structures to form, rather than the other way around. There is no need to invoke coherent structures to explain steep spectra: the steepness arises merely as a consequence of global conservation laws, molecular viscosity, and a spectrally localized forcing. As a matter of fact, a small-scale spectrum steeper than k^{-5} implies that the large scales carry virtually all of the system's enstrophy. Hence, dynamics exhibiting strong large-scale structures on a much weaker turbulent background of noise are consistent with [39]. It should be noted that the small-scale spectrum only needs to be steeper than k^{-3} for most of the enstrophy to reside in the large scales. Hence, large-scale structures may also be observed in simulations where a slope between -3 and -5 in the enstrophy range can be achieved, using a large-scale dissipation (see next section). Also, if the spectrum on the large scales is shallower than k^{-3}, one may expect vortices comparable in size to the forcing scale to form, because most of the system's enstrophy is then distributed in that spectral region. This has previously been noted by Paret and Tabeling [27].

In Section 4, we show that (18), which may be rewritten as

$$\sum_{k<s}(s^2 - k^2)k^2E(k) = \sum_{k>s}(k^2 - s^2)k^2E(k), \tag{19}$$

implies the spectral exponents satisfy $\beta > 5$ and $\alpha + \beta \geq 8$, where $-\alpha$ ($-\beta$) is the asymptotic slope of the range of wavenumbers lower (higher) than the forcing wavenumber (in the usual case where the forcing wavenumber is no greater than the geometric mean of the integral and dissipation wavenumbers). For example, if the small-scale spectrum is approximately k^{-5}, then the large-scale spectrum should scale as k^{-3} (or steeper). This would be consistent with the observed k^{-3} spectrum for the large-scale dynamics of the atmosphere (Lilly and Peterson [19]). Now, a large-scale k^{-3} spectrum means that the enstrophy scales as k^{-1}; each octave in this range contains approximately the same amount of enstrophy. Therefore, the dissipation of energy is uniformly distributed among successive octaves in the energy range, so that no inverse energy cascade is possible. Nevertheless, a spectrum steeper than k^{-3} on the large scales is allowed by (19). This is more likely to occur if the small-scale spectrum is only marginally steeper than k^{-5} (see Section 4). Unlike the KLB inverse cascade, which carries virtually all of the injected energy to ever-larger scales, this inverse cascade, if realizable, would only carry a fraction of the energy input to the largest scales.

The dynamics of a bounded fluid in equilibrium is characteristically different from the quasi-steady KLB picture. There is an infinite energy reservoir at $k = 0$ in unbounded systems that is forever available to collect the energy transfer; this feature is absent in bounded fluids (and in unbounded fluids satisfying the Poincaré inequality). Hence, the departure should not come as a surprise. What seems ironic is that the enstrophy is only weakly dissipated in bounded fluids at high Reynolds numbers but strongly dissipated in the unbounded KLB case: in the bounded case the result $P/Z = s^2$ [39] implies that the enstrophy dissipation rate $2\nu P/Z$ becomes $2\nu s^2$, while in the KLB theory it is approximately $\nu k_\nu^2/\ln(k_\nu/s)$, which is much greater than $2\nu s^2$ since $k_\nu \gg s$.

There appears to be no simple generalization from the dynamics of a bounded fluid in absolute equilibrium to that of its unbounded classical counterpart and *vice versa*. The familiar reconciliation found in the literature is that the $k^{-5/3}$ range is modified or disrupted at the large scales when the inverse cascade reaches the lowest available wavenumber. If the KLB picture applies to non-equilibrium dynamics in a bounded system, v.z. before the inverse cascade gets reflected by the spectral boundary,[2] then a dramatic adjustment of the

[2] In a finite system, an inverse energy cascade carries a non-negligible amount of enstrophy. If an inverse energy cascade carrying virtually all of the energy input rate ϵ, reaches a wavenumber k_*, it would transfer enstrophy at the rate $k_*^2\epsilon$ to wavenumber k_*. Hence, the ratio of the enstrophy accompanying the inverse energy cascade to the enstrophy input is approximately given by k_*^2/s^2, independent of the viscosity ν.

spectrum has to occur as an equilibrium is approached (see Smith and Yakhot [35] for a discussion of the so-called finite-size effects). We have not yet investigated in detail how this adjustment takes place; however, it seems plausible that as the inverse cascade hits the spectral boundary and gradually loses its strength ($dE/dt \to 0$), a substantial amount of the energy gets bounced back to the forcing scale. As there is little dissipation at the large scales, growth of the energy spectrum in the energy range is inevitable. This growth proceeds until the large-scale energy spectrum is sufficiently excited so that the enstrophy dissipation occurs mainly in the vicinity of the forcing scale, whereupon a forced-dissipative equilibrium is reached. As $dE/dt \to 0$, the enstrophy cascade (if initially present) ceases since the quantity $P - s^2 Z$ on the right-hand side of (6) decreases to zero. As a consequence, a gradual steepening of the enstrophy-range spectrum takes place (whatever the spectral slope of the enstrophy range during the transient phase). In Section 4 and Appendix A, we establish that the sum of the steady-state spectral exponents in the energy and enstrophy ranges must asymptotically approach -8.

Remark 2. It should be emphasized that the possibility of a direct enstrophy cascade in a bounded fluid during the non-equilibrium phase cannot be ruled out (not to say, however, that it actually occurs). For the case of a monoscale time-independent forcing, it is shown in Tran and Shepherd [39] that a direct enstrophy cascade is not realizable on average, whether the average be taken on a chaotic trajectory, limit cycle, or on the entire global attractor. This result leaves only the possibility of a direct enstrophy cascade in a neighborhood of the resulting monoscale stationary solution, on its unstable manifold. In this region one simultaneously has $P - s^2 Z > 0$ and $s^2 E - Z > 0$ (see Tran and Shepherd [39] and also Tran, Shepherd, and Cho [40]); the former inequality prevents one from ruling out a direct enstrophy cascade. More quantitative determination of the quantity $P - s^2 Z$ (which is viscosity-dependent, see [40]) in this region will help assess the existence of the enstrophy cascade (and how this would depend on the viscosity) as a monoscale basic flow loses its stability.

It is interesting to note that a k^{-3} spectrum (or slightly steeper), subject to experimental error, is observed in the laboratory experiments of Paret, Jullien, and Tabeling [25], and Rutgers [30]. In these experiments, mechanical friction at the bottom and top boundaries (in particular with the air, as noted in [30]) of the fluid could be sufficiently strong to outplay viscosity, so that Ekman drag alone is essentially responsible for dissipation. This might explain the observed spectra, according to the discussion in the previous section. Moreover, the analysis of [39] suggests that a combination of strong Ekman drag and weak viscosity allows for such a shallow spectrum to be realizable. Thus, there is no contradiction between the observed spectra and the predicted k^{-5} spectrum for a Navier–Stokes fluid in equilibrium.

4 Constraints on constant spectral slopes

We now derive constraints on the spectral slopes for bounded fluids in equilibrium and for a dual cascade in unbounded fluids, particularly for a persistent inverse energy cascade. We assume that the quasi-steady (steady) spectrum for the unbounded (bounded) case can be approximated by

$$E(k) = \begin{cases} ak^{-\alpha} & \text{if } k_0 \le k < s, \\ bk^{-\beta} & \text{if } s \le k \le k_\nu, \end{cases} \tag{20}$$

where $a,\ b,\ \alpha,\ \beta$ are constants, k_0 is the lowest wavenumber in the energy range ($k_0 \to 0$ as $t \to \infty$ for the unbounded case), and k_ν is the highest wavenumber in the enstrophy range, beyond which the spectrum is supposed to be steeper than $k^{-\beta}$. In Appendix A, we show that the arguments below can even be extended to the more realistic case, where the inertial-range slopes depend on wavenumber.

The quantity $P - s^2 Z$ can then be estimated as

$$\begin{aligned}\int_{k_0}^{\infty} (k^2 - s^2) k^2 E(k)\, dk &\geq a \int_{k_0}^{s} (k^2 - s^2) k^{2-\alpha}\, dk + b \int_{s}^{k_\nu} (k^2 - s^2) k^{2-\beta}\, dk \\ &= a s^{5-\alpha} \int_{k_0/s}^{1} (\kappa^2 - 1)\kappa^{2-\alpha}\, d\kappa + b s^{5-\beta} \int_{s/k_\nu}^{1} (1 - \kappa^2)\kappa^{\beta-6}\, d\kappa \\ &= a s^{5-\alpha} \left(- \int_{k_0/s}^{1} (1-\kappa^2)\kappa^{2-\alpha}\, d\kappa + \int_{s/k_\nu}^{1} (1-\kappa^2)\kappa^{\beta-6}\, d\kappa \right),\end{aligned} \tag{21}$$

where the inequality results from dropping the spectral contribution beyond k_ν (which is considerable if $\beta \leq 5$), the second line is obtained by the respective changes of variables $\kappa = k/s$ and $\kappa = s/k$ in the two integrals on the right-hand-side of the first line, and the third line is obtained using the continuity relation $a s^{-\alpha} = b s^{-\beta}$.

In the bounded case the left-hand side of (21) vanishes. Therefore,

$$\int_{k_0/s}^{1} (1-\kappa^2)\kappa^{2-\alpha}\, d\kappa \geq \int_{s/k_\nu}^{1} (1-\kappa^2)\kappa^{\beta-6}\, d\kappa. \tag{22}$$

For a strong forcing at a relatively low wavenumber s, it is reasonable to assume that $k_0/s \geq s/k_\nu$. This requires $2-\alpha \leq \beta - 6$, as the integrals in (22) decrease if the corresponding powers of κ ($\beta - 6$ and $2 - \alpha$) increase. That is,

$\alpha + \beta \geq 8$.

On the other hand, the convergence of the right-hand integral in (22), as $s/k_\nu \to 0$, requires also that $\beta > 5$. This result was derived in [39], on the basis that the dissipation of enstrophy mainly occurs in the vicinity of the forcing scale. Now if $\beta = 5 + \delta$, where it may be plausible that $0 < \delta \ll 1$ for high Reynolds numbers, then $\alpha \geq 3 - \delta$. It thus seems possible to obtain $\alpha \approx 3$. In the limits $k_0/s \to 0$ and $s/k_\nu \to 0$, as is usual for high-Reynolds-number turbulence, the inequality $\alpha + \beta \geq 8$ approaches an equality.

Remark 3. We caution that replacing the molecular viscosity νk^2 by a general viscosity $\nu_\mu k^{2\mu}$ ($\mu \geq 0$) in the previous argument leads to the result $\alpha + \beta \geq 4 + 4\mu$. The significant dependence on μ of this constraint suggests that the introduction of a hyperviscosity could seriously alter the expected steady-state spectral slopes.

We now derive a condition for a persistent inverse energy cascade in the unbounded case. We assume $\alpha < 3$, in accord with the realization of an inverse cascade toward wavenumber $k = 0$ that carries no enstrophy with it. In the limits $k_0/s \to 0$ and $s/k_\nu \to 0$, the condition $P - s^2 Z > 0$ is guaranteed if $2 - \alpha > \beta - 6$ and $\alpha < 3$, or equivalently, $\alpha + \beta < 8$ and $\alpha < 3$. These constraints admit a variety of spectra for a quasi-steady state in which an inverse energy cascade to wavenumber zero, carrying with it virtually no enstrophy, and a direct enstrophy cascade to the dissipation wavenumber k_ν, carrying with it virtually no energy, are allowed. Note that the inverse cascade scenario cannot be ruled out even if $\beta > 5$, i.e, in the absence of a direct enstrophy cascade, when the inequality $\alpha + \beta < 8$ holds. For the KLB energy-range spectrum $k^{-5/3}$, it is interesting to note that this condition requires only $\beta < 19/3$.

We therefore suggest that an inverse energy cascade in the absence of a direct enstrophy cascade can be realizable for a wide range of (modest) Reynolds numbers.

Remark 4. If the molecular viscosity νk^2 is replaced by a hyperviscosity $\nu_\mu k^{2\mu}$ for $\mu > 1$ in an unbounded fluid, the condition for a persistent inverse energy cascade $P - s^2 Z > 0$ is replaced by $\int_0^\infty (k^2 - s^2) k^{2\mu} E(k)\, dk > 0$. This leads to $\alpha + \beta < 4 + 4\mu$. Of course, the condition $\alpha < 3$ is required for a zero-enstrophy-carrying inverse energy cascade.

5 Large-scale dissipation

In this section, we examine how a large-scale dissipation could be used to obtain a dual cascade, and in particular, a direct enstrophy cascade. Consider (1) in the bounded case, with a general dissipation:

$$\frac{d}{dt} E(k) = T(k) - D(k)E(k) + F(k), \tag{23}$$

where $D(k)$ is a non-negative function of k. Systems for which $D(k)$ vanishes in the intermediate wavenumber range, including the forcing region, and for which the boundedness of energy is not guaranteed (Eyink [9]) have previously been studied in the literature.

For a general $D(k)$, (6) becomes

$$s^2 \frac{d}{dt} E - \frac{d}{dt} Z = \sum_k (k^2 - s^2) D(k) E(k), \tag{24}$$

which, in equilibrium, reduces to the balance equation

$$\sum_k (k^2 - s^2) D(k) E(k) = 0. \tag{25}$$

A slightly different form of this equation is derived in [39]. Equation (25) implies that the energy-range spectrum is related to the enstrophy-range spectrum in an intimate manner. For a given $D(k)$, an increase of the energy in one range requires an increase of the energy in the other. Thus, a steeper (shallower) energy-range spectrum corresponds to a shallower (steeper) enstrophy-range spectrum, for fixed $E(s)$. Another obvious consequence of (25) is that a nontrivial equilibrium is not possible if $D(k)$ vanishes for all $k < s$.

We are interested in an expression for $D(k)$ that retains the usual molecular viscosity and includes a large-scale dissipation. Thus, we consider $D(k) = D_\ell(k) + 2\nu k^2$, where $D_\ell(k)$ is a non-negative function of k. This dissipation includes the physically relevant case in which $D_\ell(k)$ is a positive constant representing friction from the planetary boundary layer in the geophysical context. Equation (25) then becomes

$$2\nu(P - s^2 Z) = \sum_k (s^2 - k^2) D_\ell(k) E(k). \tag{26}$$

The left-hand side of (26) is the familiar term due to viscosity, which would vanish in the absence of the large-scale dissipation. It can now become positive since the right-hand side can be made positive in a variety of ways. Two popular forms of $D_\ell(k)$ used in numerical simulations are the inverse viscosity $D_\ell(k) = 2\nu_\mu k^{2\mu}$, for $\mu < 0$, and mechanical friction restricted to the largest scales (say $k < k_\ell$): $D_\ell(k) \propto H(k_\ell - k)$, where $H(k_\ell - k)$ is the Heaviside step function (see Maltrud and Vallis [20], for example). To facilitate the formation of an enstrophy cascade, one might try to maximize the right-hand side of (26). However, since we have no *a priori* control over $E(k)$ for different $D_\ell(k)$, it is not known how to maximize the product $D_\ell(k)E(k)$ for $k < s$. Nevertheless, restricting $D_\ell(k)$ to $k < s$ by setting $D_\ell(k) = 0$ for $k \geq s$ is reasonable since any non-zero contribution to the right-hand side of (26) beyond s is negative.

If a positive value for the right-hand side of (26) can be achieved, the quantity $P - s^2 Z$ will be large for sufficiently small ν. This may help break the constraint that the enstrophy range slope must be steeper than k^{-5} and allow for a direct enstrophy cascade. An inverse cascade should be realizable, as we expect most of the energy dissipation to occur at the large scales. Thus, a dual cascade is possible. However, this cannot be achieved without a cost, as $D_\ell(k)$, which dissipates energy at the rate $\sum_k D_\ell(k)E(k)$, also dissipates enstrophy on the large scales at the rate $\sum_k D_\ell(k)k^2E(k)$. But

$$k_0^2 \sum_k D_\ell(k)E(k) \leq \sum_k D_\ell(k)k^2E(k), \tag{27}$$

where k_0 is the lowest wavenumber, corresponding to the system size. Hence, if $\sum_k D_\ell(k)E(k)$ is comparable to the energy injection rate (which is ideally sought after in the spirit of the KLB theory), then the ratio of the enstrophy dissipation rate at the large scales to the enstrophy injection rate is greater than $k_0^2/k_{\max}^2$. This fraction of the enstrophy dissipation at the large scales may be small, but not negligible. By allowing the enstrophy to be transferred to the large scales, a non-negligible amount of enstrophy may be trapped in the forcing region. If this is the case, then the spectrum in the forcing region has to adjust dramatically (since ν is small and the large-scale dissipation is assumed to be weak around the forcing scale) to balance the trapped enstrophy. In Appendix B, we emphasize the difficulty of obtaining an enstrophy-range spectrum shallower than k^{-5} with a large-scale dissipation that is well separated from the forcing region.

Numerical simulations of 2D turbulence can resolve up to a certain wavenumber, say k_T. Therefore, there is always a finite amount of energy dissipation at the small scales. This is analogous to the dissipation of enstrophy at the large scales previously considered, due to the finite size of the domain. If $D_h(k)$ represents the small-scale dissipation coefficient, the enstrophy and energy dissipations on the small scales are respectively given by $\sum_k D_h(k)k^2E(k)$ and $\sum_k D_h(k)E(k)$. These quantities satisfy

$$\sum_k D_h(k)k^2E(k) \leq k_T^2 \sum_k D_h(k)E(k). \tag{28}$$

Hence, if the dissipation of enstrophy by $D_h(k)$ is comparable to the enstrophy injection rate (which is ideally sought after in the spirit of the KLB theory), then the ratio of the energy dissipation rate at the small scales to the energy injection rate is greater than $k_{\min}^2/k_T^2$. This is true for any $D_h(k) \geq 0$, including a hyperviscosity of arbitrary degree.

Intuitively, if a large-scale dissipation is extended to s, one would expect it to absorb the reflected energy and keep the spectrum in the forcing region from growing as $\nu \to 0$. For a strong forcing and strong large-scale dissipation $D_\ell(k)$ (confined to $k \leq s$), it may be hypothesized that the value of the right-hand side of (26) is unaffected as $\nu \to 0$, given all else fixed. If this is the case, the quantity $P - s^2Z$ grows as ν^{-1} and the ratio P/Z is given by

$$\frac{P}{Z} = s^2 + \frac{1}{2\nu Z}\sum_k (s^2 - k^2)D_\ell(k)E(k). \tag{29}$$

This makes $P/Z \to \infty$ as $\nu \to 0$, a favorable limit for a direct enstrophy cascade, with a spectrum shallower than k^{-5}. However, the resulting cascade would not have the physical significance of the KLB theory since the direct enstrophy cascade (regardless of the spectral slope) might only be marginal, with a significant fraction of the enstrophy dissipated in the energy range

($k \leq s$) due to the strong large-scale dissipation, contrary to the classical theory.

6 CONCLUSION

In this paper we have analysed the classical dual cascade theory of 2D turbulence in unbounded fluids formulated by Kraichnan [13,14], Leith [15], and Batchelor [2]. The main feature of the theory—the dual cascade—is contrasted to the behaviour of 2D turbulence in a region that satisfies the Poincaré inequality, such as a doubly periodic domain. It is shown that the dual cascade picture, if realizable, would strictly be an unbounded-system phenomenon. This important point is not adequately stressed and has often led to confusion in the literature. The familiar qualitative argument that the $k^{-5/3}$ range is modified or disrupted at the large scales when the inverse energy cascade reaches the largest available scale in a bounded system (assuming the applicability of the dual-cascade dynamics to the transient phase) is inadequate. Two-dimensional turbulence either in a doubly periodic domain or in an unbounded channel with a periodic boundary condition on the across-channel dimension does not behave in the manner predicted by KLB. In particular, the spectral slopes in such systems are found to satisfy $\beta > 5$ and $\alpha + \beta \geq 8$, where $-\alpha$ ($-\beta$) is the slope of the range of wavenumbers lower (higher) than the forcing wavenumber. This result is well supported by numerical simulations, which consistently find enstrophy-range spectra steeper than the KLB prediction and dynamically dominant large-scale structures (McWilliams [21,22]; Santangelo, Benzi, and Legras [32]). It may even explain the observed large-scale k^{-3} spectrum in the atmosphere (Lilly and Peterson[19], Boer and Shepherd

[3]).

We have shown that a dual cascade in unbounded fluids is possible if β satisfies $3 < \beta < 5$; this includes both the classical $\beta = 3$ scaling as an extreme limit, and also the theories of Saffman [31], Moffat [24], and Sulem and Frisch [37], which propose $\beta = 4$, $\beta = 11/3$, and $\beta \leq 11/3$, respectively. Moreover, in the absence of a direct enstrophy cascade ($\beta > 5$), an inverse energy cascade, corresponding to flows with low Reynolds numbers (which should be relatively easy to simulate) cannot be ruled out.

The fundamental difference between bounded and unbounded fluids is that there is an infinite energy reservoir in the unbounded case, which allows a persistent inverse energy cascade to ever-larger scales to form, so that the energy eventually evades viscous dissipation altogether. Provided that the spectrum near $k = 0$ is shallower than k^{-3}, the inverse cascade asymptotically carries no enstrophy. This luxury is a consequence of both the unboundedness of the domain (in both directions) and the scale-selectivity of the molecular viscosity. In addition to the simultaneous conservation of energy and enstrophy, these properties constitute the basic building blocks of the KLB theory. Another important hypothesis is the existence of a quasi-steady state. As long as an inverse cascade is realizable and a quasi-steady state can be established in an unbounded system, the cascade dynamics are fundamentally distinct from what occurs in a bounded fluid in equilibrium. There appears to be no sound basis for extending the results from one case to the other. Of course, in an unbounded fluid it is quite possible for an inverse energy cascade to exist without a corresponding direct enstrophy cascade; in this case, there might then be certain similarities between the dynamics of the bounded and unbounded systems.

The dissipation operator plays an important role in the spectral distribution of energy. This is especially apparent in the balance equation (25) for a fluid in a doubly periodic domain: the product of the energy spectrum and the spectral dissipation function in the energy and enstrophy ranges are intimately related. This information is important for numerical 2D turbulence simulations, where various dissipation mechanisms are employed: it should help researchers rule out certain spectral distributions and anticipate possible outcomes for a given dissipation mechanism. Finally, we showed that a large-scale dissipation could give rise to a direct enstrophy cascade since the quantity $P - s^2 Z$ could grow as $1/\nu$, until $P/Z \gg s^2$.

A Constraints on general spectral slopes

Strictly speaking, the spectrum (20) is too simplistic; actual spectral slopes will tend to vary monotonically with wavenumber (particularly in the enstrophy range, as one approaches the onset of the dissipation range). The arguments of Section 4 can be readily extended to more general spectra. Normalizing all wavenumbers so that $s = 1$, we express the energy spectrum as

$$E(k) = a \begin{cases} k^{-\alpha(k)} & \text{if } k_0 \le k < 1, \\ k^{-\beta(k)} & \text{if } 1 \le k < k_1, \\ k^{-\gamma(k)} & \text{if } k_1 \le k \le k_T, \end{cases} \tag{A.1}$$

with $\beta(k_1) = \gamma(k_1)$, where k_0 is the lower spectral cutoff wavenumber (determined by the domain size), k_1 is the highest wavenumber in the enstrophy range, and k_T is the highest retained (truncation) wavenumber. Equation (18), or equivalently (25), then appears as

$$\int_{k_0}^{1}(1-k^2)k^{2-\alpha(k)}dk = \int_{1}^{k_1}(k^2-1)k^{2-\beta(k)}dk + \int_{k_1}^{k_T}(k^2-1)k^{2-\gamma(k)}\,dk. \tag{A.2}$$

The change of variable $\kappa = 1/k$ in the integrals on the right-hand side yields

$$\int_{k_0}^{1}(1-\kappa^2)\kappa^{2-\alpha(\kappa)}d\kappa = \int_{1/k_1}^{1}(1-\kappa^2)\kappa^{\beta(1/\kappa)-6}d\kappa + \epsilon_\nu, \tag{A.3}$$

where $\epsilon_\nu = \int_{1/k_T}^{1/k_1}(1-\kappa^2)\kappa^{\gamma(1/\kappa)-6}\,dk \geq 0$. For bounded turbulence, we restrict our attention to the usual case where $k_1 \geq 1/k_0$. Since $\int_{1/k_1}^{1}(1-\kappa^2)\kappa^{\theta}d\kappa$ is a strictly decreasing function of θ, we find that the maximum slopes $\overline{\alpha} = \sup_k \alpha(k)$ and $\overline{\beta} = \sup_k \beta(k)$ satisfy

$$\begin{aligned}\int_{1/k_1}^{1}(1-\kappa^2)\kappa^{2-\overline{\alpha}}\,d\kappa &\geq \int_{1/k_1}^{1}(1-\kappa^2)\kappa^{2-\alpha(\kappa)}\,d\kappa \geq \int_{k_0}^{1}(1-\kappa^2)\kappa^{2-\alpha(\kappa)}\,d\kappa\\ &\geq \int_{1/k_1}^{1}(1-\kappa^2)\kappa^{\beta(1/\kappa)-6}\,d\kappa \geq \int_{1/k_1}^{1}(1-\kappa^2)\kappa^{\overline{\beta}-6}\,d\kappa.\end{aligned}$$

Hence $2-\overline{\alpha} \leq \overline{\beta}-6$; that is, $\overline{\alpha}+\overline{\beta} \geq 8$.

One can also obtain estimates for the minimum slopes $\underline{\alpha} = \inf_k \alpha(k)$ and $\underline{\beta} = \inf_k \beta(k)$, assuming $\gamma(k) \geq \underline{\beta}$:

$$\begin{aligned}\int_{k_0}^{1}(1-\kappa^2)\kappa^{2-\underline{\alpha}}\,d\kappa &\leq \int_{k_0}^{1}(1-\kappa^2)\kappa^{2-\alpha(\kappa)}\,d\kappa = \int_{1/k_1}^{1}(1-\kappa^2)\kappa^{\beta(1/\kappa)-6}\,d\kappa + \epsilon_\nu\\ &\leq \int_{1/k_T}^{1}(1-\kappa^2)\kappa^{\underline{\beta}-6}\,d\kappa.\end{aligned} \tag{A.4}$$

In the asymptotic limit as $k_0 \to 0$ and $k_T \to \infty$, we deduce $\underline{\alpha}+\underline{\beta} \leq 8$.

It is instructive to specialize these results to the case of constant slopes, where $\underline{\alpha} = \overline{\alpha} = \alpha$ and $\underline{\beta} = \overline{\beta} = \beta$ (e.g. if $k_1 \ll k_\nu$). For a bounded fluid satisfying

$k_1 \geq 1/k_0$, we then see that $\lim_{k_0 \to 0} \alpha + \beta = 8$.

B Spectral slopes for a system with an inverse viscosity

It was suggested in Section 4 that a large-scale dissipation, well separated from the forcing scale, may not give rise to the desired direct enstrophy cascade and the corresponding k^{-3} spectrum. To demonstrate this point, we consider the special form $D(k) = 2\nu' k^{-2} + 2\nu k^2$, with $\nu' s^{-2} = \nu s^2$, so that at the forcing scale, the inverse viscosity and the usual molecular viscosity have the same strength. Equation (25) becomes

$$\sum_k (k^2 - s^2)(\nu' k^{-2} + \nu k^2) E(k) = 0. \tag{B.1}$$

Assuming the spectral scaling (20) and following the steps leading to (22), we obtain

$$\int_{k_0/s}^{1} (1-\kappa^2)(\kappa^{-2}+\kappa^2)\kappa^{-\alpha}\, d\kappa \geq \int_{s/k_\nu}^{1} (1-\kappa^2)(\kappa^2+\kappa^{-2})\kappa^{\beta-4}\, d\kappa. \tag{B.2}$$

In the case $k_0/s \geq s/k_\nu$ it follows that $\alpha+\beta \geq 4$. In the limit $s/k_\nu \to 0$, we still require $\beta > 5$. For the KLB enstrophy-range spectrum to be realizable it is necessary that $\alpha \geq 1$; moreover, s/k_ν cannot be much smaller than k_0/s. The former condition does not seem to be plausible in the presence of an inverse viscosity, while the latter condition requires an unphysically narrow enstrophy range for a forcing at relatively small wavenumbers.

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