**Math 655: Statistical Theories of Turbulence**

Fall, 2015 Assignment 3

October 23, 2015 due November 20, 2015

1. (a) Let $f$ and $g$ be functions from $\mathbb{R}^n \to \mathbb{R}$ with Fourier transforms $f_{\boldsymbol{k}}$ and $g_{\boldsymbol{k}}$, respectively. Prove in $n$ dimensions that

$$\frac{1}{(2\pi)^n}\int \langle f_{\boldsymbol{k}} g_{\boldsymbol{k}}^* \rangle \, e^{i\boldsymbol{k}\cdot\boldsymbol{\ell}}\, d\boldsymbol{k} = \int \langle f(\boldsymbol{r}+\boldsymbol{\ell})g(\boldsymbol{r})\rangle \, d\boldsymbol{r}.$$

$$\frac{1}{(2\pi)^n}\int \langle f_{\boldsymbol{k}} g_{\boldsymbol{k}}^* \rangle \, e^{i\boldsymbol{k}\cdot\boldsymbol{\ell}}\, d\boldsymbol{k} = \frac{1}{(2\pi)^n}\int\!\!\int\!\!\int \langle f(\boldsymbol{r}')g(\boldsymbol{r})\rangle \, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'+i\boldsymbol{k}\cdot\boldsymbol{r}+i\boldsymbol{k}\cdot\boldsymbol{\ell}}\, d\boldsymbol{r}'\, d\boldsymbol{r}\, d\boldsymbol{k} = \int \langle f(\boldsymbol{r}+\boldsymbol{\ell})g(\boldsymbol{r})\rangle \, d\boldsymbol{r}.$$

(b) Consider the case where $f$ and $g$ are periodic on an $n$-dimensional cube $V$ of side $2\pi$ and introduce the Fourier coefficients

$$f_{\boldsymbol{k}} = \frac{1}{(2\pi)^n}\int_V f(\boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, d\boldsymbol{x}.$$

Use the discrete formulation of part(a) to derive the $n$-dimensional version of the *Wiener-Khinchin* formula [Frisch, p. 54].

Hints: The inverse Fourier theorem tells us that

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}} f_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}};$$

this result can be expressed either as

$$\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} = (2\pi)^n \delta(\boldsymbol{x}-\boldsymbol{x}').$$

or

$$\int_V e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}\, d\boldsymbol{x} = (2\pi)^n \delta_{\boldsymbol{k},\boldsymbol{k}'}.$$

The discrete Fourier version of part(a) reads

$$\sum_{\boldsymbol{k}} \langle f_{\boldsymbol{k}} g_{\boldsymbol{k}}^* \rangle \, e^{i\boldsymbol{k}\cdot\boldsymbol{\ell}} = \frac{1}{(2\pi)^n}\int_V \langle f(\boldsymbol{r}+\boldsymbol{\ell})g(\boldsymbol{r})\rangle \, d\boldsymbol{r}.$$

On setting $f = g$ and taking the discrete Fourier transform, we find

$$\frac{1}{(2\pi)^n}\sum_{\boldsymbol{k}}\int_V \langle |f_{\boldsymbol{k}}|^2\rangle \, e^{i\boldsymbol{k}\cdot\boldsymbol{\ell}+i\boldsymbol{p}\cdot\boldsymbol{\ell}}\, d\boldsymbol{\ell} = \langle |f_{\boldsymbol{p}}|^2\rangle = \frac{1}{(2\pi)^{2n}}\int_V\int_V \langle f(\boldsymbol{r}+\boldsymbol{\ell})f(\boldsymbol{r})\rangle \, e^{i\boldsymbol{p}\cdot\boldsymbol{\ell}}\, d\boldsymbol{r}\, d\boldsymbol{\ell}$$

Under the assumption of homogeneity, we have $\langle f(\boldsymbol{r}+\boldsymbol{\ell})f(\boldsymbol{r})\rangle = \Gamma(\boldsymbol{\ell})$. Since $\int_V d\boldsymbol{r} = (2\pi)^n$, we deduce

$$\langle |f_{\boldsymbol{p}}|^2\rangle = \frac{1}{(2\pi)^n}\int \Gamma(\boldsymbol{\ell})e^{i\boldsymbol{p}\cdot\boldsymbol{\ell}}\, d\boldsymbol{\ell}.$$

2. Show that even though Burgers equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2},$$

is nonlinear, it can be reduced to the solution of a (linear) heat equation by the transformation

$$u(x,t) = -2\nu\frac{\partial}{\partial x}\log h(x,t).$$

First, rewrite Burgers equation as

$$u_t + \frac{1}{2}(u^2)_x = \nu u_{xx}.$$

We are asked to try to express the solution in terms of a potential $\phi = -2\nu\log h$ (which we assume to be $C^2$), such that $u = \phi_x$:

$$\phi_{xt} + \frac{1}{2}(\phi_x^2)_x = \nu\phi_{xxx}.$$

On integrating with respect to $x$, we find

$$\phi_t + \frac{1}{2}(\phi_x^2) = \nu\phi_{xx} + C(t),$$

where $C(t)$ is an unknown function of $t$. Now $\phi_t = -2\nu h_t/h$, $\phi_x = -2\nu h_x/h$ and

$$\phi_{xx} = \frac{2\nu}{h^2}h_x^2 - \frac{2\nu}{h}h_{xx}.$$

Thus

$$C(t) = \phi_t + \frac{1}{2}(\phi_x^2) - \nu\phi_{xx} = -\frac{2\nu h_t}{h} + \frac{2\nu^2 h_x^2}{h^2} - \frac{2\nu^2}{h^2}h_x^2 + \frac{2\nu^2}{h}h_{xx} = -\frac{2\nu h_t}{h} + \frac{2\nu^2}{h}h_{xx},$$

which simplifies to the linear equation

$$h_t + \frac{C(t)}{2\nu}h = \nu h_{xx}$$

If we let $H(x,t) = h(x,t)e^{\int C(t)\,dt/(2\nu)}$, we obtain a heat equation for $H(x,t)$,

$$H_t = \nu H_{xx},$$

which can then be solved to find $u(x,t)$:

$$u(x,t) = -2\nu\frac{\partial}{\partial x}\log h(x,t) = -2\nu\frac{\partial}{\partial x}\log H(x,t).$$

Note that $u$ does not actually depend on the function $C(t)$.

This remarkable result is known as the Cole-Hopf transformation (Forsyth 1906, Cole 1950, Hopf 1952).

3. Let $v_x$ represent any particular component of the velocity for homogeneous isotropic three-dimensional turbulence. Show that

$$\left\langle v_x^2 \right\rangle = \frac{2E}{3}, \qquad \left\langle \left( \frac{\partial v_x}{\partial x} \right)^2 \right\rangle = \frac{2}{15} Z,$$

$$\lambda^2 = \frac{5E}{Z}, \qquad R_\lambda = \sqrt{\frac{10}{3}} \frac{E}{Z^{1/2} \nu},$$

where $E$ is the turbulent energy, $Z$ is the turbulent enstrophy,

$$\lambda = \left( \frac{\langle v_x^2 \rangle}{\left\langle \left( \frac{\partial v_x}{\partial x} \right)^2 \right\rangle} \right)^{1/2}$$

is the *Taylor microscale*, and $R_\lambda = \lambda \sqrt{\langle v_x^2 \rangle} / \nu$ is the *Taylor Reynolds number*.

Hints: Use the properties of the partial derivatives of the homogeneous isotropic correlation tensor $\Gamma_{kk'}(\boldsymbol{\ell}) = \langle v_k(\boldsymbol{x}) v_{k'}(\boldsymbol{x} + \boldsymbol{\ell}) \rangle$. For example:

$$\left\langle \frac{\partial v_k(\boldsymbol{x})}{\partial x_j} \frac{\partial v_{k'}(\boldsymbol{x} + \boldsymbol{\ell})}{\partial x_{j'}} \right\rangle = \frac{\partial}{\partial \ell_{j'}} \left\langle \frac{\partial v_k(\boldsymbol{x})}{\partial x_j} v_{k'}(\boldsymbol{x} + \boldsymbol{\ell}) \right\rangle = \frac{\partial}{\partial \ell_{j'}} \left\langle \frac{\partial v_k(\boldsymbol{x} - \boldsymbol{\ell})}{\partial x_j} v_{k'}(\boldsymbol{x}) \right\rangle$$
$$= -\frac{\partial}{\partial \ell_j} \frac{\partial}{\partial \ell_{j'}} \langle v_k(\boldsymbol{x} - \boldsymbol{\ell}) v_{k'}(\boldsymbol{x}) \rangle = -\frac{\partial}{\partial \ell_j} \frac{\partial}{\partial \ell_{j'}} \Gamma_{kk'}.$$

Also, parity requires that $\Gamma_{kj}(\boldsymbol{\ell}) = \langle v_k(0) v_j(-\boldsymbol{\ell}) \rangle = \langle v_j(0) v_k(\boldsymbol{\ell}) \rangle = \Gamma_{jk}(\boldsymbol{\ell})$ and incompressibility implies that

$$\frac{\partial}{\partial \ell_k} \Gamma_{jk}(\boldsymbol{\ell}) = \frac{\partial}{\partial \ell_k} \Gamma_{kj}(\boldsymbol{\ell}) = 0.$$

For isotropic turbulence $\left\langle v_x^2 \right\rangle = \left\langle v_y^2 \right\rangle = \left\langle v_z^2 \right\rangle$. We then see immediately that $2E = 3 \left\langle v_x^2 \right\rangle$ and hence $\left\langle v_x^2 \right\rangle = 2E/3$.

On denoting $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{\ell}$, $\boldsymbol{v}' = \boldsymbol{v}(\boldsymbol{x}')$, and $\boldsymbol{\omega}' = \boldsymbol{\omega}(\boldsymbol{x}')$, we find

$$2Z = \left\langle \omega_i \omega_i' \right\rangle = \epsilon_{ijk} \epsilon_{ij'k'} \left\langle \frac{\partial v_k}{\partial x_j} \frac{\partial v_{k'}'}{\partial x_{j'}'} \right\rangle = \left\langle \frac{\partial v_k}{\partial x_j} \frac{\partial v_k'}{\partial x_j} \right\rangle - \left\langle \frac{\partial v_k}{\partial x_j} \frac{\partial v_j'}{\partial x_k} \right\rangle$$
$$= -\frac{\partial}{\partial \ell_j} \frac{\partial}{\partial \ell_j} \Gamma_{kk}(\boldsymbol{\ell}) + \frac{\partial}{\partial \ell_j} \frac{\partial}{\partial \ell_k} \Gamma_{kj}(\boldsymbol{\ell}) = -\frac{\partial}{\partial \ell_j} \frac{\partial}{\partial \ell_j} \Gamma_{kk}(\boldsymbol{\ell}).$$

The most general isotropic form for $\Gamma_{ij}$ is

$$\Gamma_{ij}(\boldsymbol{\ell}) = A(\ell) \delta_{ij} + B(\ell) \ell_i \ell_j,$$

but incompressibility implies that

$$0 = \frac{\partial}{\partial \ell_i}\Gamma_{ij}(\boldsymbol{\ell}) = A'(\ell)\frac{\ell_i}{\ell}\delta_{ij} + B'(\ell)\ell\ell_j + B(\ell)(\delta_{ii}\ell_j + \ell_i\delta_{ij}) = \ell_j\left(\frac{A'}{\ell} + B'\ell + 4B\right).$$

On multiplying by $\ell_j$ and summing over $j$, we find that $0 = A' + B'\ell^2 + 4B\ell$. By differentiating this result with respect to $\ell$ and taking the limit as $\ell \to 0$, we deduce

$$0 = A''(0) + 4B(0).$$

We need to find the Laplacian of $\Gamma_{ii}(\boldsymbol{\ell}) = A(\ell)\delta_{ii} + B(\ell)\ell_i\ell_i = 3A(\ell) + B(\ell)\ell^2$.

That is, we want to compute

$$\begin{aligned} -2Z &= \lim_{\ell\to 0}\frac{1}{\ell^2}\frac{\partial}{\partial \ell}\ell^2\frac{\partial \Gamma_{ii}}{\partial \ell} = \lim_{\ell\to 0}\frac{1}{\ell^2}\frac{\partial}{\partial \ell}\left(3A'\ell^2 + B'\ell^4 + 2B\ell^3\right) \\ &= \lim_{\ell\to 0}\frac{1}{\ell^2}\left(3A''\ell^2 + 6A'\ell + B''\ell^4 + 4B'\ell^3 + 2B'\ell^3 + 6B\ell^2\right) = 9A''(0) + 6B(0) = \frac{15}{2}A''(0), \end{aligned}$$

on noting that $A'(0) = 0$, since $A(\ell)$ is an even function, and $\lim\limits_{\ell\to 0} A'(\ell)/\ell = A''(0)$.

Finally, we note that since

$$-\left\langle\left(\frac{\partial v_x}{\partial x}\right)^2\right\rangle = \lim_{\ell\to 0}\frac{\partial^2}{\partial \ell^2}\Gamma_{00}(\ell\hat{\boldsymbol{x}}) = \lim_{\ell\to 0}\frac{\partial^2}{\partial \ell^2}\left(A + B\ell^2\right) = A''(0) + 2B(0) = \frac{A''(0)}{2},$$

it follows that $2Z = 15\left\langle\left(\frac{\partial v_x}{\partial x}\right)^2\right\rangle$.

The square of the Taylor microscale is then given by

$$\lambda^2 = \frac{\langle v_x^2\rangle}{\frac{2Z}{15}} = \frac{5E}{Z},$$

so that the Taylor Reynolds number is given by

$$\frac{1}{\nu}\sqrt{\frac{5E}{Z}}\sqrt{\frac{2E}{3}} = \sqrt{\frac{10}{3Z}}\frac{E}{\nu}.$$

4. A three-dimensional fluid subject to periodic boundary conditions is stirred by an isotropic Gaussian *white-noise* solenoidal force $\boldsymbol{f}$:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f},$$

such that the Fourier transform $\boldsymbol{f}_{\boldsymbol{k}}$ of $\boldsymbol{f}$ satisfies $\boldsymbol{k}\cdot\boldsymbol{f}_{\boldsymbol{k}}(t) = 0$ and

$$\langle \boldsymbol{f}_{\boldsymbol{k}}(t)\cdot\boldsymbol{f}^*_{\boldsymbol{k}'}(t')\rangle = F^2_{\boldsymbol{k}}\delta_{\boldsymbol{k}\boldsymbol{k}'}\delta(t - t'),$$

Compute the mean rate of energy injection per unit volume $\epsilon$ in terms of $F_{\boldsymbol{k}}$.

Hint: A solenoidal white-noise Gaussian forcing $\boldsymbol{f}_{\boldsymbol{k}}$ has the form $\boldsymbol{f}_{\boldsymbol{k}}(t) = N_{\boldsymbol{k}}\big(\mathbf{1} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}\big)\boldsymbol{\cdot}\boldsymbol{\xi}_{\boldsymbol{k}}(t)$, where $N_{\boldsymbol{k}}$ is real and $\boldsymbol{\xi}_{\boldsymbol{k}}$ is a unit central complex Gaussian random 3-vector: $\langle \boldsymbol{\xi}_{\boldsymbol{k}}(t)\boldsymbol{\xi}^*_{\boldsymbol{k}'}(t')\rangle = \delta_{\boldsymbol{k}\boldsymbol{k}'}\mathbf{1}\,\delta(t-t')$. This means that

$$\begin{aligned}\langle \boldsymbol{f}_{\boldsymbol{k}}(t)\boldsymbol{\cdot}\boldsymbol{f}^*_{\boldsymbol{k}'}(t')\rangle &= N_{\boldsymbol{k}}N_{\boldsymbol{k}'}\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)\langle \xi_{\boldsymbol{k}j}(t)\xi^*_{\boldsymbol{k}'j'}(t')\rangle\left(\delta_{j'i} - \frac{k'_{j'}k'_i}{k'^2}\right)\\ &= N^2_{\boldsymbol{k}}\delta_{\boldsymbol{k},\boldsymbol{k}'}\delta(t-t')\left(\mathbf{1} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}\right)\boldsymbol{:}\left(\mathbf{1} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}\right) = 2N^2_{\boldsymbol{k}}\delta_{\boldsymbol{k},\boldsymbol{k}'}\delta(t-t').\end{aligned}$$

We then see that $2N_k^2 = F_k^2$ (the factor of 2 signifies that there are only two independent directions, once the incompressibility constraint has been taken into account).

Since the nonlinear (advective and pressure) terms conserve energy, the energy balance reads

$$\left\langle \frac{1}{2V}\int \frac{\partial\,|\boldsymbol{u}|^2}{\partial t}\,d\boldsymbol{x}\right\rangle = \left\langle \frac{1}{V}\int \frac{\partial \boldsymbol{u}}{\partial t}\boldsymbol{\cdot}\boldsymbol{u}\,d\boldsymbol{x}\right\rangle = \epsilon - \left\langle \frac{1}{V}\int \nu\,|\boldsymbol{u}|^2\,d\boldsymbol{x}\right\rangle,$$

where $V$ denotes the volume of the periodic box and

$$\begin{aligned}\epsilon &= \frac{1}{V}\int \langle \boldsymbol{f}(\boldsymbol{x},t)\boldsymbol{\cdot}\boldsymbol{u}(\boldsymbol{x},t)\rangle\,d\boldsymbol{x} = \mathrm{Re}\sum_{\boldsymbol{k}}\langle \boldsymbol{f}_{\boldsymbol{k}}(t)\boldsymbol{\cdot}\boldsymbol{u}^*_{\boldsymbol{k}}(t)\rangle\\ &= \mathrm{Re}\sum_{\boldsymbol{k},\boldsymbol{k}'}\int \langle \boldsymbol{f}_{\boldsymbol{k}}(t)\boldsymbol{f}^*_{\boldsymbol{k}'}(t')\rangle\boldsymbol{:}\left\langle \frac{\delta \boldsymbol{u}^*_{\boldsymbol{k}}(t)}{\delta \boldsymbol{f}^*_{\boldsymbol{k}'}(t')}\right\rangle\,dt'.\end{aligned}$$

Following Novikov [1964], we express

$$\boldsymbol{u}_{\boldsymbol{k}}(t) = \boldsymbol{u}_{\boldsymbol{k}}(t') + \int_{t'}^{t} A_{\boldsymbol{k}}[\boldsymbol{u}(\bar{t})]\,d\bar{t} + \int_{t'}^{t}\boldsymbol{f}_{\boldsymbol{k}}(\bar{t})\,d\bar{t}$$

in terms of some unknown functional $A_{\boldsymbol{k}}$ of the velocity field $\boldsymbol{u}(t)$ at time $t$. If $A_{\boldsymbol{k}}$ has a bounded functional derivative, we can compute the nonlinear Green's function

$$\frac{\delta \boldsymbol{u}_{\boldsymbol{k}}(t)}{\delta \boldsymbol{f}_{\boldsymbol{k}'}(t')} = \int_{t'}^{t}\frac{\delta A_{\boldsymbol{k}}[\boldsymbol{u}(\bar{t})]}{\delta \boldsymbol{f}_{\boldsymbol{k}'}(t')}\,d\bar{t} + \int_{t'}^{t}\delta_{\boldsymbol{k}\boldsymbol{k}'}\mathbf{1}\delta(\bar{t}-t')\,d\bar{t} = \int_{t'}^{t}\frac{\delta A_{\boldsymbol{k}}[\boldsymbol{u}(\bar{t})]}{\delta \boldsymbol{f}_{\boldsymbol{k}'}(t')}\,d\bar{t} + \delta_{\boldsymbol{k}\boldsymbol{k}'}\mathbf{1}H(t-t').$$

Since $H(0) = \frac{1}{2}$, we obtain

$$\epsilon = \sum_{\boldsymbol{k}} N^2_{\boldsymbol{k}}\left(\mathbf{1} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}\right)\boldsymbol{:}\left(\mathbf{1} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}\right)H(0) = \frac{1}{2}\sum_{\boldsymbol{k}} 2N^2_{\boldsymbol{k}} = \frac{1}{2}\sum_{\boldsymbol{k}} F^2_{\boldsymbol{k}}.$$

5. Consider the two-point velocity increment correlation tensor

$$B_{ij}(\boldsymbol{\ell}) = \langle (v_i(\boldsymbol{r}) - v_i(\boldsymbol{r}+\boldsymbol{\ell}))(v_j(\boldsymbol{r}) - v_j(\boldsymbol{r}+\boldsymbol{\ell}))\rangle$$

of an incompressible three-dimensional homogeneous and isotropic turbulent velocity field.

(a) Using the fact that the most general isotropic second-rank tensor can be written in the form

$$B_{ij}(\boldsymbol{\ell}) = A(\ell)\delta_{ij} + B(\ell)n_i n_j,$$

where $\boldsymbol{n} = \boldsymbol{\ell}/\ell$ is the unit vector in the direction of $\boldsymbol{\ell}$, show that

$$B_{ij} = B_{TT}(\delta_{ij} - n_i n_j) + B_{LL} n_i n_j,$$

where $B_{LL}$ is the longitudinal mean-square velocity increment (corresponding to the direction of $\boldsymbol{\ell}$) and $B_{TT}$ is the transverse mean-square velocity increment (corresponding to the directions transverse to $\boldsymbol{\ell}$).

Since $n_L = 1$ and $n_T = 0$ we see that $B_{LL} = A + B$ and $B_{TT} = A$. Hence

$$B_{ij} = B_{TT}\delta_{ij} + (B_{LL} - B_{TT})n_i n_j,$$

(b) Show that

$$B_{ij}(\boldsymbol{\ell}) = \frac{2}{3}\left\langle v^2\right\rangle \delta_{ij} - 2\left\langle v_i(\boldsymbol{r})v_j(\boldsymbol{r}+\boldsymbol{\ell})\right\rangle$$

and hence

$$\frac{\partial B_{ij}}{\partial \ell_j} = 0.$$

Using homogeneity and the parity symmetry $\boldsymbol{x} \to -\boldsymbol{x}$, $\boldsymbol{v} \to -\boldsymbol{v}$, we simplify

$$\begin{aligned} B_{ij}(\boldsymbol{\ell}) &= \langle v_i(\boldsymbol{r})v_j(\boldsymbol{r})\rangle - \langle v_i(\boldsymbol{r}+\boldsymbol{\ell})v_j(\boldsymbol{r})\rangle - \langle v_i(\boldsymbol{r})v_j(\boldsymbol{r}+\boldsymbol{\ell})\rangle + \langle v_i(\boldsymbol{r}+\boldsymbol{\ell})v_j(\boldsymbol{r}+\boldsymbol{\ell})\rangle \\ &= 2\left\langle v_i(\boldsymbol{0})v_j(\boldsymbol{0})\right\rangle - 2\left\langle v_i(\boldsymbol{r})v_j(\boldsymbol{r}+\boldsymbol{\ell})\right\rangle \end{aligned}$$

Furthermore, the parity symmetry $x \to -x$, $v_x \to -v_x$ implies that $\langle v_x(\boldsymbol{0})v_y(\boldsymbol{0})\rangle = -\langle v_x(\boldsymbol{0})v_y(\boldsymbol{0})\rangle$, etc., so that $\langle v_i(\boldsymbol{0})v_j(\boldsymbol{0})\rangle = 2E\delta_{ij}/3$ (no summation), using our first result from Question 3. Since $E = \frac{1}{2}\left\langle \boldsymbol{v}^2\right\rangle$, we conclude that

$$B_{ij}(\boldsymbol{\ell}) = \frac{2}{3}\left\langle v^2\right\rangle \delta_{ij} - 2\left\langle v_i(0)v_j(\boldsymbol{\ell})\right\rangle$$

It is now clear that the incompressibility condition $\partial v_j(\boldsymbol{\ell})/\partial \ell_j = 0$ implies that $\partial B_{ij}/\partial \ell_j = 0$.

(c) Show that

$$\frac{1}{\ell}\frac{\partial}{\partial \ell}(\ell^2 B_{LL}) = 2B_{TT}.$$

Show that your result agrees with Eq. (12) in Kolmogorov's first 1941 paper, *The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers*. Note: The notation that Kolmogorov uses for the transverse and longitudinal components is defined in his third paper, *Dissipation of energy in the locally isotropic turbulence*. Kolmogorov used this result to calculate an expression for the mean energy dissipation rate in terms of the viscosity and the second derivative of $B_{LL}$.

On multiplying our result in part (a) by $n_i$ and summing over $i$, we find

$$n_i B_{ij} = B_{TT}(n_j - n_j) + B_{LL} n_j = B_{LL} n_j$$

On differentiating both sides with respect to $\ell_j$ we find

$$\frac{\partial n_i}{\partial \ell_j} B_{ij} + n_i \frac{\partial B_{ij}}{\partial \ell_j} = \frac{\partial}{\partial \ell_j}(B_{LL} n_j).$$

From part(c), we know that the second term on the left-hand side vanishes. Also,

$$\frac{\partial n_i}{\partial \ell_j} = \frac{\delta_{ij}}{\ell} - \frac{\ell_i}{\ell^2}\frac{\ell_j}{\ell} = \frac{1}{\ell}(\delta_{ij} - n_i n_j).$$

In particular $\partial n_j / \partial \ell_j = 2/\ell$. Using part(a), we then find that

$$\frac{1}{\ell}(\delta_{ij} - n_i n_j)[B_{TT}(\delta_{ij} - n_i n_j) + B_{LL} n_i n_j] = \frac{\partial B_{LL}}{\partial \ell} n_j n_j + \frac{2}{\ell} B_{LL}.$$

On using the facts that

$$\sum_{ij}(\delta_{ij} - n_i n_j)(\delta_{ij} - n_i n_j) = \sum_i \left(1 - n_i^2 - n_i^2\right) + \sum_i n_i^2 \sum_j n_j^2 = 3 - 1 - 1 + 1 = 2$$

and

$$\sum_{ij}(\delta_{ij} - n_i n_j) n_i n_j = \sum_i n_i^2 - \sum_i n_i^2 \sum_j n_j^2 = 1 - 1 = 0,$$

we finally obtain Kolmogorov's result ($-\bar{B}$ in Kolmogorov's notation translates to $B_{TT} - B_{LL}$):

$$\frac{2}{\ell} B_{TT} = \frac{\partial B_{LL}}{\partial \ell} + \frac{2}{\ell} B_{LL} = \frac{1}{\ell^2}\frac{\partial}{\partial \ell}(\ell^2 B_{LL}).$$