Math 655: Statistical Theories of Turbulence

Fall, 2015 Assignment 2

September 25 2015, due October 26

1. Use the phenomenological arguments of Kolmogorov and Kraichnan to determine the exponent of the inertial-range energy spectrum power law consistent with a cascade characterized by k-independent helicity transfer.

   The helicity transfer is proportional to the quantity

   $$\bar{\Pi}_H(k) \doteq \left[\int_0^k \bar{k}^2 E(\bar{k})\, d\bar{k}\right]^{1/2} k^2 E(k).$$

   Let $f(k) = k^2 E(k)$, so that

   $$\frac{\bar{\Pi}}{f^2} = \int_0^k f\, dk.$$

   Differentiate this expression with respect to k to obtain

   $$-2\bar{\Pi}^2 \frac{f'}{f^4} = 1.$$

   Let k_0 be the smallest wavenumber in the inertial range. Integrate between k_0 and k to obtain

   $$E(k) = k^{-7/3}\left[\frac{3}{2\bar{\Pi}_H^2}\left(1 - \frac{k_0}{k}\right) + \left(\frac{k_0}{k}\right) k_0^{-7} E^{-3}(k_0)\right]^{-1/3} \qquad (k \geq k_0).$$

   We can rewrite this as

   $$E(k) = \left(\frac{2}{3}\right)^{\frac{1}{3}} \bar{\Pi}_H^{2/3} k^{-7/3} \chi^{-1/3}(k) \qquad (k \geq k_0),$$

   where

   $$\chi(k) \doteq 1 - \frac{k_0}{k}(1 - \chi_0)$$

   and $\chi_0 \doteq 2\bar{\Pi}_H^2 k_0^{-7} E^{-3}(k_0)/3 = \chi(k_0) > 0$. Notice that for $k \gg k_0\,|1 - \chi_0|$ we have $E(k) \sim k^{-7/3}$.

2. (a) Show that for two-dimensional unforced incompressible turbulence, the pressure P is related to the stream function ψ by

   $$\nabla^2 P = 2 \det \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{yx} & \psi_{yy} \end{pmatrix}.$$

The velocity field is $\boldsymbol{u} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\psi = (-\psi_y, \psi_x)$. The Laplacian of the pressure satisfies

$$
\begin{aligned}
\nabla^2 P = -\boldsymbol{\nabla}\boldsymbol{\cdot}[(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{u}] &= -\frac{\partial}{\partial x}[\psi_y(\psi_{yx}) - \psi_x\psi_{yy}] - \frac{\partial}{\partial y}[-\psi_y(\psi_{xx}) + \psi_x\psi_{xy}] \\
&= -\psi_{yx}^2 - \psi_y\psi_{yxx} + \psi_{xx}\psi_{yy} + \psi_x\psi_{yyx} + \psi_{yy}\psi_{xx} + \psi_y\psi_{xxy} - \psi_{xy}^2 - \psi_x\psi_{xyy} \\
&= 2(\psi_{xx}\psi_{yy} - \psi_{xy}^2) \\
&= 2(\psi_{xx}\psi_{yy} - \psi_{xy}\psi_{yx}).
\end{aligned}
$$

(b) What is the pressure field required to maintain the incompressibility of the velocity field $\boldsymbol{u} = (\sin y, \sin x, 0)$?

Let $u = \sin y$, $v = \sin x$. From part (a), we see that

$$\nabla^2 P = 2(-v_x u_y - u_x^2) = -2\cos x \cos y$$

from which we deduce $P(x, y) = \cos x \cos y$ (plus any solution of Laplaces equation that satisfies the boundary conditions; for periodic boundary conditions, this means to within a constant).

3. In two dimensions, the statistical equipartition theory predicts that the ensemble-averaged energy for the inviscid unforced incompressible Navier–Stokes (Euler) equation should be distributed according to

$$E_{\boldsymbol{k}} = \frac{1}{2}\left(\frac{1}{\alpha + \beta k^2}\right),$$

where the constants α and β are related to the total energy $E = \sum_{\boldsymbol{k}} E_{\boldsymbol{k}}$ and enstrophy Z in the flow. Here the sum is over all excited (nonzero) Fourier modes.

(a) Given α and β, it is straighforward to calculate E and Z. What is the formula for Z?

$$Z = \frac{1}{2}\sum_{\boldsymbol{k}}\left(\frac{k^2}{\alpha + \beta k^2}\right),$$

(b) Given E and Z, the inverse problem of determining α and β is more difficult. Suppose that only a finite number $2N$ of Fourier modes are excited. Show that the problem of determining (α, β) from (E, Z) may be reduced to the problem of solving for the root of a single nonlinear equation.

The constants α and β may be determined from the initial energy E and enstrophy Z by expressing the ratio $r \doteq \frac{Z}{E}$ in terms of $\rho \doteq \frac{\alpha}{\beta}$, using the relation

$$Z = \frac{1}{2\beta}\sum_{\boldsymbol{k}}\left(1 - \frac{\alpha}{\alpha + \beta k^2}\right) = \frac{1}{\beta}(N - \alpha E).$$

We find that

$$r = \frac{N}{\beta E} - \rho,$$

or

$$r = 2N\left[\sum_{\boldsymbol{k}} \frac{1}{\rho + k^2}\right]^{-1} - \rho.$$

Upon inverting the last equation for $\rho(r)$ with a numerical root solver, we may determine α and β from the relations

$$\beta = \frac{N}{\rho E + Z}, \qquad \alpha = \rho\beta.$$

(c) Does a solution to (b) exist for all possible combinations of E and Z? Why or why not?

No, because if $k_{\min} \le |\boldsymbol{k}| \le k_{\max}$, then

$$\frac{1}{2}k_{\min}^2 \sum_{\boldsymbol{k}} \left(\frac{1}{\alpha + \beta k^2}\right) \le \frac{1}{2}\sum_{\boldsymbol{k}} \left(\frac{k^2}{\alpha + \beta k^2}\right) \le \frac{1}{2}k_{\max}^2 \sum_{\boldsymbol{k}} \left(\frac{1}{\alpha + \beta k^2}\right),$$

so that

$$k_{\min}^2 E \le Z \le k_{\max}^2 E.$$

Therefore if the quantity $r = Z/E$ lies outside the interval $[k_{\min}^2, k_{\max}^2]$, then no solution can exist.

(d) In three-dimensional inviscid turbulence, one obtains an equipartition of the modal energies $E_{\boldsymbol{k}}$, since the Lagrange multiplier β corresponding to the enstrophy is zero. What quantity is in equipartition in two dimensions, when α and β are both nonzero?

$$(\alpha + \beta k^2)\,|\boldsymbol{u}_{\boldsymbol{k}}|^2$$

4. Consider two-dimensional flow in a plane perpendicular to $\hat{\boldsymbol{z}}$.

(a) Show that the tensor

$$\epsilon_{\boldsymbol{kpq}} \doteq (\hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{p}\times\boldsymbol{q})\delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},0}$$

is antisymmetric under interchange of any two indices.

The antisymmetry with respect to interchange of the last two indices follows from the antisymmetry of the cross product. Also,

$$\epsilon_{\boldsymbol{pkq}} = (\hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{k}\times\boldsymbol{q})\delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},0} = -(\hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{p}\times\boldsymbol{q})\delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},0} = -\epsilon_{\boldsymbol{kpq}}$$

and

$$\epsilon_{\boldsymbol{qpk}} = -\epsilon_{\boldsymbol{pqk}} = \epsilon_{\boldsymbol{pkq}} = -\epsilon_{\boldsymbol{kpq}}.$$

(b) Prove that the two-dimensional Euler equation

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} = \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}$$

may be written as the *noncanonical Hamiltonian system*

$$\dot{\omega}_{\boldsymbol{k}} = J_{\boldsymbol{kq}} \frac{\partial H}{\partial \omega_{\boldsymbol{q}}},$$

(where H is the Hamiltonian, in this case the total energy), by showing that that the *symplectic tensor*

$$J_{\boldsymbol{kq}} \doteq \epsilon_{\boldsymbol{kpq}} \omega^*_{\boldsymbol{p}}$$

obeys both the *antisymmetry*

$$J_{\boldsymbol{kq}} = -J_{\boldsymbol{qk}}$$

and the *Jacobi identity*

$$J_{\boldsymbol{k}\boldsymbol{\ell}} \frac{\partial J_{\boldsymbol{pq}}}{\partial \omega_{\boldsymbol{\ell}}} + J_{\boldsymbol{p}\boldsymbol{\ell}} \frac{\partial J_{\boldsymbol{qk}}}{\partial \omega_{\boldsymbol{\ell}}} + J_{\boldsymbol{q}\boldsymbol{\ell}} \frac{\partial J_{\boldsymbol{kp}}}{\partial \omega_{\boldsymbol{\ell}}} = 0.$$

The Euler equation can be put into the above Hamiltonian form since

$$H = \frac{1}{2} \sum_{\boldsymbol{k}} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2} = \frac{1}{2} \sum_{\boldsymbol{k}} \frac{\omega_{\boldsymbol{k}} \boldsymbol{\omega}_{-k}}{k^2} \Rightarrow \frac{\partial H}{\partial \omega_{\boldsymbol{q}}} = \frac{1}{2} \frac{\omega^*_{\boldsymbol{q}}}{q^2} + \frac{1}{2} \frac{\boldsymbol{\omega}_{-\boldsymbol{q}}}{q^2} = \frac{\omega^*_{\boldsymbol{q}}}{q^2}.$$

To prove that the result is indeed a Hamiltonian, we first note that $J_{\boldsymbol{kq}}$ inherits the antisymmetry of $\epsilon_{\boldsymbol{kpq}}$ that we established in part(a). To establish the Jacobi symmetry, we first compute

$$J_{\boldsymbol{k}\boldsymbol{\ell}} \frac{\partial J_{\boldsymbol{pq}}}{\partial \omega_{\boldsymbol{\ell}}} = \epsilon_{\boldsymbol{kj\ell}} \omega^*_{\boldsymbol{j}} \epsilon_{\boldsymbol{prq}} \frac{\partial \omega_{-\boldsymbol{r}}}{\partial \omega_{\boldsymbol{\ell}}} = \epsilon_{\boldsymbol{kj\ell}} \epsilon_{\boldsymbol{p}(-\boldsymbol{\ell})\boldsymbol{q}} \omega^*_{\boldsymbol{j}} = \hat{\boldsymbol{z}} \boldsymbol{\cdot} \boldsymbol{k} \times (\boldsymbol{p} + \boldsymbol{q})\, \hat{\boldsymbol{z}} \boldsymbol{\cdot} \boldsymbol{p} \times \boldsymbol{q}\, \delta(\boldsymbol{k} + \boldsymbol{j} + \boldsymbol{p} + \boldsymbol{q}) \omega^*_{\boldsymbol{j}}$$

Since $\epsilon_{\boldsymbol{kpq}}$ is invariant under cyclic permutations of its indices, the sum of the cyclic permutations of $J_{\boldsymbol{k}\boldsymbol{\ell}} \frac{\partial J_{\boldsymbol{pq}}}{\partial \omega_{\boldsymbol{\ell}}}$ may be written as $A_{\boldsymbol{kpq}} \omega_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}}$, where

$$A_{\boldsymbol{kpq}} = \hat{\boldsymbol{z}} \boldsymbol{\cdot} \boldsymbol{k} \times (\boldsymbol{p} + \boldsymbol{q})\, \hat{\boldsymbol{z}} \boldsymbol{\cdot} \boldsymbol{p} \times \boldsymbol{q} + \hat{\boldsymbol{z}} \boldsymbol{\cdot} \boldsymbol{p} \times (\boldsymbol{q} + \boldsymbol{k})\, \hat{\boldsymbol{z}} \boldsymbol{\cdot} \boldsymbol{q} \times \boldsymbol{k} + \hat{\boldsymbol{z}} \boldsymbol{\cdot} \boldsymbol{q} \times (\boldsymbol{k} + \boldsymbol{p})\, \hat{\boldsymbol{z}} \boldsymbol{\cdot} \boldsymbol{k} \times \boldsymbol{p}.$$

Of the six terms in the above expression, the first and last, the second and third, and the fourth and fifth cancel each other pairwise, so that $A_{\boldsymbol{kpq}} = 0$.

5. (a) Prove the Gaussian integration by parts formula

$$\langle vf(v)\rangle = \langle v^2\rangle \left\langle \frac{\partial f}{\partial v}\right\rangle$$

for a (scalar) centered Gaussian random variable v and a continuously differentiable (C^1) function $f : \mathbb{R} \to \mathbb{R}$ that vanishes at $\pm\infty$.

$$\begin{aligned}\langle vf(v)\rangle = \int vf(v)dP &= \frac{1}{\sqrt{2\pi}\sigma}\int f(v)ve^{-\frac{v^2}{2\sigma^2}}\,dv\\ &= -\frac{\sigma^2}{\sqrt{2\pi}\sigma}\int f(v)\frac{\partial}{\partial v}e^{-\frac{v^2}{2\sigma^2}}\,dv\\ &= \frac{\sigma^2}{\sqrt{2\pi}\sigma}\int \frac{\partial f}{\partial v}e^{-\frac{v^2}{2\sigma^2}}\,dv\\ &= \sigma^2\int\frac{\partial f}{\partial v}\,dP = \sigma^2\left\langle\frac{\partial f}{\partial v}\right\rangle.\end{aligned}$$

Upon setting $f(v) = v$ we see that the second moment of v is just the variance of v:

$$\langle v^2\rangle = \sigma^2\langle 1\rangle = \sigma^2.$$

Hence

$$\langle vf(v)\rangle = \langle v^2\rangle\left\langle\frac{\partial f}{\partial v}\right\rangle.$$

(b) Use part (a) to show that that the odd-order moments of a centered Gaussian distribution are zero.

First, we note that

$$\langle v\rangle = 0$$

since v is centered.

Part (a) implies that

$$\left\langle v^{2n+1}\right\rangle = \langle v^2\rangle\left\langle\frac{\partial v^{2n}}{\partial v}\right\rangle = 2n\langle v^2\rangle\left\langle v^{2n-1}\right\rangle.$$

Therefore, by induction, $\left\langle v^{2n-1}\right\rangle = 0$ for all $n \in \mathbb{N}$.