Math 655: Statistical Theories of Turbulence

Fall, 2004 Assignment 4

November 23, 2015 due December 14, 2015

1. Shell models are reduced models of turbulence formulated in Fourier space, where typical velocity Fourier amplitudes on each shell n are represented by a *single quantity* u_n. The shell wavenumbers k_n are often taken to scale geometrically, say $k_n = \lambda^n$:

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n = i k_n \sum_{l,m} A_{l,m} u_{n+l}^* u_{n+m}^* + F.$$

Here ν is the viscosity and F is an external forcing.

Suppose we choose N successive shells labelled $n = 0, 1, \ldots, N-1$ and adopt the Fourier space "boundary" conditions $u_0 = u_{N-1} = 0$. If one restricts the couplings to nearest neighbours only and enforces conservation of the energy

$$E \doteq \frac{1}{2} \sum_{n=0}^{N-1} |u_n|^2,$$

one obtains this generalization of the Desnyansky and Novikov [1974] (DN) model:

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n = i k_n (a_n u_{n-1}^{*2} - \lambda a_{n+1} u_n^* u_{n+1}^*) + i k_n (b_n u_{n-1}^* u_n^* - \lambda b_{n+1} u_{n+1}^{*2}),$$

where a_n and b_n are arbitrary coefficients.

(a) When $\nu = F = 0$ and a_n and b_n are independent of n, show that

$$u_n = A k_n^{-1/3}$$

is a fixed point of the generalized DN model.

(b) Show that even though this is a fixed point of the nonlinearity, in the absence of forcing and dissipation, the associated energy spectrum agrees with the Kolmogorov Law for forced-dissipative D-dimensional turbulence! What is the value of D?

2. Calculate appropriate parameters, run the simulations, and answer the questions described in the two-dimensional turbulence computer lab manual at `http://www.math.ualberta.ca/~bowman/m655/lab.pdf`