Math 655: Statistical Theories of Turbulence

Fall, 2015 Assignment 1

September 11, due September 28

1. For any vector fields $\boldsymbol{u}, \boldsymbol{v} : \mathbb{R}^3 \to \mathbb{R}^3$ with continuous second derivatives show that if $\boldsymbol{u}$ and $\boldsymbol{v}$ vanish sufficiently fast at infinity then

   (a)
$$\int \boldsymbol{u}\boldsymbol{\cdot}(\boldsymbol{\nabla}\times\boldsymbol{v})\,d\boldsymbol{x} = \int \boldsymbol{v}\boldsymbol{\cdot}(\boldsymbol{\nabla}\times\boldsymbol{u})\,d\boldsymbol{x},$$

$$\int u_k\epsilon_{ijk}\frac{\partial v_j}{\partial x_i}\,d\boldsymbol{x} = -\int v_j\epsilon_{ijk}\frac{\partial u_k}{\partial x_i}\,d\boldsymbol{x} = -\int v_k\epsilon_{ikj}\frac{\partial u_j}{\partial x_i}\,d\boldsymbol{x} = \int v_k\epsilon_{ijk}\frac{\partial u_j}{\partial x_i}\,d\boldsymbol{x}$$

   (b)
$$\int \boldsymbol{u}\boldsymbol{\cdot}\nabla^2\boldsymbol{v}\,d\boldsymbol{x} = -\int(\boldsymbol{\nabla}\times\boldsymbol{u})\boldsymbol{\cdot}(\boldsymbol{\nabla}\times\boldsymbol{v})\,d\boldsymbol{x} \quad \text{if } \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} = 0.$$

   This follows from 1(a):

$$\int(\boldsymbol{\nabla}\times\boldsymbol{u})\boldsymbol{\cdot}(\boldsymbol{\nabla}\times\boldsymbol{v})\,d\boldsymbol{x} = \int \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{v})\,d\boldsymbol{x} = \int \boldsymbol{u}\boldsymbol{\cdot}[\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}) - \nabla^2\boldsymbol{v}]\,d\boldsymbol{x} = -\int \boldsymbol{u}\boldsymbol{\cdot}\nabla^2\boldsymbol{v}\,d\boldsymbol{x}.$$

   Alternatively, we can establish the identity directly:

$$\begin{aligned}-\int(\boldsymbol{\nabla}\times\boldsymbol{u})\boldsymbol{\cdot}(\boldsymbol{\nabla}\times\boldsymbol{v})\,d\boldsymbol{x} &= -\int \epsilon_{ijk}\frac{\partial u_j}{\partial x_i}\epsilon_{\bar{i}\bar{j}k}\frac{\partial v_{\bar{j}}}{\partial x_{\bar{i}}}\,d\boldsymbol{x} = -\int\left(\frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} - \frac{\partial u_j}{\partial x_i}\frac{\partial v_i}{\partial x_j}\right)d\boldsymbol{x} \\ &= \int\left(u_j\frac{\partial^2 v_j}{\partial x_i^2} - u_j\frac{\partial}{\partial x_j}\frac{\partial v_i}{\partial x_i}\right)d\boldsymbol{x} = \int \boldsymbol{u}\boldsymbol{\cdot}\nabla^2\boldsymbol{v}\,d\boldsymbol{x}\end{aligned}$$

   since $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} = 0$.

2. (a) For any vector fields $\boldsymbol{u}, \boldsymbol{v} : \mathbb{R}^3 \to \mathbb{R}^3$, where $\boldsymbol{u}$ is differentiable, prove that

$$(\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{v} = \boldsymbol{v}\boldsymbol{\cdot}\big[\boldsymbol{\nabla}\boldsymbol{u} - (\boldsymbol{\nabla}\boldsymbol{u})^T\big].$$

$$\epsilon_{ijk}(\boldsymbol{\nabla}\times\boldsymbol{u})_i v_j\hat{\boldsymbol{x}}_k = \epsilon_{ijk}\left(\epsilon_{i\bar{j}\bar{k}}\partial_{\bar{j}}u_{\bar{k}}\right)v_j\hat{\boldsymbol{x}}_k = v_j(\partial_j u_k - \partial_k u_j)\hat{\boldsymbol{x}}_k = \boldsymbol{v}\boldsymbol{\cdot}\big[\boldsymbol{\nabla}\boldsymbol{u} - (\boldsymbol{\nabla}\boldsymbol{u})^T\big].$$

   (b) Use part (a) to show that the vortex-stretching term $\boldsymbol{\omega}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}$ can be written in the form $\boldsymbol{D}\boldsymbol{\cdot}\boldsymbol{\omega}$, where

$$\boldsymbol{D} = \frac{1}{2}\big[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\big].$$

   In the special case where $\boldsymbol{v} \doteq \boldsymbol{\nabla}\times\boldsymbol{u} = \boldsymbol{\omega}$, we find from part (a) that

$$0 = \boldsymbol{\omega}\times\boldsymbol{\omega} = \boldsymbol{\omega}\boldsymbol{\cdot}\big[\boldsymbol{\nabla}\boldsymbol{u} - (\boldsymbol{\nabla}\boldsymbol{u})^T\big].$$

Hence $\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u} = \boldsymbol{\omega}\cdot(\boldsymbol{\nabla}\boldsymbol{u})^T$, so

$$\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u} = \frac{1}{2}\boldsymbol{\omega}\cdot\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\right] = \boldsymbol{\omega}\cdot\boldsymbol{D} = \boldsymbol{D}\cdot\boldsymbol{\omega}.$$

In the last step, we used the symmetry of $\boldsymbol{D}$:

$$\boldsymbol{\omega}\cdot\boldsymbol{D} = \boldsymbol{\omega}_i\boldsymbol{D}_{ij}\hat{\boldsymbol{x}}_j = \boldsymbol{\omega}_i\boldsymbol{D}_{ji}\hat{\boldsymbol{x}}_j = \hat{\boldsymbol{x}}_j\boldsymbol{D}_{ji}\boldsymbol{\omega}_i = \boldsymbol{D}\cdot\boldsymbol{\omega}.$$

3. A simple one-dimensional model for turbulence is Burgers equation,

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu\frac{\partial^2}{\partial x^2}v.$$

What global integral invariants does the inviscid version of Burgers equation have?

The spatial integral of any continuously differentiable function f of v is an invariant of

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0,$$

given either periodic boundary conditions or the condition $\lim_{x\to\pm\infty} v = 0$. Let

$$g(v) = \int_0^v f'(\bar{v})\bar{v}\,d\bar{v},$$

(this integral always exists, since f' is continuous). Note that $g'(v) = f'(v)v$. Then

$$\frac{d}{dt}\int f(v)\,dx = \int f'(v)\frac{\partial v}{\partial t}\,dx = -\int f'(v)v\frac{\partial v}{\partial x}\,dx = -\int \frac{\partial g(v)}{\partial x}\,dx = 0.$$

Note: One does not assume the incompressibility condition $\partial v/\partial x = 0$ for Burgers equation since this leads to the trivial solution that v is independent of both space and time.

4. Prove that the helicity

$$H \doteq \frac{1}{2}\int \boldsymbol{u}\cdot\boldsymbol{\omega}\,d\boldsymbol{x}$$

is conserved by the three-dimensional inviscid incompressible Navier–Stokes equation.

From 1(a) we see that

$$\begin{aligned}
\frac{1}{2}\frac{dH}{dt} &= -\frac{1}{2}\frac{d}{dt}\int \boldsymbol{u}\cdot\boldsymbol{\omega}\,d\boldsymbol{x}\\
&= \frac{1}{2}\int \frac{\partial \boldsymbol{u}}{\partial t}\cdot\boldsymbol{\nabla}\times\boldsymbol{u}\,d\boldsymbol{x} + \frac{1}{2}\int \boldsymbol{u}\cdot\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\boldsymbol{u}\,d\boldsymbol{x}\\
&= \frac{1}{2}\int \boldsymbol{u}\cdot\boldsymbol{\nabla}\times\frac{\partial \boldsymbol{u}}{\partial t}\,d\boldsymbol{x} + \frac{1}{2}\int \boldsymbol{u}\cdot\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\boldsymbol{u}\,d\boldsymbol{x}\\
&= \int \boldsymbol{u}\cdot\frac{\partial \boldsymbol{\omega}}{\partial t}\,d\boldsymbol{x} = \int \boldsymbol{u}\cdot[\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{\omega}] = \int \boldsymbol{u}\cdot\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{\omega})\\
&= \int \boldsymbol{\nabla}\times\boldsymbol{u}\cdot(\boldsymbol{u}\times\boldsymbol{\omega}) = \int \boldsymbol{\omega}\cdot(\boldsymbol{u}\times\boldsymbol{\omega}) = 0
\end{aligned}$$

since $\boldsymbol{\omega}$ and $\boldsymbol{u}\times\boldsymbol{\omega}$ are perpendicular to each other.

5. Show that

$$\frac{1}{2}\int \boldsymbol{A}\cdot\boldsymbol{\omega}\, d\boldsymbol{x},$$

is an invariant of the three-dimensional inviscid incompressible Navier–Stokes equation, where $\boldsymbol{A}$ is any vector potential for the velocity $\boldsymbol{u}$ and $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$.

From 1(a) we see that

$$\frac{1}{2}\int \boldsymbol{A}\cdot\boldsymbol{\omega}\, d\boldsymbol{x} = \frac{1}{2}\int \boldsymbol{A}\cdot\boldsymbol{\nabla}\times\boldsymbol{u}\, d\boldsymbol{x} = \frac{1}{2}\int \boldsymbol{u}\cdot\boldsymbol{\nabla}\times\boldsymbol{A}\, d\boldsymbol{x} = \frac{1}{2}\int \boldsymbol{u}\cdot\boldsymbol{u}\, d\boldsymbol{x};$$

hence this is just the total energy in the flow, which we have seen to be an invariant of the nonlinear terms of the Navier–Stokes equation.

Alternatively, we can show the invariance directly by first noting that, in the Coulomb gauge, $\boldsymbol{\omega}$ and $\boldsymbol{A}$ are related by

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} = -\nabla^2\boldsymbol{A}.$$

Thus

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int \boldsymbol{A}\cdot\boldsymbol{\omega}\, d\boldsymbol{x} &= -\frac{1}{2}\frac{d}{dt}\int \boldsymbol{A}\cdot\nabla^2\boldsymbol{A}\, d\boldsymbol{x} \\
&= -\frac{1}{2}\int \frac{\partial \boldsymbol{A}}{\partial t}\cdot\nabla^2\boldsymbol{A}\, d\boldsymbol{x} - \frac{1}{2}\int \boldsymbol{A}\cdot\frac{\partial \nabla^2\boldsymbol{A}}{\partial t}\, d\boldsymbol{x} \\
&= -\frac{1}{2}\int \frac{\partial \nabla^2\boldsymbol{A}}{\partial t}\cdot\boldsymbol{A}\, d\boldsymbol{x} - \frac{1}{2}\int \boldsymbol{A}\cdot\frac{\partial \nabla^2\boldsymbol{A}}{\partial t}\, d\boldsymbol{x} \\
&= \int \boldsymbol{A}\cdot\frac{\partial \boldsymbol{\omega}}{\partial t}\, d\boldsymbol{x} = \int \boldsymbol{A}\cdot(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \boldsymbol{A}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}\, d\boldsymbol{x} \\
&= \int -\boldsymbol{A}\cdot(\nabla^2\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\times\boldsymbol{A} \;+ \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}\cdot\boldsymbol{\nabla})\nabla^2\boldsymbol{A}\, d\boldsymbol{x} \\
&= \int -A_k\nabla^2 A_l\epsilon_{ijk}\frac{\partial A_j}{\partial x_l x_i} + A_l\epsilon_{ijk}\frac{\partial A_j}{\partial x_i}\frac{\partial \nabla^2 A_l}{\partial x_k}\, d\boldsymbol{x} \\
&= \int \frac{\partial A_k}{\partial x_l}\epsilon_{ijk}\frac{\partial A_j}{\partial x_i}\nabla^2 A_l - \frac{\partial A_l}{\partial x_k}\epsilon_{ijk}\frac{\partial A_j}{\partial x_i}\nabla^2 A_l\, d\boldsymbol{x}.
\end{aligned}$$

The terms where $l = k$ cancel each other, so we only need to consider the terms where $l = i$ and $l = j$. Hence

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int \boldsymbol{A}\cdot\boldsymbol{\omega}\, d\boldsymbol{x} \\
&= \int \left(\frac{\partial A_k}{\partial x_i}\epsilon_{ijk}\frac{\partial A_j}{\partial x_i}\nabla^2 A_i - \frac{\partial A_i}{\partial x_k}\epsilon_{ijk}\frac{\partial A_j}{\partial x_i}\nabla^2 A_i + \frac{\partial A_k}{\partial x_j}\epsilon_{ijk}\frac{\partial A_j}{\partial x_i}\nabla^2 A_j - \frac{\partial A_j}{\partial x_k}\epsilon_{ijk}\frac{\partial A_j}{\partial x_i}\nabla^2 A_j\right) d\boldsymbol{x} \\
&= 0
\end{aligned}$$

since the first and last terms vanish and the second term, after making the substitution $i \to j \to k \to i$, cancels the third term.