Math 538:
Asymptotic Methods

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Preface

These notes were developed for a graduate applied mathematics course on Asymptotic Methods at the University of Alberta. We thank Andy Hammerlindl and Tom Prince for coauthoring the high-level graphics language `Asymptote` (freely available at `http://asymptote.sourceforge.net`) that was used to draw the mathematical figures in this text.

Chapter 1

Asymptotic Series

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever. N. H. Abel (1828)

The differential equations encountered in applied mathematics, science, and engineering research are only rarely soluble in terms of familiar mathematical functions. When an exact solution is lacking, it is often desirable to use *local analysis* to determine the approximate behaviour of a solution near a point of interest (which could even be ∞). Asymptotic series provide a powerful technique for constructing such approximations.

1.A A Simple Example

To illustrate what an asymptotic series is, suppose we want to evaluate the Laplace transform of $\cos t$:

$$I(x) = \int_0^\infty e^{-xt} \cos t \, dt \quad (x > 0).$$

If we didn't know how to integrate this result directly, we might be tempted to evaluate I by substituting in the Taylor series of $\cos t$:

$$I(x) = \int_0^\infty e^{-xt} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \, dt \quad (x > 0).$$

If we can justify the interchange of the two limit processes (the integral and infinite sum), we would then obtain, on substituting $u = xt$,

$$\begin{aligned}
I(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int_0^\infty e^{-xt} t^{2n}\, dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\; x^{2n+1}} \int_0^\infty e^{-u} u^{2n}\, du \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\; x^{2n+1}} \Gamma(2n+1) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\; x^{2n+1}} (2n)! \\
&= \frac{1}{x} \sum_{n=0}^{\infty} \left(-\frac{1}{x^2}\right)^n,
\end{aligned}$$

where $\Gamma(z)$ is defined by $\int_0^\infty e^{-u}\, u^{z-1}\, du$ for $\operatorname{Re} z > 0$ and elsewhere by $\Gamma(z+1) = z\Gamma(z)$ (cf. Sect. 2.A). The resulting geometric series, with ratio $-1/x^2$, converges for $x > 1$ to the value

$$\frac{1}{x} \cdot \frac{1}{1 + \frac{1}{x^2}} = \frac{x}{x^2 + 1}.$$

In fact, on directly integrating $I(x)$ by parts twice, we can quickly verify that the final formula $I(x) = x/(1 + x^2)$ is valid also for $x \in (0, 1]$ even though the geometric series we encountered above diverges on this interval.

Now suppose we attempt to apply the same technique to the compute the Laplace transform of $1/(1 + t)$, whose Taylor series is just the geometric series $\sum_{n=0}^{\infty} (-1)^n t^n$:

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-xt}}{1+t}\, dt \\
&= \int_0^\infty e^{-xt} \sum_{n=0}^{\infty} (-1)^n t^n\, dt \\
&= \sum_{n=0}^{\infty} (-1)^n \int_0^\infty e^{-xt} t^n\, dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{x^{n+1}} \int_0^\infty e^{-u} u^n\, du \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{x^{n+1}} \Gamma(n+1) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}}.
\end{aligned} \tag{1.1}$$

Unfortunately, we see that the resulting series diverges for all x (for any fixed x the terms do not even go to zero as $n \to \infty$).

Why did our procedure fail for the second example? The problem is that the Taylor series for $1/(1+t)$ only converges for $|t| < 1$ and we integrated for $t \in [0, \infty)$. Somewhat surprisingly, even though our final result is a divergent series, it can still be useful for computing approximations to $f(x)$ for sufficiently large x. In terms of the Nth partial sums[1]

$$f_N(x) \doteq \sum_{n=0}^{N} \frac{(-1)^n n!}{x^{n+1}},$$

we define the pointwise *error* or *remainder*

$$R_N(x) \doteq f(x) - f_N(x).$$

On making use of the Nth partial sums of the previously encountered geometric series,

$$\sum_{n=0}^{N} (-1)^n t^n = \frac{1}{1+t} - \frac{(-t)^{N+1}}{1+t}.$$

we can express

$$\begin{aligned} f(x) &= \int_0^\infty e^{-xt} \frac{1}{1+t}\, dt \\ &= \int_0^\infty e^{-xt} \left[\sum_{n=0}^{N} (-1)^n t^n + \frac{(-t)^{N+1}}{1+t} \right] dt \\ &= f_N(x) + \int_0^\infty e^{-xt} \frac{(-t)^{N+1}}{1+t}\, dt. \end{aligned}$$

Then

$$|R_N(x)| = \int_0^\infty e^{-xt} \frac{t^{N+1}}{1+t}\, dt \leqslant \int_0^\infty e^{-xt}\, t^{N+1}\, dt = \frac{(N+1)!}{x^{N+2}}. \tag{1.2}$$

Thus, the partial sum f_N of the series approximates $f(x)$ with an error that is less than or equal to the first term it neglects, even though the series itself diverges! For large x, this remainder is small, even for a few terms. For example, for $N = 3$, we have

$$|R_3(x)| \leqslant \frac{24}{x^5},$$

so for $x = 10$, and $x = 100$ say, we have

$$|R_3(10)| \leqslant 2.4 \times 10^{-4} \quad \text{and} \quad |R_3(100)| \leqslant 2.4 \times 10^{-9}.$$

The series we found for $f(x)$ is called an *asymptotic series*. The key distinction here is the order in which the limits $N \to \infty$ and $x \to \infty$ are taken:

[1] we use the symbol $\doteq$ to emphasize a definition, although the notation $:=$ is more common.

- The series we found for $I(x)$ is convergent since $\lim\limits_{N\to\infty} R_N(x) = 0$ for fixed $x > 1$.

- The series we found for $f(x)$ is asymptotic since $\lim\limits_{x\to\infty} R_N(x) = 0$ for fixed N.

1.B Order Symbols

Let f and g be functions defined on $D \subset X \to Y$, where the sets X and Y could represent either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Let A be a subset of D.

Definition: We say f is in the *order of* g on A if $\exists M > 0 \ni$

$$|f(x)| \leqslant M\,|g(x)| \text{ for all } x \in A.$$

We write $f(x) = \mathcal{O}(g(x))$. The order of g can be thought of as the class of all functions that are asymptotically smaller than or equal to (some positive multiple of) g on A. Equivalently, if $g(x)$ is nonzero on A,

$$\sup_{x\in A}\left|\frac{f(x)}{g(x)}\right| < \infty;$$

(i.e. $\left|\frac{f}{g}\right|$ is bounded on A).

- Let $A = X = Y = \mathbb{R}$, $f(x) = \sin x$, $g(x) = 1$. Then $\sin x = \mathcal{O}(1)$ on $\mathbb{R}$ since $|\sin x| \leqslant 1$ for all $x \in \mathbb{R}$.

- Let $A = X = Y = \mathbb{R}$, $f(x) = \sin x$, $g(x) = x$. Then $\sin x = \mathcal{O}(x)$ on $\mathbb{R}$ since $|\sin x| \leqslant |x|$ for all $x \in \mathbb{R}$.

- Let $A = X = Y = \mathbb{R}$, $f(x) = \sinh x$, $g(x) = \cosh x$. Then $\sinh x = \mathcal{O}(\cosh(x))$ on $\mathbb{R}$ since $|\sinh x| = \left|\frac{e^x - e^{-x}}{2}\right| \leqslant \frac{e^x + e^{-x}}{2} = \cosh x$ for all $x \in \mathbb{R}$.

- Let $A = X = Y = \mathbb{R}$, $f(x) = 10x$, $g(x) = x$. Then $f(x) = \mathcal{O}(g(x))$ on $\mathbb{R}$.

- Let $X = Y = \mathbb{C}$, $A = B_r(0) \doteq \{z \in \mathbb{C} : |z| < r\}$, $f(z) = z^2$, $g(z) = z$. Then $z \in B_r(0) \Rightarrow |z| < r \Rightarrow |z^2| \leqslant r\,|z|$. Hence $z^2 = \mathcal{O}(z)$ on $B_r(0)$.

- Let $A = X = \mathbb{R}$, $Y = \mathbb{C}$, $f(x) = e^{ix}$, $g(x) = 1$. Then $e^{ix} = \mathcal{O}(1)$ on $\mathbb{R}$.

Remark: Of the first two statements, $\sin x = \mathcal{O}(1)$ is more useful if $|x|$ is large and $\sin x = \mathcal{O}(x)$ is more useful if $|x|$ is small.

Remark: Frequently one is interested in an order relation near some distinguished point z_0 (which could include ∞). It is therefore convenient to introduce a local definition of asymptotic ordering.

Let X_∞ be one of the sets $\mathbb{R}_\infty = \mathbb{R} \cup \{-\infty, \infty\}$ or $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Suppose $z_0 \in \bar{A} \subset X_\infty$ (i.e. z_0 is a limit point of A).

Definition: We say f is in the *order of* g as $z \to z_0$ on $A \subset X_\infty$ if X_∞ contains a neighbourhood U of z_0 such that for some M

$$z \in U \cap A \Rightarrow |f(z)| \leqslant M\, |g(z)|\, .$$

We write $f(z) = \mathcal{O}(g(z))$ as $z \to z_0$. Equivalently, if g is nonzero near $z_0 \in \bar{A}$,

$$\limsup_{\substack{z \to z_0 \\ z \in A}} \left| \frac{f(z)}{g(z)} \right| < \infty$$

(i.e. $\left|\dfrac{f}{g}\right|$ is bounded on A near z_0). Recall $\limsup\limits_{z \to z_0} f(z) \doteq \inf\limits_{\delta > 0} \sup\limits_{\substack{z \in B_\delta(z_0) \\ z \neq z_0}} f(z)$ is the supremum of values of $f(z)$ near z_0. In particular, if $f(z)$ is continuous at z_0 then $\limsup\limits_{z \to z_0} f(z) = \lim\limits_{z \to z_0} f(z) = f(z_0)$.

- $X = Y = \mathbb{C}$, $A = B_r(0)$, $z_0 = 0 \in \bar{A}$. Then $|z| < r \Rightarrow |z^2| \leqslant r\,|z| \Rightarrow$

$$z^2 = \mathcal{O}(z) \quad (z \to 0 \text{ in } A).$$

- $X = Y = \mathbb{C}$, $A = B_r(0)^c$, $z_0 = \infty \in \bar{A}$. Then $0 < r \leqslant |z| \Rightarrow |z| \leqslant \frac{1}{r}\,|z^2| \Rightarrow$

$$z = \mathcal{O}(z^2) \quad (z \to \infty \text{ in } A).$$

Definition: We say f is in the *little order of* g as $z \to z_0$ in $A \subset X_\infty$ if for all $\varepsilon > 0$, X_∞ contains an neighbourhood U_ε of z_0 such that

$$z \in U_\varepsilon \cap A \Rightarrow |f(z)| \leqslant \varepsilon\, |g(z)|\, .$$

We write $f(z) = o(g(z))$ as $z \to z_0$. Equivalently, if g is nonzero near z_0 (and $f(z_0) = 0$ when $z_0 \in A$),

$$\lim_{\substack{z \to z_0 \\ z \in A}} \frac{f(z)}{g(z)} = 0.$$

- $A = X = Y = \mathbb{C}$. Considering the limits points 0 and ∞ of $\bar{A}$, we see that $\lim_{z\to 0} \frac{z^2}{z} = 0 \Rightarrow z^2 = o(z)$ $(z \to 0$ in $A)$ and $\lim_{z\to\infty} \frac{z}{z^2} = 0 \Rightarrow z = o(z^2)$ $(z \to \infty$ in $A)$.

Remark: A less common but sometimes more convenient notation for $f = \mathcal{O}(g)$ is $f \preccurlyeq g$. The latter emphasizes that the (somewhat misleading) notation $f = \mathcal{O}(g)$ actually represents a logical binary relation between the functions f and g and not an equality. Likewise, an alternative notation for $f = o(g)$ is $f \prec g$ (although the notation $f \ll g$, or equivalently, $g \gg f$, is much more common in applied fields). These alternative notations are convenient for stating the following theorem.

Theorem 1.1 (Order Properties): *The following implications hold:*

(i) $f \prec g$ $(z \to z_0$ *in* $A) \Rightarrow f \preccurlyeq g$ $(z \to z_0$ *in* $A)$;

(ii) $f \preccurlyeq g$ $(z \to z_0$ *in* $A), \alpha \in \mathbb{R}^+ \Rightarrow |f|^\alpha \preccurlyeq |g|^\alpha$ $(z \to z_0$ *in* $A)$;

(iii) $f \prec g$ $(z \to z_0$ *in* $A), \alpha \in \mathbb{R}^+ \Rightarrow |f|^\alpha \prec |g|^\alpha$ $(z \to z_0$ *in* $A)$;

(iv) $f \preccurlyeq g \preccurlyeq h$ $(z \to z_0$ *in* $A) \Rightarrow f \preccurlyeq h$ $(z \to z_0$ *in* $A)$;

(v) $f \preccurlyeq g \prec h$ $(z \to z_0$ *in* $A) \Rightarrow f \prec h$ $(z \to z_0$ *in* $A)$;

(vi) $f \prec g \preccurlyeq h$ $(z \to z_0$ *in* $A) \Rightarrow f \prec h$ $(z \to z_0$ *in* $A)$;

(vii) $f \preccurlyeq \phi,\ g \preccurlyeq \psi$ $(z \to z_0$ *in* $A) \Rightarrow fg \preccurlyeq \phi\psi$ $(z \to z_0$ *in* $A)$;

(viii) $f \preccurlyeq \phi,\ g \prec \psi$ $(z \to z_0$ *in* $A) \Rightarrow fg \prec \phi\psi$ $(z \to z_0$ *in* $A)$;

(ix) $f \preccurlyeq \phi,\ g \preccurlyeq \phi$ $(z \to z_0$ *in* $A) \Rightarrow f + g \preccurlyeq \phi$ $(z \to z_0$ *in* $A)$;

(x) $f \prec \phi,\ g \prec \phi$ $(z \to z_0$ *in* $A) \Rightarrow f + g \prec \phi$ $(z \to z_0$ *in* $A)$.

Proof: These results follow easily from the definitions. □

Problem 1.1: Prove Theorem 1.1.

Definition: If $f - g = o(g)$ $(z \to z_0$ in $A)$, we say f is *asymptotic to* g and write $f \sim g$ $(z \to z_0$ in $A)$.

Remark: If $g(z) \neq 0$ near z_0 (and $f(z_0) = g(z_0)$ when $z_0 \in A$), then $f \sim g$ ($z \to z_0$ in A) is equivalent to $\lim\limits_{\substack{z\to z_0\\ z\in A}} \dfrac{f(z)}{g(z)} = 1$.

In Problem 1.2 we show that $f \sim g$ is an equivalence relation. The following theorem establishes a connection between this asymptotic equivalence and asymptotic ordering.

Theorem 1.2 (Asymptotic Functions Have Same Order): *$f \sim g$ ($z \to z_0$ in A) $\Rightarrow$ $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$ ($z \to z_0$ in A).*

Proof: If $f \sim g$ ($z \to z_0$ in A) then $f - g = o(g)$ ($z \to z_0$ in A), which implies that $\forall \varepsilon > 0$, $\exists$ a neighbourhood U_ε of $z_0 \ni$

$$z \in U_\varepsilon \cap A \Rightarrow |f(z) - g(z)| \leqslant \varepsilon |g(z)|.$$

But by the triangle inequality we have

$$|f(z)| - |g(z)| \leqslant |f(z) - g(z)| \leqslant \varepsilon |g(z)|,$$

$$|g(z)| - |f(z)| \leqslant |f(z) - g(z)| \leqslant \varepsilon |g(z)|.$$

The first inequality yields $|f(z)| \leqslant (1+\varepsilon)|g(z)|$, which implies that $f = \mathcal{O}(g)$ ($z \to z_0$ in A). For the special case $\varepsilon = 1/2$, the second inequality implies that $|g(z)| \leqslant 2\,|f(z)|$, so we see that $g = \mathcal{O}(f)$ ($z \to z_0$ in A). $\square$

Remark: The converse of the above theorem is not true. Consider $f(x) = x$ and $g(x) = 2x$. We have

$$|f(x)| = |x| \leqslant 2|x| = |g(x)| \text{ hence } f = \mathcal{O}(g) \ (x \to 0),$$

$$|g(x)| = 2|x| \leqslant 2|f(x)| \text{ hence } g = \mathcal{O}(f) \ (x \to 0).$$

But $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = \lim\limits_{x\to 0} \frac{x}{2x} = \frac{1}{2} \neq 1$. Therefore $f \not\sim g$ ($x \to 0$).

Q. Does $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$ ($z \to z_0$ in A) imply $\exists K \neq 0$ (const.) $\ni$ $f \sim Kg$ ($z \to z_0$ in A)?

A. Exercise.

Problem 1.2: Show that the definition of $f \sim g$ satisfies

(i) $f \sim f$,

(ii) $f \sim g \Rightarrow g \sim f$,

(iii) $f \sim g,\ g \sim h \Rightarrow f \sim h$,

making it an equivalence relation.

Problem 1.3: If f and g are continuous at $z = z_0$, is it necessarily true that $f = \mathcal{O}(g)$? Prove or provide a counterexample.

1.C Sequences and Series

Suppose

(i) $\phi_n \colon D \subset X \to Y$ for $n = 0, 1, \ldots, N$, where $X, Y \in \{\mathbb{R}, \mathbb{C}\}$;

(ii) $A \subset D$;

(iii) $z_0 \in X_\infty$ is a limit point of A.

Definition: The set $\{\phi_n\}_{n=0}^{N}$ is called an *asymptotic sequence* ($z \to z_0$ in A) if $\phi_{n+1} = o(\phi_n)$ ($z \to z_0$ in A) for $n = 0, 1, \ldots, N-1$.

- If $A = X = Y = \mathbb{C}$, $z_0 \in \mathbb{C}$, and $\phi_n(z) = (z - z_0)^n$, then

$$\lim_{z\to z_0} \frac{\phi_{n+1}(z)}{\phi_n(z)} = \lim_{z\to z_0} (z - z_0) = 0 \Rightarrow \phi_{n+1} = o(\phi_n) \ (z \to z_0 \text{ in } A).$$

  Hence $\{\phi_n\}_{n=0}^{\infty}$ is an asymptotic sequence.

- If $A = X = Y = \mathbb{C}$, $z_0 = \infty$, $\phi_n(z) = \frac{1}{z^n}$, then

$$\lim_{z\to \infty} \frac{\phi_{n+1}(z)}{\phi_n(z)} = \lim_{z\to \infty} \frac{1}{z} = 0 \Rightarrow \phi_{n+1} = o(\phi_n) \ (z \to \infty \text{ in } A).$$

  Hence $\{\phi_n\}_{n=0}^{\infty}$ is an asymptotic sequence.

- If $A = X = Y = \mathbb{C}$, $z_0 = \infty$, $\phi_n(z) = \frac{\alpha(z)}{z^n}$, then

$$\lim_{z\to \infty} \frac{\phi_{n+1}(z)}{\phi_n(z)} = \lim_{z\to \infty} \frac{1}{z} = 0 \Rightarrow \phi_{n+1} = o(\phi_n) \ (z \to z_0 \text{ in } A).$$

  Hence $\{\phi_n\}_{n=0}^{\infty}$ is an asymptotic sequence.

- If $X = Y = \mathbb{C}$, $A = \{z \in \mathbb{C};\, |\arg z| \le \theta_0 < \frac{\pi}{2}\}$, $z_0 = \infty$, $\phi_n(z) = e^{-nz}$, then

$$\lim_{z\to z_0} \frac{\phi_{n+1}(z)}{\phi_n(z)} = \lim_{z\to z_0} e^{-z} = \lim_{x\to +\infty} (e^{-x} e^{-iy}) = 0 \Rightarrow \phi_{n+1} = o(\phi_n) \ (z \to z_0 \text{ in } A).$$

  Hence $\{\phi_n\}_{n=0}^{\infty}$ is an asymptotic sequence.

- If $X = Y = \mathbb{R}$, $A = (0, \infty)$, $x_0 = \infty$, $\phi_n(x) = e^{x/n}$, then

$$\lim_{x\to\infty} \frac{\phi_{n+1}(x)}{\phi_n(x)} = \lim_{x\to\infty} e^{\frac{-x}{n(n+1)}} = 0 \Rightarrow \phi_{n+1} = o(\phi_n) \ (x \to \infty).$$

  Hence $\{\phi_n\}_{n=0}^{\infty}$ is an asymptotic sequence. This example shows that it is not necessary for the individual elements of an asymptotic series go to zero, or even remain bounded. The "asymptoticness" of a sequence is determined solely from the ratio of consecutive terms.

- Let $X = Y = \mathbb{R}$, $A = (0, 1)$, $x_0 = 0$. Consider the two sequences:

$$\{\phi_n(x)\}_{n=0}^{\infty} = \{\log x,\ 1,\ x\log x,\ x,\ x^2\log^2 x,\ x^2\log x,\ x^2,\ x^3\log^3 x,\ x^3\log^2 x,\ x^3\log x,\ x^3, \ldots\},$$

$$\psi_n(x) = x^n, \quad n = 0, 1, 2, \ldots.$$

  It is easily seen that $\lim\limits_{x\to 0} \frac{\phi_{n+1}(x)}{\phi_n(x)} = 0$ for all $n \in \mathbb{N}_0$, so that these are both asymptotic sequences, with $\{\psi_n\}$ being a subsequence of $\{\phi_n\}$. One might say that $\{\phi_n\}$ is a more "refined" asymptotic sequence than $\{\psi_n\}$.

Remark: Asymptotic sequences satisfy the following properties:

1. Any subsequence of an asymptotic sequence is an asymptotic sequence.
2. If $\{\phi_n\}$ is an asymptotic sequence and $\alpha > 0$ then $\{|\phi_n|^\alpha\}$ is an asymptotic sequence.
3. If $\{\phi_n\}$ and $\{\psi_n\}$ are asymptotic sequences then $\{\phi_n\psi_n\}$ is an asymptotic sequence.

Definition: Two sequences $\{\phi_n\}_{n=0}^{N}$ and $\{\psi_n\}_{n=0}^{N}$ (not necessarily asymptotic) are said to be *asymptotically equivalent* if $\phi_n = \mathcal{O}(\psi_n)$ and $\psi_n = \mathcal{O}(\phi_n)$ for $n = 0, 1, \ldots, N$.

Remark: If $\lim\limits_{z\to z_0} \left|\frac{\phi_n}{\psi_n}\right| = L_n$ where $0 < L_n < \infty$, for each $n = 0, 1, \ldots, N$, then $\{\phi_n\}_{n=0}^{N}$ and $\{\psi_n\}_{n=0}^{N}$ are asymptotically equivalent as $z \to z_0$.

Lemma 1.1: *If $\{\phi_n\}$ and $\{\psi_n\}$ are asymptotically equivalent sequences and $\{\phi_n\}$ is an asymptotic sequence then $\{\psi_n\}$ is an asymptotic sequence.*

Proof: $\psi_{n+1} \preccurlyeq \phi_{n+1} \prec \phi_n \preccurlyeq \psi_n$. Therefore $\psi_{n+1} \prec \psi_n$ (i.e. $\psi_{n+1} = o(\psi_n)$). $\square$

Definition: A sum $\sum_{n=0}^{N} a_n\phi_n(z)$, ($a_n$ const.) is called an *asymptotic series* if $\{\phi_n\}_{n=0}^{N}$ is an asymptotic sequence.

In the case $N = \infty$, nothing is implied about the convergence of this series. In fact, the question of convergence is of no particular interest in asymptotic theory.

We now define how an asymptotic series can represent a function near some distinguished point z_0.

Definition: If $f(z) = \sum_{n=0}^{N} a_n\phi_n(z) + o(\phi_N)$ $(z \to z_0 \text{ in } A)$ for some $N \in \mathbb{N}_0$, the asymptotic series $\sum_{n=0}^{N} a_n\phi_n(z)$ is said to be an *asymptotic expansion to N terms* for f $(z \to z_0 \text{ in } A)$. We write $f(z) \sim \sum_{n=0}^{N} a_n\phi_n(z)$ $(z \to z_0 \text{ in } A)$.

Definition: The *remainder* after N terms is $R_N \doteq f(z) - \sum_{n=0}^{N} a_n\phi_n(z)$.

Remark: If we let

$$F_N(z) = \begin{cases} \frac{R_N(z)}{\phi_N(z)} & \text{for } \phi_N(z) \neq 0, \\ 0 & \text{for } \phi_N(z) = 0, \end{cases}$$

we see that the statement $f(z) = \sum_{n=0}^{N} a_n\phi_n(z) + o(\phi_N)$ $(z \to z_0 \text{ in } A)$ is equivalent to $f(z) = \sum_{n=0}^{N} a_n\phi_n(z) + F_N(z)\phi_N$, where $\lim_{\substack{z\to z_0\\ z\in A}} F_N(z) = 0$.

Definition: If for each $N \in \mathbb{N}_0$, $\sum_{n=0}^{N} a_n\phi_n(z)$ is an asymptotic expansion to N terms for f then the series is an *asymptotic expansion* of f. We write $f(z) \sim \sum_{n=0}^{\infty} a_n\phi_n(z)$ $(z \to z_0 \text{ in } A)$.

We now return to the example of the previous section.

- Consider the function f given in Eq. (1.1). Our naïve attempt to obtain a series representation for f led to the divergent series $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}}$. There are two questions to be considered here: (i) is this an asymptotic series; and (ii) if so, is this an asymptotic expansion of f?

For this series we have $\phi_n(x) = x^{-(n+1)}$ and

$$\lim_{x\to\infty} \frac{\phi_{n+1}(x)}{\phi_n(x)} = \lim_{x\to\infty} \frac{1}{x} = 0$$

so that series is indeed an asymptotic series. The remainder R_N satisfies (cf. Eq. (1.2))

$$|R_N(x)| \leqslant \frac{(N+1)!}{x^{N+2}} = (N+1)!\ \phi_{N+1}(x),$$

so that

$$\frac{|R_N(x)|}{\phi_N(x)} \leqslant (N+1)!\ \frac{\phi_{N+1}(x)}{\phi_N(x)}, \text{and hence} \lim_{x\to\infty} \frac{|R_N(x)|}{\phi_N(x)} = 0,\ \forall N.$$

Thus we have

$$f(x) = \int_0^\infty \frac{e^{-xt}}{1+t}\, dt \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}}\ (x \to \infty).$$

The following result indicates the degree to which asymptotic expansions are unique.

Theorem 1.3 (Uniqueness): *Let $\{\phi_n\}_{n=0}^N$ be an asymptotic sequence. Then*

$$f(z) \sim \sum_{n=0}^{N} a_n \phi_n(z)\ \ (z \to z_0 \text{ in } A) \iff$$

$$a_0 = \lim_{\substack{z\to z_0\\ z\in A}} \frac{f(z)}{\phi_0(z)}, \qquad a_n = \lim_{\substack{z\to z_0\\ z\in A}} \frac{f(z) - \sum_{j=0}^{n-1} a_j \phi_j(z)}{\phi_n(z)},\ n = 1, 2, \ldots, N.$$

Proof: This equivalence follows directly from the fact that

$$\lim_{\substack{z\to z_0\\ z\in A}} \left(\frac{f(z)}{\phi_0(z)} - a_0 \right) = \lim_{\substack{z\to z_0\\ z\in A}} \frac{f(z) - a_0\phi_0(z)}{\phi_0(z)} = \lim_{\substack{z\to z_0\\ z\in A}} \frac{R_0(z)}{\phi_0(z)} = 0$$

and

$$\lim_{\substack{z\to z_0\\ z\in A}} \left(\frac{f(z) - \sum_{j=0}^{n-1} a_j\phi_j(z)}{\phi_n(z)} - a_n \right) = \lim_{\substack{z\to z_0\\ z\in A}} \frac{f(z) - \sum_{j=0}^{n-1} a_j\phi_j(z) - a_n\phi_n(z)}{\phi_n(z)}$$

$$= \lim_{\substack{z\to z_0\\ z\in A}} \frac{R_n(z)}{\phi_n(z)} = 0. \quad \square$$

Remark: This result shows that, given a function f and an asymptotic sequence $\{\phi_n\}_{n=0}^{N}$, the asymptotic expansion of f, if it exists, is unique.

Remark: In particular, for the asymptotic sequence $\phi_n(z) = (z-z_0)^n$, the corresponding asymptotic expansion of an analytic function f is in fact just its Taylor series, with the unique coefficients (cf. Prob. 1.5)

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, \ldots.$$

Remark: On the other hand, as the following example illustrates, a function f may have asymptotic expansions relative to different asymptotic sequences, and furthermore, these sequences need not be asymptotically equivalent.

Problem 1.4: Consider the function $\frac{1}{1+x}$ and for $n = 1, 2, \ldots$ the sequences

$$\phi_n(x) = \frac{1}{x^n}, \quad \psi_n(x) = \frac{x-1}{x^{2n}}, \quad \theta_n(x) = \frac{x^2-x+1}{x^{3n}}.$$

Verify that:

(i) these sequences are asymptotic as $(x \to +\infty)$;

(ii) these sequences are not asymptotically equivalent;

(iii) the statements

$$\frac{1}{1+x} \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^n}, \quad \frac{1}{1+x} \sim \sum_{n=1}^{\infty} \frac{x-1}{x^{2n}}, \quad \frac{1}{1+x} \sim \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x^2-x+1)}{x^{3n}}$$

are valid asymptotic representations as $(x \to +\infty)$.

Remark: In addition to a single function having different asymptotic expansions relative to different asymptotic sequences, a single asymptotic series can be a valid asymptotic representation for more than one function: if two functions f and g differ by an amount that is asymptotically smaller than any element of the sequence $\{\phi_n\}_{n=0}^{\infty}$, the series doesn't "see the difference".

Theorem 1.4 (Nonuniqueness): *If* $f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$ *and* $f - g = o(\phi_n)$ $(z \to z_0$ *in* $A)$ $\forall n$, *then* $g(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$ $(z \to z_0$ *in* $A)$.

Proof: Exercise. $\square$

- Consider the functions $f(x) \doteq \frac{1}{1+x}$ and $g(x) \doteq \frac{1+e^{-x}}{1+x}$. From the previous example we know that $\frac{1}{1+x} \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^n} (x \to +\infty)$. But, on examining the difference between f and g we get

$$f(x) - g(x) = \frac{1}{1+x} - \frac{1+e^{-x}}{1+x} = \frac{-e^{-x}}{1+x} = o\left(\frac{1}{x^n}\right) \quad (x \to +\infty) \ \forall n.$$

Hence, by Theorem 1.4 we have $\dfrac{1+e^{-x}}{1+x} \sim \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^n}$ as $x \to +\infty$.

Problem 1.5: Let f be a function that is infinitely differentiable in a neighbourhood of x_0 and $a_k = f^{(k)}(x_0)/k!$. Define $\phi_n(x) = (x - x_0)^n$ and $S_n(x) = \sum_{j=0}^{n} a_j \phi_j(x)$.

(a) Evaluate the kth derivative $\phi_n^{(k)}(x)$ of ϕ_n at $x = x_0$.

(b) Evaluate the kth derivative $S_n^{(k)}(x)$ of S_n at $x = x_0$.

(c) Evaluate $\lim_{x \to x_0} [f^{(k)}(x) - S_n^{(k)}(x)]$ for $n = 0, 1, 2, \ldots$.

(d) Use L'Hôpitals Rule to compute for $n = 1, 2, \ldots$

$$\lim_{x \to x_0} \frac{f(x) - S_{n-1}(x)}{\phi_n(x)}.$$

Note: always check that L'Hôpitals Rule applies before using it!

(e) Recall Taylor's Remainder theorem:

$$f(x) = \sum_{n=0}^{N-1} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + R_N(x, x_0),$$

where $R_N(x, x_0) = (x - x_0)^N f^{(N)}(c_N)/N!$ for some number c_N between x and x_0. Use the continuity of $f^{(N)}$ to conclude that $f^{(N)}(c_N)$ is bounded by some constant M_N. Deduce that

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \quad (x \to x_0),$$

(f) Evaluate the limit in part (d) directly from part (e) and Theorem 1.3.

1.D Operations on Asymptotic Series

Suppose that

(i) $\phi_n : D \subset X \to Y, n = 0, 1, \ldots, N$, where $X, Y \in \{\mathbb{R}, \mathbb{C}\}$;

(ii) $A \subset D$;

(iii) $z_0 \in X_\infty$ is a limit point of A;

(iv) $\{\phi_n\}_{n=0}^{\infty}$ $(z \to z_0$ in $A)$ is an asymptotic sequence.

We want to determine under what conditions is it legitimate to add, multiply, divide, differentiate, and integrate asymptotic series. Term by term addition of asymptotic series is always possible:

Theorem 1.5 (Addition): *If $f(z) \sim \sum_{n=0}^{\infty} a_n\phi_n(z)$ and $g(z) \sim \sum_{n=0}^{\infty} b_n\phi_n(z)$ $(z \to z_0$ in $A)$, then $\alpha f(z) + \beta g(z) \sim \sum_{n=0}^{\infty}(\alpha a_n + \beta b_n)\phi_n(z)$ $(z \to z_0$ in $A)$.*

Proof: We are given

$$f(z) \sim \sum_{n=0}^{\infty} a_n\phi_n(z) \Rightarrow f(z) = \sum_{n=0}^{N} a_n\phi_n(z) + F_N(z)\phi_N(z),$$

$$g(z) \sim \sum_{n=0}^{\infty} b_n\phi_n(z) \Rightarrow g(z) = \sum_{n=0}^{N} b_n\phi_n(z) + G_N(z)\phi_N(z),$$

with $\lim\limits_{z\to z_0} F_N(z) = \lim\limits_{z\to z_0} G_N(z) = 0$. Thus

$$\alpha f(z) + \beta g(z) = \sum_{n=0}^{N}(\alpha a_n + \beta b_n)\phi_n(z) + [\alpha F_N(z) + \beta G_N(z)]\phi_N(z).$$

The desired result follows from the fact that $\lim\limits_{z\to z_0}[\alpha F_N(z) + \beta G_N(z)] = 0 + 0 = 0$. □

Remark: Taking the formal product of $\sum_m a_m\phi_m$ and $\sum_n b_n\phi_n$ yields an expression containing all possible products $\phi_m\phi_n,\ m, n \in \mathbb{N}_0$. It is not possible, in general, to arrange the functions $\{\phi_m\phi_n\}_{m,n=0}^{\infty}$ into an asymptotic sequence. However, we will mostly restrict our attention to asymptotic sequences satisfying the property

$$\phi_m(z)\phi_n(z) = \alpha(z)\phi_{m+n}(z), \tag{1.3}$$

where without loss of generality one takes $\alpha(z) = 0$ whenever $\phi_0(z) = 0$. Sequences which satisfy Eq. (1.3) can be multiplied in a manner reminiscent of polynomial multiplication. Such sequences are necessarily of a certain form:

Lemma 1.2: *If $\{\phi_n\}_{n=0}^{\infty}$ $(z \to z_0$ in $A)$ is an asymptotic sequence that satisfies Eq. (1.3), then ϕ_n can be expressed as $\phi_n(z) = \alpha(z)\beta^n(z)$, where $\lim\limits_{\substack{z\to z_0\\ z\in A}} \beta(z) = 0$. Here, we interpret $\beta^0(z) = 1$ for all $\beta(z)$.*

Proof: Exercise. □

Theorem 1.6 (Multiplication/Division): *If*

(i) the sequence $\{\phi_n\}_{n=0}^{\infty}$ satisfies Eq. (1.3);

(ii) $f(z) \sim \displaystyle\sum_{n=0}^{\infty} a_n\phi_n(z) \quad (z \to z_0 \text{ in } A);$

(iii) $g(z) \sim \displaystyle\sum_{n=0}^{\infty} b_n\phi_n(z) \quad (z \to z_0 \text{ in } A);$

with $b_0 \neq 0$, then the product fg and the quotient f/g satisfy

$$f(z)g(z) \sim \alpha(z)\sum_{n=0}^{\infty} c_n\phi_n(z) \quad (z \to z_0 \text{ in } A)$$

$$\frac{f(z)}{g(z)} \sim \frac{1}{\alpha(z)}\sum_{n=0}^{\infty} d_n\phi_n(z) \quad (z \to z_0 \text{ in } A),$$

where

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}$$

and

$$d_0 = \frac{a_0}{b_0}, \qquad d_n = \frac{1}{b_0}\left(a_n - \sum_{j=0}^{n-1} d_j b_{n-j}\right) \quad n \geqslant 1.$$

Proof: We are given

$$f(z) = \sum_{n=0}^{N} a_n\phi_n(z) + F_N(z)\phi_N(z), \qquad g(z) = \sum_{n=0}^{N} b_n\phi_n(z) + G_N(z)\phi_N(z),$$

where $\lim\limits_{z\to z_0} F_N(z) = \lim\limits_{z\to z_0} G_N(z) = 0$.

Omitting the z arguments for brevity, we obtain, introducing the indices $n = j+k$ (which runs from 0 to $2N$) and $m = n - N$,

$$
\begin{aligned}
fg &= \sum_{j=0}^{N}\sum_{k=0}^{N} a_j b_k \phi_j \phi_k + \sum_{j=0}^{N} (a_j G_N + b_j F_N)\phi_j \phi_N + F_N G_N \phi_N \phi_N \\
&= \left(\sum_{n=0}^{N}\sum_{j=0}^{n} + \sum_{n=N+1}^{2N}\sum_{j=n-N}^{N} \right) a_j b_{n-j} \alpha \phi_n + \left\{ \sum_{j=0}^{N} (a_j G_N + b_j F_N)\phi_j + F_N G_N \phi_N \right\} \phi_N \\
&= \sum_{n=0}^{N}\sum_{j=0}^{n} a_j b_{n-j} \alpha \phi_n + \sum_{m=1}^{N}\sum_{j=m}^{N} a_j b_{N+m-j} \phi_m \phi_N \\
&\quad + \left\{ \sum_{j=0}^{N} (a_j G_N + b_j F_N)\phi_j + F_N G_N \phi_N \right\} \phi_N \\
&= \alpha \left(\sum_{n=0}^{N} c_n \phi_n + E_N \phi_N \right),
\end{aligned}
$$

where each term of

$$
E_N \doteq \sum_{m=1}^{N}\sum_{j=m}^{N} a_j b_{N+m-j} \beta^m + \sum_{j=0}^{N} (a_j G_N + b_j F_N)\beta^j + F_N G_N \beta^N
$$

is seen to approach 0 as $z \to z_0$ since

$$
\lim_{z \to z_0} \beta^j(z) = \begin{cases} 1 & \text{for } j = 0, \\ 0 & \text{for } j > 0, \end{cases}
$$

and $\lim\limits_{z \to z_0} F_N(z) = \lim\limits_{z \to z_0} G_N(z) = 0$.

The formula for division follows immediately upon inverting the product formula for $\frac{f(z)}{g(z)} \cdot g(z)$,

$$
a_n = \sum_{j=0}^{n} d_j b_{n-j},
$$

to obtain d_n for $n = 0, 1, 2, \ldots$. $\square$

Remark: In the following two theorems, it will be helpful to recall that if f is holomorphic in a set A then it either has isolated zeros in A or is identically zero on A.

Theorem 1.7 (Termwise Integration): *If*

(i) $\{\phi_n\}_{n=0}^{\infty}$ $(z \to z_0$ *in* $A)$ *is an asymptotic sequence;*

(ii) $\forall n$, ϕ_n *is holomorphic in* A, *with antiderivative* Φ_n *satisfying* $\lim\limits_{\substack{z\to z_0\\ z\in A}} \Phi_n(z) = 0$;

(iii) f *is holomorphic in* A;

(iv) $f(z) \sim \sum\limits_{n=0}^{\infty} a_n\phi_n(z)$ $(z \to z_0$ *in* $A)$;

Then the antiderivatives $\Phi_n(z)$ *form an asymptotic sequence and*

$$\int_{z_0}^{z} f(\zeta)\, d\zeta \sim \sum_{n=0}^{\infty} a_n\Phi_n(z) \quad (z \to z_0 \text{ in } A),$$

provided the path of integration (except possibly for z_0*) lies in* A.

Proof: Without loss of generality, we may assume that none of the ϕ_n are identically zero on A. It follows that these holomorphic functions must have isolated zeros; that is, there exists a $\delta_n > 0$ such that $0 < |z - z_0| < \delta_n \Rightarrow \phi_n(z) \neq 0$.

From (i) and (ii), we see from L'Hôpitals Rule that

$$\lim_{z\to z_0} \frac{\Phi_{n+1}(z)}{\Phi_n(z)} = \lim_{z\to z_0} \frac{\Phi'_{n+1}(z)}{\Phi'_n(z)} = \lim_{z\to z_0} \frac{\phi_{n+1}(z)}{\phi_n(z)} = 0.$$

Also, (iv) implies

$$f(z) = \sum_{n=0}^{N} a_n\phi_n(z) + F_N(z)\phi_N(z),$$

where $\lim\limits_{z\to z_0} F_N(z) = 0$ for each N. On integrating each side between z_0 and z, we find

$$\int_{z_0}^{z} f(\zeta)\, d\zeta = \sum_{n=0}^{N} a_n\Phi_n(z) + E_N(z)\Phi_N(z),$$

where

$$E_N(z) = \frac{\int_{z_0}^{z} F_N(\zeta)\phi_N(\zeta)\, d\zeta}{\Phi_N(z)}.$$

The desired result then follows upon taking the limit $z \to z_0$:

$$\lim_{z\to z_0} E_N(z) = \lim_{z\to z_0} \frac{\int_{z_0}^{z} F_N(\zeta)\phi_N(\zeta)\, d\zeta}{\Phi_N(z)} = \lim_{z\to z_0} \frac{F_N(z)\phi_N(z)}{\Phi'_N(z)} = \lim_{z\to z_0} F_N(z) = 0. \quad \square$$

Remark: By far the most difficult theorem to prove, and the one with the most conditions, is the following theorem on term-by-term differentiation of asymptotic series.

Theorem 1.8 (Termwise Differentiation): *Let $z_0 \in \mathbb{C}$. If*

(i) ϕ_n is holomorphic in an open convex set A such that $z_0 \in \bar{A}$;

(ii) $\{\phi_n\}_{n=0}^{\infty}$, and $\{\phi_n'\}_{n=0}^{\infty}$ are asymptotic sequences as ($z \to z_0$ in A)

(iii) $\dfrac{\phi_n}{\phi_n'} = \begin{cases} \mathcal{O}(z - z_0) & (z \to z_0 \text{ in } A), \text{ if } z_0 \neq \infty, \\ \mathcal{O}(z) & (z \to \infty \text{ in } A), \text{ if } z_0 = \infty; \end{cases}$

(iv) f is holomorphic in A;

(v) $f(z) \sim \displaystyle\sum_{n=0}^{\infty} a_n \phi_n(z) \quad (z \to z_0 \text{ in } A),$

then

$$f'(z) \sim \sum_{n=0}^{\infty} a_n \phi_n'(z) \quad (z \to z_0 \text{ in } A).$$

Proof:

Case $z_0 \neq \infty$:

Consider the circle $C_\lambda(z) = \{w \in \mathbb{C} : |w - z| = \lambda\,|z - z_0|\}$ of radius $\lambda\,|z - z_0|$ about z_0 for a fixed $\lambda \in (0, 1)$ such that $C_\lambda \subset A$. We are given that

$$f(z) = \sum_{n=0}^{N} a_n \phi_n(z) + F_N(z)\phi_N(z),$$

where $\lim\limits_{\substack{z \to z_0 \\ z \in A}} F_N(z) = 0$. Note from (i) and (iv) that F_N is holomorphic in A for all N and

$$\begin{aligned} f'(z) &= \sum_{n=0}^{N} a_n \phi_n'(z) + F_N(z)\phi_N'(z) + F_N'(z)\phi_N(z) \\ &= \sum_{n=0}^{N} a_n \phi_n'(z) + \left[F_N(z) + F_N'(z)\frac{\phi_N(z)}{\phi_N'(z)} \right] \phi_N'(z). \end{aligned}$$

The conditions that the functions ϕ_n are holomorphic and form an asymptotic series guarantee that ϕ_N' is nonzero and bounded near z_0. It thus suffices to show that

$$\lim_{\substack{z \to z_0 \\ z \in A}} F_N'(z)\frac{\phi_N(z)}{\phi_N'(z)} = 0.$$

Let $M_N(z)$ be the maximum value achieved by the continuous function $|F_N|$ on the compact set $C_\lambda(z)$. Then since all points of $C_\lambda(z)$ remain inside A and approach z_0 as $z \to z_0$, we see from $\lim\limits_{\substack{z\to z_0\\ z\in A}} |F_N(z)| = 0$ that $\lim\limits_{\substack{z\to z_0\\ z\in A}} M_N(z) = 0$. Hence Cauchy's residue formula implies that

$$|F_N'(z)| = \left|\frac{1}{2\pi i}\int_{C_\lambda(z)} \frac{F_N(w)}{(w-z)^2}\,dw\right| \leqslant \frac{1}{2\pi} M_N(z) \frac{2\pi\lambda\,|z-z_0|}{\lambda^2\,|z-z_0|^2} = \frac{M_N(z)}{\lambda\,|z-z_0|}.$$

Finally, from (iii), we know that there exists a $K_N > 0$ and $\delta_N > 0$ such that

$$0 < |z - z_0| < \delta_N,\ z \in A \Rightarrow \left|\frac{\phi_N(z)}{\phi_N'(z)}\right| \leqslant K_N\,|z - z_0|\,.$$

Now given $\varepsilon > 0$, choose $\delta \in (0, \delta_N)$ such that

$$\begin{aligned} 0 < |z - z_0| < \delta,\ z \in A &\Rightarrow M_N(z) < \frac{\varepsilon\lambda}{K_N} \\ &\Rightarrow \left|F_N'(z)\frac{\phi_N(z)}{\phi_N'(z)}\right| \leqslant \frac{M_N(z)}{\lambda\,|z-z_0|} K_N\,|z - z_0| < \varepsilon, \end{aligned}$$

as desired.

Case $z_0 = \infty$:

The proof proceeds similarly, but with $C_\lambda(z) = \{w \in \mathbb{C} : |w - z| = \lambda\,|z|\}$ and $|z - z_0| < \delta_N$ replaced by $|z| > \delta_N$. Note for $w \in C_\lambda(z)$ that $w \to \infty$ as $z \to \infty$ since

$$(1-\lambda)\,|z| = |z| - |w - z| \leqslant |z + w - z| = |w|\,. \quad \square$$

Remark: For some applications, it is convenient to choose the set A in Theorem 1.8 to be an open subset of the pie-shaped sector $S(z_0, \alpha, \beta) \doteq \{z \in \mathbb{C};\ z \neq z_0, \alpha < \operatorname{Arg}(z - z_0) < \beta\}$.

Remark: In this chapter we have given the basic theoretical foundation of asymptotic sequences and series needed for the remainder of the course. The examples presented thus far have been relatively simple in that an asymptotic expansion to a given function f relative to a given asymptotic sequence $\{\phi_n\}_{n=0}^{\infty}$ was not difficult to obtain. In most applications, however, the situation is not so straightforward. The functions for which asymptotic expansions are sought are usually unknown in advance, typically being the solution to some initial or boundary value problem. Even the appropriate asymptotic sequence to be used in the expansion of the solution is not usually known and must be determined as part of the overall solution procedure. These problems can be quite challenging.

Chapter 2

Expansion of Integrals

2.A The Gamma Function

For $\mathrm{Re}(z) > 0$, define

$$\Gamma_+(z) \doteq \int_{0^+}^{\infty} e^{-t} t^{z-1}\, dt,$$

where the integration is performed along the positive real axis. Then Γ_+ is holomorphic in the right half plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. A single integration by parts yields the following recurrence relation

$$\begin{aligned} \Gamma_+(z+1) &= \int_{0^+}^{\infty} e^{-t} t^{z}\, dt = -e^{-t} t^{z}\Big|_0^{\infty} + z \int_{0^+}^{\infty} e^{-t} t^{z-1}\, dt \\ &= z\Gamma_+(z). \end{aligned} \tag{2.1}$$

Since $\Gamma_+(1) = \int_0^{\infty} e^{-t}\, dt = 1 = 0!$, we see that $\Gamma_+(n+1) = n!$ for $n \in \mathbb{N}_0$. Continuing in this manner we find that $\Gamma_+(z+n) = (z+n-1)\dots(z+1)z\Gamma_+(z)$. On rearranging this formula,

$$\Gamma_+(z) = \frac{\Gamma_+(z+n)}{z(z+1)\dots(z+n-1)},$$

it is possible to analytically continue the function to the left-half plane:

$$\Gamma(z) \doteq \begin{cases} \Gamma_+(z) & \mathrm{Re}(z) > 0, \\ \dfrac{\Gamma_+(z+n)}{z(z+1)\dots(z+n-1)} & -n < \mathrm{Re}(z) \leqslant -n+1, z \neq -n+1, n = 1, 2, 3, \dots \end{cases}$$

The resulting function $\Gamma(z)$ is holomorphic in the complex plane except at $z = 0, -1, -2, \dots,$ where it has simple poles. The graph of $\Gamma(x)$ for $x \in \mathbb{R}$ is shown in Figure 2.1 and an interactive three-dimensional plot of the surface $\Gamma(z)$ for $z \in \mathbb{C}$ is shown in Figure 2.2.

We proceed to derive a few useful relationships involving the Γ function.

- For $\alpha \in (0,1)$ we have

$$\Gamma(\alpha) = \int_{0^+}^{\infty} e^{-t} t^{\alpha-1}\, dt = 2\int_{0^+}^{\infty} e^{-y^2} y^{2\alpha-1}\, dy \qquad (\text{letting } t = y^2),$$

which leads to

$$\begin{aligned}
\Gamma(\alpha)\Gamma(1-\alpha) &= \left(2\int_{0^+}^{\infty} e^{-y^2} y^{2\alpha-1}\, dy\right)\left(2\int_{0^+}^{\infty} e^{-x^2} x^{1-2\alpha}\, dx\right)\\
&= 4\int_{0^+}^{\infty}\int_{0^+}^{\infty} e^{-(x^2+y^2)} \left(\frac{y}{x}\right)^{2\alpha-1} dx\, dy\\
&= 4\int_0^{\pi/2} \tan^{2\alpha-1}\theta \int_0^{\infty} e^{-r^2} r\, dr\, d\theta\\
&= 2\int_0^{\pi/2} \tan^{2\alpha-1}\theta\, d\theta.
\end{aligned}$$

In particular, we see for $\alpha = 1/2$ that $\Gamma^2(1/2) = 2\int_0^{\pi/2} d\theta = \pi$ and

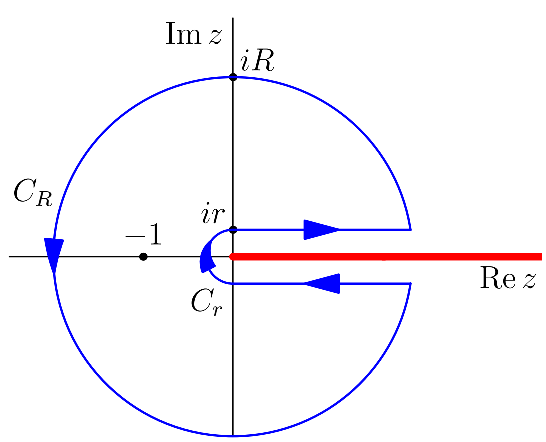
$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = 2\int_0^{\infty} e^{-x^2}\, dx = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

A substitution then leads to the important result $\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\pi/a}$ for $a > 0$.

For arbitrary $\alpha \in (0,1)$, we find, on substituting $z = \tan^2\theta$,

$$I(\alpha) \doteq \Gamma(\alpha)\Gamma(1-\alpha) = 2\int_{0^+}^{\pi/2} \tan^{2\alpha-1}\theta\, d\theta = \int_{0^+}^{\infty} \frac{z^{\alpha-1}}{1+z}\, dz.$$

The integral here can be evaluated by a contour integration in the complex plane, noting that the function $z^{\alpha-1} = e^{(\alpha-1)\log z}$ is holomorphic on the star-shaped domain obtained by slicing the complex plane along the positive real axis. This *branch cut* is shown in red in the following figure. In other words we choose the antiderivative $\log z = \log|z| + i\arg z$ of the function $z \mapsto 1/z$, where $\arg z \in [0, 2\pi)$.



Here the large circular contour C_R is chosen to have radius $R \geqslant 2$, so that $|1+z| \geqslant R/2$ on C_R, and the small semicircular contour C_r is chosen to have radius $r \leqslant 1/2$, so that $|1+z| \geqslant 1/2$ on C_r. On denoting

$$f(z) \doteq \frac{z^{\alpha-1}}{1+z} = \frac{e^{(\alpha-1)\log z}}{1+z},$$

we then see, accounting for the residue from the pole of f at $z=-1$, that

$$2\pi i e^{(\alpha-1)i\pi} = \int_{ir}^{R+ir} f + \int_{C_R} f + \int_{R-ir}^{-ir} f + \int_{C_r} f.$$

Since $\alpha < 1$, we see that the contribution from the circular arc C_R is

$$\left|\int_{C_R} f\right| \leqslant \frac{R^{\alpha-1}}{\frac{R}{2}} \cdot 2\pi R = 4\pi R^{\alpha-1} \underset{R\to\infty}{\to} 0.$$

Likewise, since $\alpha > 0$, the contribution from the semicircular contour C_r is

$$\left|\int_{C_r} f\right| \leqslant \frac{r^{\alpha-1}}{\frac{1}{2}} \cdot \pi r = 2\pi r^{\alpha} \underset{r\to 0}{\to} 0.$$

We thus deduce that

$$\begin{aligned}
2\pi i e^{(\alpha-1)i\pi} &= \lim_{\substack{r\to 0\\ R\to\infty}} \left[\int_{ir}^{R+ir} f - \int_{-ir}^{R-ir} f\right] \\
&= \int_{0^+}^{\infty} \frac{e^{(\alpha-1)\log|z|}}{1+z}\, dz - \int_{0^+}^{\infty} \frac{e^{(\alpha-1)(\log|z|+i2\pi)}}{1+z}\, dz \\
&= I(\alpha)\big(1 - e^{(\alpha-1)2\pi i}\big).
\end{aligned}$$

Thus

$$\pi = I(\alpha) \cdot \frac{e^{-(\alpha-1)\pi i} - e^{(\alpha-1)\pi i}}{2i} = I(\alpha) \cdot \frac{-e^{-\alpha\pi i} + e^{\alpha\pi i}}{2i},$$

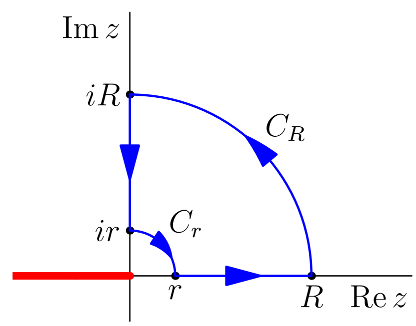
from which we see that

$$I(\alpha) = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\pi\alpha},$$

On extending this result by analytic continuation, one finds for all $z \in \mathbb{C} \setminus \mathbb{Z}$ that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}. \tag{2.2}$$

- For $\alpha \geqslant 1$ and positive x and λ, another frequently encountered integral can be expressed in terms of Γ using the substitution $u = xt^\lambda$:
$$\int_0^\infty e^{-xt^\lambda} t^{\alpha-1}\,dt = \frac{1}{\lambda x^{\frac{\alpha}{\lambda}}}\int_0^\infty e^{-u} u^{\frac{\alpha}{\lambda}-1}\,du = \frac{\Gamma\left(\frac{\alpha}{\lambda}\right)}{\lambda x^{\frac{\alpha}{\lambda}}}. \tag{2.3}$$
For the special case $\alpha = x = 1$, this result simplifies to
$$\int_0^\infty e^{-t^\lambda}\,dt = \frac{1}{\lambda}\Gamma\left(\frac{1}{\lambda}\right) = \Gamma\left(1+\frac{1}{\lambda}\right).$$
For $0 < \alpha < 1$ and $x > 0$, a related integral is
$$\int_0^\infty e^{ixt} t^{\alpha-1}\,dt = \frac{i^\alpha \Gamma(\alpha)}{x^\alpha}, \tag{2.4}$$
where we introduce a branch cut (shown in red) along the negative real axis:



We note that $f(z) = e^{ixz} z^{\alpha-1}$ is holomorphic inside the blue contour. Cauchy's Integral Theorem thus implies that
$$0 = \int_r^R f(t)\,dt + \int_{C_R} f + i\int_R^r f(it)\ dt + \int_{C_r} f.$$
Since $\alpha < 1$, we see that
$$\begin{aligned}\left|\int_{C_R} f\right| &\leqslant \int_0^{\pi/2} e^{-xR\sin\theta} R^{\alpha-1} R\ d\theta \\ &\leqslant R^{\alpha-1}\int_0^{\pi/2} e^{-2xR\theta/\pi} R\ d\theta = R^{\alpha-1}\frac{\pi}{2x}\left(1-e^{-xR}\right) \underset{R\to\infty}{\to} 0.\end{aligned}$$
Likewise, since $\alpha > 0$, we see that
$$\left|\int_{C_r} f\right| \leqslant r^\alpha \frac{\pi}{2x}\left(\frac{1-e^{-xr}}{r}\right) \underset{r\to 0}{\to} 0.$$
Hence
$$\int_0^\infty f(t)\,dt = -\int_\infty^0 f(it)\ idt = i^\alpha \int_0^\infty e^{-xt} t^{\alpha-1}\ dt = \frac{i^\alpha\Gamma(\alpha)}{x^\alpha},$$
as claimed.

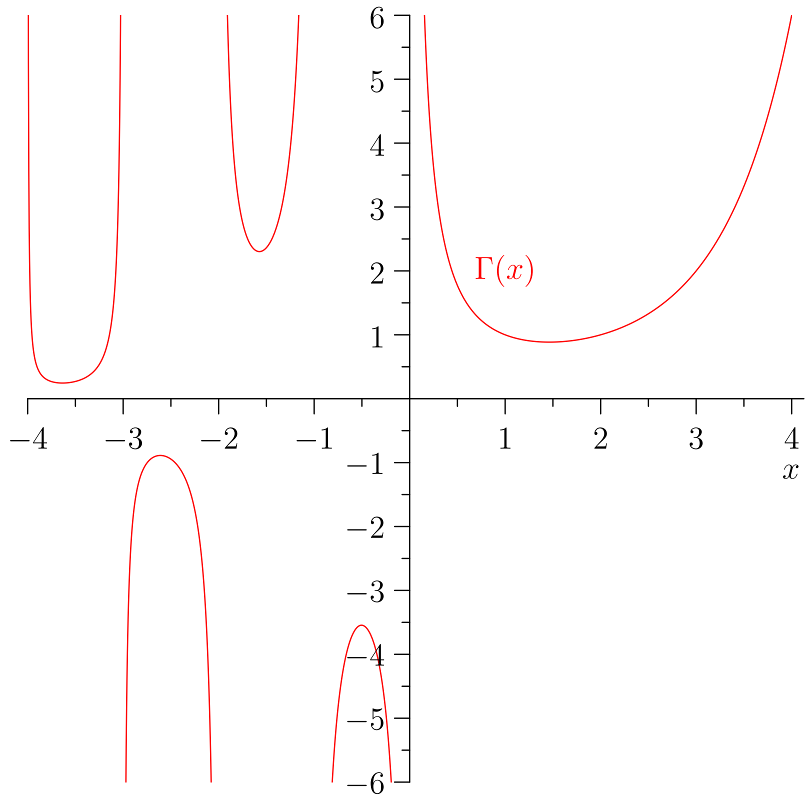


Figure 2.1: Graph of $\Gamma(x)$ on the real line.

Gamma Function and Binomial Expansions

The Γ function gives us a convenient way to write general binomial expansions. For arbitrary $\alpha, z \in \mathbb{C}$ consider the general binomial expansion

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-k)(\alpha-k+1)}{k!}z^k + \cdots.$$

It is convenient to introduce a generalized binomial coefficient.

Definition: The *generalized binomial coefficient* is defined as follows:

$$\binom{\alpha}{k} \doteq \frac{1}{k!}\prod_{j=0}^{k-1}(\alpha - j).$$

We can now write the general binomial expansion in a more compact form

$$(1+z)^\alpha = 1 + \sum_{k=1}^{\infty}\binom{\alpha}{k}z^k.$$

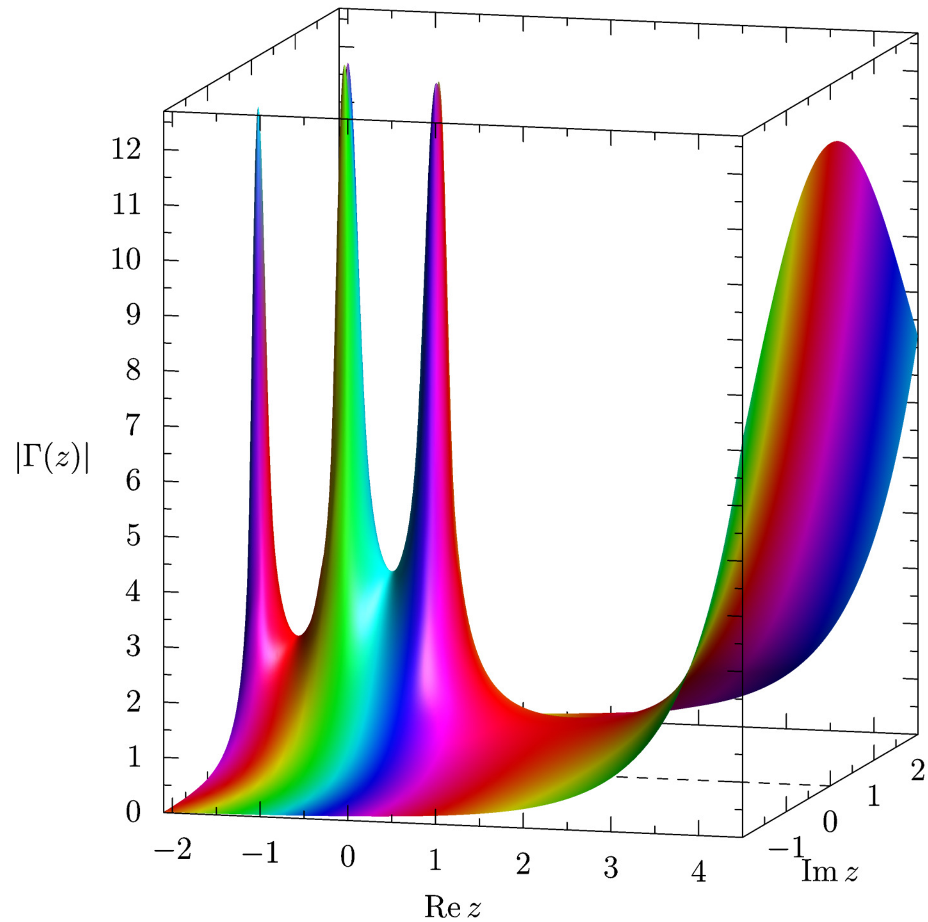


Figure 2.2: Surface plot of $\Gamma(z)$ in the complex plane, using an RGB color wheel to represent the phase. Red indicates real positive values. The poles at the negative integers and 0 are evident.

For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, we re-write the genearlized binomial coefficient as follows:

$$\begin{aligned}
\binom{\alpha}{k} &= \frac{1}{k!}\prod_{j=0}^{k-1}(\alpha - j) = \frac{\alpha(\alpha-1)\cdots(\alpha-k)(\alpha-k+1)}{k!} \\
&= \frac{\alpha(\alpha-1)\cdots(\alpha-k)(\alpha-k+1)}{k!}\cdot\frac{\Gamma(\alpha-k+1)}{\Gamma(\alpha-k+1)} \\
&= \frac{\alpha(\alpha-1)\cdots(\alpha-k)\Gamma(\alpha-k)}{k!\,\Gamma(\alpha-k+1)} \quad \text{(using Eq. (2.1))} \\
&= \frac{\alpha\Gamma(\alpha)}{k!\,\Gamma(\alpha-k+1)} = \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}. \qquad (2.5)
\end{aligned}$$

The above formula is a convenient way to represent the general binomial coefficient in terms of the Γ function. However, there is a slight variation of this formula which is also convenient. To arrive at this other formula we need to make use of certain properties of the Γ function. Using Eq. (2.2) with $z = \alpha + 1$, we get

$$\Gamma(\alpha+1)\Gamma(-\alpha) = \frac{\pi}{\sin\pi(\alpha+1)}.$$

Again using Eq. (2.2), but with $z = \alpha - k + 1$, we get

$$\Gamma(\alpha-k+1)\Gamma(k-\alpha) = \frac{\pi}{\sin\pi(\alpha-k+1)} = (-1)^k \frac{\pi}{\sin\pi(\alpha+1)}.$$

Combining these, for $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, allows us to write an equivalent form for the binomial coefficient as

$$\binom{\alpha}{k} = (-1)^k \frac{\Gamma(k-\alpha)}{k!\ \Gamma(-\alpha)}. \tag{2.6}$$

Note that $\binom{\alpha}{0} = 1$, so, for $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, the binomial expansion becomes

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\ \Gamma(\alpha-k+1)} z^k, \tag{2.7}$$

or, equivalently

$$(1+z)^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k-\alpha)}{k!\ \Gamma(-\alpha)} z^k. \tag{2.8}$$

For negative integers $\alpha = -n$ with $n \in \mathbb{N}$, formula Eq. (2.8) can be used directly. For non-negative integer values of α, we take the appropriate limits. Since the Γ function has simple poles at non-positive integer values of its argument, it follows that

$$\lim_{z\to -n} \Gamma(z) = \infty, \quad \text{or equivalently} \quad \lim_{z\to -n} \frac{1}{\Gamma(z)} = 0 \quad \text{for } n \in \mathbb{N}_0,$$

so that for non-negative integers $\alpha = n \in \mathbb{N}_0$, taking the limit in Eq. (2.5) yields

$$\begin{aligned}\binom{n}{k} = \lim_{\alpha\to n}\binom{\alpha}{k} &= \lim_{\alpha\to n} \frac{\Gamma(\alpha+1)}{k!\ \Gamma(\alpha-k+1)} \\ &= \begin{cases} \dfrac{\Gamma(n+1)}{k!\ \Gamma(n-k+1)} \\ 0 \end{cases} = \begin{cases} \dfrac{n!}{k!\ (n-k)!}, & \text{for } k = 0,1,2,\ldots,n, \\ 0, & \text{for } k \geqslant n+1, \end{cases}\end{aligned}$$

and, thus we recover the usual expression

$$(1+z)^n = \sum_{k=0}^{n} \frac{n!}{k!\ (n-k)!} z^k.$$

Problem 2.1: Show that
$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{2^{2n}\, n!}\sqrt{\pi}.$$

Problem 2.2: (Alternate derivation of the binomial expansion in Eq. (2.7).)
Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. Consider the infinite sum

$$f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha+1-n)} z^n. \tag{2.9}$$

(a) What is the radius of convergence R of the series in Eq. (2.9)?

(b) Compute $f'(z)$ for $|z| < R$.

(c) Use your answer in part (b) to find a first-order differential equation with solution $f(z)$ for $|z| < R$. Hint: compute $(1+z)f'(z)$.

(d) Solve the first-order differential equation in part (c) subject to the boundary condition $f(0) = 1$ to determine $f(z)$.

(e) Confirm that we chose the right boundary condition in part (d) by verifying that all of the coefficients in the Taylor series of $f(z)$ about 0 agree with those in Eq. (2.9).

(f) Use part (d) to find the asymptotic expansion of $(t^2+2t)^{-1/2}$ for the asymptotic sequence $\{t^n\}_{n=0}^{\infty}$ as $t \to 0$.

2.B Some Elementary Examples

In this section we give a few examples to illustrate, in a relatively *ad-hoc* manner, some of the techniques used to obtain asymptotic approximations to integrals of the form
$$I(x) = \int_{x_0}^{x} f(\xi)\, d\xi, \quad \text{or} \quad I(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta.$$

If $f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$ for some asymptotic sequence $\{\phi_n\}_{n=0}^{\infty}$ then, provided the conditions of Theorem 1.7 hold, we can integrate the asymptotic series term by term.

- Calculate an asymptotic expansion for $I(x) = \int_x^\infty e^{-t^4}\,dt$ as $(x \to 0^+)$.

If we try to integrate $e^{-t^4} \sim \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{n!}$ $(t \to 0^+)$ term by term we encounter a problem arising from the upper limit of integration:

$$\int_x^\infty e^{-t^4}\,dt \sim \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+1}}{n!\,(4n+1)}\bigg|_x^\infty = *@\star?$$

The correct approach is to first peel off an infinite definite integral:

$$I(x) = \int_x^\infty e^{-t^4}\,dt = \int_0^\infty e^{-t^4}\,dt - \int_0^x e^{-t^4}\,dt \sim \Gamma\left(\frac{5}{4}\right) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{n!\,(4n+1)} \ (x \to 0^+).$$

Note the form of the terms in the asymptotic series:

$$\phi_n(x) = x^{4n+1} = \alpha(x)\beta^n(x), \text{ where } \alpha(x) = x,\ \beta(x) = x^4.$$

- Find an asymptotic series as $(z \to 0 \text{ in } A)$ for the *exponential integral*

$$E_1(z) = \int_z^\infty \frac{e^{-\zeta}}{\zeta}\,d\zeta, \text{ where } A = \mathcal{S}(0, -\theta_0, \theta_0), \quad \theta_0 < \frac{\pi}{2}.$$

Since E_1 has a singularity at $z = 0$, we cannot split $\int_z^\infty = \int_0^\infty - \int_0^z$. Instead, consider its derivative:

$$E_1'(z) = -\frac{e^{-z}}{z} \sim -\sum_{n=0}^{\infty} \frac{(-1)^n z^{n-1}}{n!} \sim -\frac{1}{z} - \sum_{n=1}^{\infty} \frac{(-1)^n z^{n-1}}{n!}.$$

On moving the first term to the left-hand side and integrating from 1 to z, we find

$$E_1(z) + \log z \sim -\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n\,n!} + C \quad (z \to 0 \text{ in } A).$$

where the constant C can be evaluated as

$$\begin{aligned}
C &= \lim_{\substack{z\to 0\\ z\in A}} (E_1(z) + \log z) \\
&= \lim_{\substack{z\to 0\\ z\in A}} \left[\int_z^\infty \frac{e^{-\zeta}}{\zeta}\,d\zeta + \log z - \log(1+z)\right] \\
&= \lim_{\substack{z\to 0\\ z\in A}} \lim_{T\to\infty} \int_z^T \frac{e^{-\zeta}}{\zeta} - \frac{1}{\zeta} + \frac{1}{1+\zeta}\,d\zeta \\
&= \int_{0^+}^\infty \frac{e^{-\zeta}}{\zeta} - \frac{1}{\zeta(1+\zeta)}\,d\zeta.
\end{aligned}$$

On using the fact that

$$\log n = \int_1^n \frac{1}{k}\,dk = \int_1^n \int_0^\infty e^{-kx}\,dx\,dk = \int_{0^+}^\infty \int_1^n e^{-kx}\,dk\,dx = \int_{0^+}^\infty \frac{e^{-x}-e^{-nx}}{x}\,dx,$$

we can show that $C = -\gamma$, where γ is Euler's constant:

$$\begin{aligned}
\gamma &\doteq \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \log n\right)\\
&= \lim_{n\to\infty}\left(\sum_{k=1}^n \int_{0^+}^\infty e^{-kx}\,dx - \int_{0^+}^\infty \frac{e^{-x}-e^{-nx}}{x}\,dx\right)\\
&= \int_{0^+}^\infty \left(\frac{e^{-x}}{1-e^{-x}} - \frac{e^{-x}}{x}\right)dx\\
&= \lim_{\delta\to 0^+}\int_\delta^\infty \left(\frac{1}{e^x-1} - \frac{e^{-x}}{x}\right)dx\\
&= \lim_{\delta\to 0^+}\left(\int_{e^\delta-1}^\infty \frac{1}{y(y+1)}\,dy - \int_\delta^\infty \frac{e^{-x}}{x}\,dx\right)\\
&= \int_{0^+}^\infty \frac{1}{x}\left(\frac{1}{1+x} - e^{-x}\right)dx\\
&= 0.5772\ldots,
\end{aligned}$$

using the substitution $y = e^x - 1$.

Therefore

$$E_1(z) = \int_z^\infty \frac{e^{-\zeta}}{\zeta}\,d\zeta \sim -\log z - \gamma - \sum_{n=1}^\infty \frac{(-1)^n z^n}{n\,n!} \quad (z \to 0 \text{ in } A).$$

Incidentally, the fact that

$$\lim_{\substack{z\to 0\\ z\in A}}(E_1(z) + \log z) = -\gamma$$

implies a close connection between the constant γ and the Γ function, namely:

$$\begin{aligned}
\Gamma'(1) &= \int_{0^+}^\infty e^{-t}\log t\,dt\\
&= \lim_{z\to 0^+}\left([-e^{-t}\log t]_z^\infty + \int_z^\infty \frac{1}{t}e^{-t}\,dt\right)\\
&= \lim_{z\to 0^+}(\log z + E_1(z))\\
&= -\gamma.
\end{aligned}$$

A slight reformulation of the definition for γ in terms of a telescoping series,

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)+\lim_{n\to\infty}\log\left(\frac{n+1}{n}\right)\\ &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log(n+1)\right)\\ &= \sum_{k=1}^{\infty}\left[\frac{1}{k}-\log\left(\frac{k+1}{k}\right)\right],\end{aligned}$$

has a simple geometric interpretation as the sum of the areas of the green regions in Figure 2.3. The total shaded (green+red) areas is just the area 1 of the rectangle $[1,2]\times[0,1]$ (since the telescoping sum $\sum_{k=1}^{\infty}[\frac{1}{k}-\frac{1}{k+1}]=1$), so γ is the fraction of the shaded area that is coloured green. The convexity of the graph of $f(x)=1/x$ on $(0,\infty)$ then establishes that $1/2<\gamma<1$.

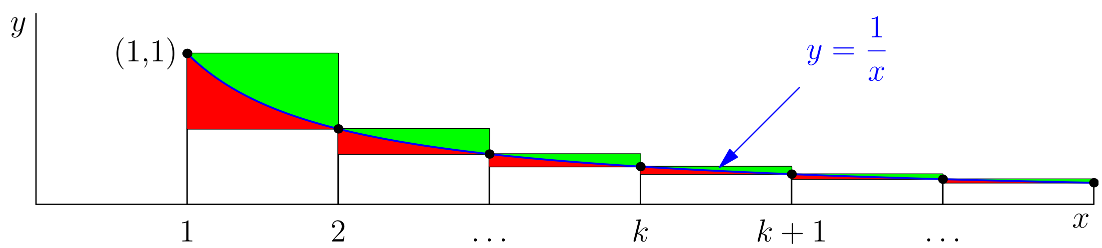


Figure 2.3: Geometrical interpretation of Euler's constant.

2.C Integration by Parts

One of the most useful techniques for assisting us in the asymptotic approximation of integrals is integration by parts. This is easily illustrated with a few examples.

- Find the asymptotic behaviour of $I(x)=\displaystyle\int_x^{\infty}e^{-t^4}\,dt$ as $(x\to\infty)$. Clearly $\displaystyle\lim_{x\to\infty}I(x)=0$. But how fast does $I(x)\to 0$? Consider

$$I(x)=-\frac{1}{4}\int_x^{\infty}\frac{1}{t^3}\frac{d}{dt}\left(e^{-t^4}\right)dt=-\left.\frac{1}{4t^3}e^{-t^4}\right|_x^{\infty}-\frac{3}{4}\int_x^{\infty}\frac{e^{-t^4}}{t^4}\,dt=\frac{e^{-x^4}}{4x^3}-\frac{3}{4}I_1(x),$$

where

$$I_n(x)\doteq\int_x^{\infty}\frac{e^{-t^4}}{t^{4n}}\,dt\leqslant\frac{1}{x^4}\int_x^{\infty}\frac{e^{-t^4}}{t^{4n-4}}\,dt=\frac{I_{n-1}(x)}{x^4}.$$

From this inequality it follows that $I_{n+1} = o(I_n)$ $(x \to \infty)$. Noting that $I(x) = I_0(x)$, we then see that $\dfrac{e^{-x^4}}{4x^3} = I_0(x) + \dfrac{3}{4}I_1(x) \sim I_0(x) = I(x)$ as $x \to \infty$. We have thus obtained the one term asymptotic expansion

$$I(x) \sim \frac{e^{-x^4}}{4x^3} \quad (x \to \infty).$$

Continuing along these lines, we find for $n = 0, 1, 2, \ldots$

$$\begin{aligned} I_n(x) &= -\frac{1}{4}\int_x^\infty \frac{1}{t^{4n+3}}\frac{d}{dt}\left(e^{-t^4}\right) dt \\ &= \frac{e^{-x^4}}{4x^{4n+3}} - \left(\frac{4n+3}{4}\right) I_{n+1}(x) \\ &= \frac{1}{4}[\phi_n(x) - (4n+3)I_{n+1}(x)], \end{aligned}$$

where $\phi_n(x) \doteq \dfrac{e^{-x^4}}{x^{4n+3}}$ forms an asymptotic sequence: $\lim\limits_{x\to\infty} \dfrac{\phi_{n+1}(x)}{\phi_n(x)} = 0$. Note for $(x \to \infty)$ that $I_{n+1} = o(I_n)$ implies that $I_{n+1} = o(\phi_n)$ as well. Hence

$$I(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \quad (x \to \infty),$$

where $a_0 = \frac{1}{4}$, $a_{n+1} = -(\frac{4n+3}{4})a_n = -(-1)^n \dfrac{3 \cdot 7 \cdot 11 \cdot \ldots \cdot (4n+3)}{4^{n+2}}$, $n \geqslant 0$. The terms in the asymptotic series may thus be written in the form

$$\phi_n(x) = \alpha(x)\beta^n(x), \text{ with } \alpha(x) = \frac{e^{-x^4}}{x^3},\ \beta(x) = \frac{1}{x^4}.$$

The integral thus has the asymptotic representation

$$\int_x^\infty e^{-t^4}\, dt \sim \frac{e^{-x^4}}{x^3}\sum_{n=0}^{\infty}\frac{a_n}{x^{4n}} \sim \frac{e^{-x^4}}{4x^3}\left[1 - \frac{3}{4x^4} + \frac{3\cdot 7}{4^2x^8} - \ldots\right] \quad (x \to \infty).$$

Remark: Comparing to the example on p. 27, we now have asymptotic expansions for $I(x)$ for both small and large x:

$$\int_x^\infty e^{-t^4}\, dt \sim \Gamma\left(\frac{5}{4}\right) - \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{n!\,(4n+1)} \qquad (x \to 0^+) \qquad \text{(series converges)},$$

$$\int_x^\infty e^{-t^4}\, dt \sim \frac{e^{-x^4}}{x^3}\sum_{n=0}^{\infty}\frac{a_n}{x^{4n}} \qquad (x \to \infty) \qquad \text{(series diverges)}.$$

The expansion for $x \to 0^+$ is just the Taylor's series for $I(x)$. Although this series converges for all x, the convergence is slow for large x, as seen in Figure 2.4. In contrast, the first few terms of the divergent series for $I(x)$ (as $x \to \infty$) accurately approximate the exact value of $I(x)$ at large x.

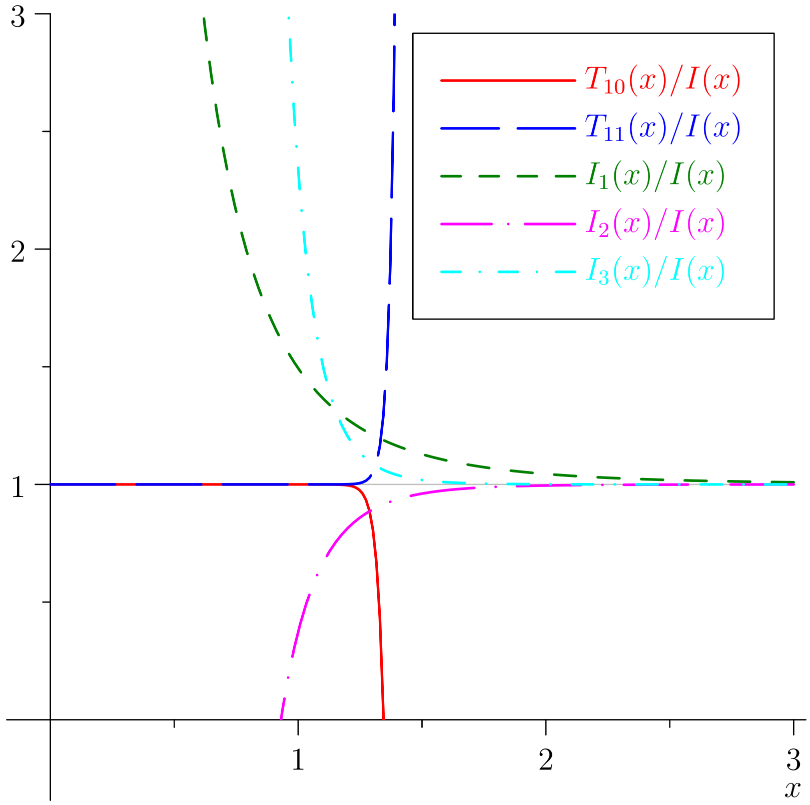


Figure 2.4: Comparison of the 10 term (T_{10}) and 11 term (T_{11}) Taylor series expansions of $I(x)$ about $x = 0$ with the N-term asymptotic expansion of $I(x) = \int_x^\infty e^{-t^4}\,dt$ $(x \to \infty)$ for $N = 1, 2, 3$.

- Behaviour of $I(x) = \displaystyle\int_0^x e^{t^2}\,dt \quad (x \to \infty)$.

Wrong approaches:

$$I(x) = \int_0^x \sum_{n=0}^\infty \frac{t^{2n}}{n!}\,dt = \sum_{n=0}^\infty \frac{x^{2n+1}}{n!\,(2n+1)} \qquad \text{not an asymptotic series } (x \to \infty)$$

$$I(x) = \int_0^\infty e^{t^2}\,dt - \int_x^\infty e^{t^2}\,dt \qquad \text{form } \infty - \infty$$

$$I(x) = \frac{1}{2}\int_0^x \frac{1}{t}\frac{d}{dt}e^{t^2}\,dt = \left.\frac{e^{t^2}}{2t}\right|_0^x + \frac{1}{2}\int_0^x \frac{e^{t^2}}{t^2}\,dt \qquad \text{singular at } t = 0$$

Correct approach:

Introduce a cutoff parameter $a \in (0, x)$:

$$\begin{aligned} I(x) &= \int_0^a e^{t^2}\,dt + \int_a^x e^{t^2}\,dt \\ &= I(a) + J_0(a, x), \end{aligned}$$

where $J_n(a, x) = \displaystyle\int_a^x \frac{e^{t^2}}{t^{2n}}\,dt$. Then

$$\begin{aligned} J_0(a, x) &= \frac{1}{2}\int_a^x \frac{1}{t}\frac{d}{dt}e^{t^2}\,dt = \left.\frac{e^{t^2}}{2t}\right|_a^x + \frac{1}{2}\int_a^x \frac{e^{t^2}}{t^2}\,dt \\ &= \frac{e^{x^2}}{2x} - \frac{e^{a^2}}{2a} + \frac{1}{2}J_1(a, x). \end{aligned}$$

We now show that $J_1(a, x) = o(e^{x^2}/2x)$:

$$\lim_{x\to\infty} \frac{J_1(a, x)}{\frac{e^{x^2}}{2x}} = \lim_{x\to\infty} \frac{\displaystyle\int_a^x \frac{e^{t^2}}{t^2}\,dt}{\frac{e^{x^2}}{2x}} = \lim_{x\to\infty} \frac{\frac{e^{x^2}}{x^2}}{(1 - \frac{1}{2x^2})e^{x^2}} = \lim_{x\to\infty} \frac{2}{2x^2 - 1} = 0.$$

Thus

$$\lim_{x\to\infty} \frac{I(x)}{\frac{e^{x^2}}{2x}} = \lim_{x\to\infty} \left\{ \frac{I(a)}{\frac{e^{x^2}}{2x}} + 1 - \frac{\frac{e^{a^2}}{2a}}{\frac{e^{x^2}}{2x}} + \frac{\frac{1}{2}J_1(a, x)}{\frac{e^{x^2}}{2x}} \right\} = 1.$$

That is,

$$I(x) \sim \frac{e^{x^2}}{2x} \quad (x \to \infty).$$

Notice that this one-term asymptotic series for $I(x)$ is independent of the cuttoff parameter a (why?).

Continuing in this manner leads to

$$\int_0^x e^{t^2}\,dt \sim \frac{e^{x^2}}{2x}\left[1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{(2x^2)^n}\right] \ (x \to \infty),$$

again independent of a.

Remark: Integration by parts "looks like"

$$\begin{aligned} I(z) &= a_0\phi_0(z) + I_1(z) \\ &= a_0\phi_0(z) + a_1\phi_1(z) + I_2(z) \\ &= \ldots \\ &= \sum_{n=0}^{N} a_n\phi_n(z) + I_{N+1}(z). \end{aligned}$$

Remark: The method will work if one can show that $I_{N+1} = o(\phi_N)$ $(z \to z_0)$.

Remark: Usually one finds $\phi_n(z) = \alpha(z)\beta^n(z)$, so that

$$I(z) \sim \alpha(z) \sum_{n=0}^{\infty} a_n \beta^n(z).$$

2.D Laplace Integrals

Until now we have considered integrals like $\int_0^x e^{t^2}\,dt$, with the variable x appearing only in the limit of integration. In this section, we consider integrals where x appears in the integrand:

$$I(x) = \int_a^b f(x,t)\,dt.$$

In particular, we will use *Laplace's method* to study the behaviour of *Laplace's integral*,

$$I(x) = \int_a^b e^{xh(t)} f(t)\,dt,$$

as $x \to \infty$.

The underlying heuristic argument is that $e^{\alpha x} = o(e^{\beta x})$ as $x \to \infty$ for $\alpha < \beta$. Therefore, one would expect the leading-order asymptotic behaviour of $I(x)$ as $x \to \infty$ to be determined in a neighbourhood of a point c at which h has a maximum in $[a, b]$. As x gets larger, this maximum becomes increasingly steep, so the dominant contribution to the integral comes from the immediate neighbourhood of the peak.

We begin by looking at a special type of Laplace integral that is of practical importance, the *Laplace Transform*:

$$\int_0^{\pi/2} e^{-x\sin t}\,dt \sim \frac{1}{x} \quad (x \to \infty).$$

If h has an interior maximum at c, with $h'(c) = 0$ and $h''(c) < 0$, and $f(c) \neq 0$, one might expect the dominant contribution to be given by

$$\begin{aligned} I(x) &\sim \int_a^b e^{x[h(c)+\frac{1}{2}h''(c)(t-c)^2]} f(c)\,dt \sim e^{xh(c)} f(c) \int_{-\infty}^{\infty} e^{\frac{1}{2}xh''(c)\tau^2}\,d\tau \\ &= e^{xh(c)} f(c) \sqrt{\frac{2\pi}{-xh''(c)}} \quad (x \to \infty). \end{aligned}$$

- Laplace's method predicts the leading-order behaviour

$$\int_{-\pi/2}^{\pi/2} e^{x\cos t}\,dt \sim e^x\sqrt{\frac{2\pi}{x}} \quad (x\to\infty).$$

If h has an enpoint maximum, say at $c=a$, with $h'(a)<0$ and $f(a)\neq 0$, the above argument suggests that

$$I(x)\sim\int_a^b e^{x[h(a)+h'(a)(t-a)]}f(a)\,dt \sim e^{xh(a)}f(a)\int_0^\infty e^{xh'(a)\tau}\,d\tau = \frac{e^{xh(a)}f(a)}{-xh'(a)} \quad (x\to\infty).$$

- Laplace's method predicts the leading-order behaviour

$$\int_0^{\pi/2} e^{-x\sin t}\,dt \sim \frac{1}{x} \quad (x\to\infty).$$

- One may need to keep additional terms in the Taylor series expansion of h or f:

$$\int_0^{\pi/2} e^{x\cos t}\,\sin t\,dt \sim \int_0^\infty e^{x(1-t^2/2)}t\,dt = e^x\left[\frac{e^{-xt^2/2}}{-x}\right]_0^\infty = \frac{e^x}{x} \quad (x\to\infty).$$

In this case we can actually compute the exact behaviour of the original integral:

$$\int_0^{\pi/2} e^{x\cos t}\,\sin t\,dt = \left[\frac{e^{x\cos t}}{-x}\right]_0^{\pi/2} = \frac{e^x-1}{x}.$$

Remark: In the previous example we see that Laplace's procedure to lowest order predicts the first term of the exact result correctly, but not the second. Clearly we need a more rigourous justification of the procedure, to help us know how many terms we need to retain in the Taylor series to obtain a prescribed number of terms in the asymptotic expansion.

- A special case of practical importance is the *Laplace Transform*,

$$I(x)=\int_0^\infty e^{-xt}f(t)\,dt.$$

In this case $h(t)=-t$ has a maximum in $[0,\infty)$ at $t=0$, with $h'(t)=-1$.

The main tool that will allow us to determine the behaviour of Laplace integrals is *Watson's Lemma.* However, in order to prove Watson's Lemma we will need the following integral, which can be easily expressed in terms of the Γ function, using the substitution $\xi = xt$:

$$\int_0^\infty e^{-xt}t^\alpha\,dt = \frac{1}{x^{\alpha+1}}\int_0^\infty e^{-\xi}\xi^\alpha\,d\xi = \frac{\Gamma(\alpha+1)}{x^{\alpha+1}},$$

and the following lemma.

Lemma 2.1 (Small Laplace Tail): *Let $\delta > 0$. Given a function $r(t)$ and some $\bar{x} \in \mathbb{R}$ such that*

$$\int_0^\infty e^{-\bar{x}t} r(t)\, dt$$

converges, the function

$$J(x) \doteq \int_\delta^\infty e^{-xt} r(t)\, dt = o(x^{-\mu}) \qquad (x \to \infty)$$

for all $\mu \in \mathbb{R}$.

Proof: Let

$$K(T) \doteq \int_\delta^T e^{-\bar{x}t} r(t)\, dt.$$

Since $J(\bar{x}) = \lim\limits_{T\to\infty} K(T)$ converges, we know that there exists a $T_0 > 0$ such that

$$T > T_0 \Rightarrow |J(\bar{x}) - K(T)| < \varepsilon.$$

Also, K is continuous and therefore bounded on $[\delta, T_0]$. We conclude that K is bounded on $[\delta, \infty)$. Let $M \doteq \sup\limits_{[\delta,\infty]} |K|$.

For $x > \bar{x}$ we have

$$\begin{aligned} J(x) &= \int_\delta^\infty e^{-(x-\bar{x})t} e^{-\bar{x}t} r(t)\, dt \\ &= \int_\delta^\infty e^{-(x-\bar{x})t} K'(t)\, dt \\ &= e^{-(x-\bar{x})t} K(t)\Big|_\delta^\infty + (x-\bar{x}) \int_\delta^\infty e^{-(x-\bar{x})t} K(t)\, dt \\ &= 0 + (x-\bar{x}) \int_\delta^\infty e^{-(x-\bar{x})t} K(t)\, dt, \end{aligned}$$

so that

$$|J(x)| \leqslant (x-\bar{x}) \int_\delta^\infty e^{-(x-\bar{x})t} |K(t)|\, dt \leqslant (x-\bar{x}) M \int_\delta^\infty e^{-(x-\bar{x})t}\, dt = M e^{-(x-\bar{x})\delta}.$$

Then, since $\lim\limits_{x\to\infty} x^\mu e^{-x\delta} = 0$ for any μ, we see that $\lim\limits_{x\to\infty} |x^\mu J(x)| = 0$, as desired. □

Theorem 2.1 (Watson's Lemma): *If $f(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^{\beta n}$ $(t \to 0^+)$, where $\alpha > -1$ and $\beta > 0$, then*

$$\int_0^\infty e^{-xt} f(t)\, dt \sim \sum_{n=0}^\infty \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}} \quad (x \to \infty),$$

provided the integral converges for all sufficiently large x.

Proof: We are given that

$$f(t) = t^\alpha \sum_{n=0}^{N} a_n t^{\beta n} + r_N(t),$$

where $r_N(t) = o(t^{\alpha+\beta N})$ as $(t \to 0^+)$. Then

$$\int_0^\infty e^{-xt} f(t)\, dt = \sum_{n=0}^{N} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}} + R_N(x),$$

where $R_N(x) = \int_0^\infty e^{-xt} r_N(t)\, dt$. We know that there exists a number $\bar{x}$ such that $R_N(x)$, like the integral on the left-hand side of the above equation, converges on $[\bar{x}, \infty)$. We need to show for all $N \in \mathbb{N}_0$ that $R_N(x) = o\big(\frac{1}{x^{\alpha+\beta N+1}}\big) \quad (x \to \infty)$.

Given $\varepsilon > 0$, there exists $\delta_N > 0$ such that

$$0 < t < \delta_N \Rightarrow |r_N(t)| < \varepsilon t^{\alpha+\beta N}.$$

Decompose $R_N(x) = I_N(x) + J_N(x)$, where

$$I_N(x) \doteq \int_0^{\delta_N} e^{-xt} r_N(t)\, dt,$$

$$J_N(x) \doteq \int_{\delta_N}^\infty e^{-xt} r_N(t)\, dt.$$

We see immediately that $I_N = o\big(\frac{1}{x^{\alpha+\beta N+1}}\big)$ $(x \to \infty)$:

$$|I_N(x)| \leqslant \int_0^{\delta_N} e^{-xt}\, |r_N(t)|\, dt \leqslant \varepsilon \int_0^{\delta_N} e^{-xt} t^{\alpha+\beta N}\, dt < \varepsilon \frac{\Gamma(\alpha + \beta N + 1)}{x^{\alpha+\beta N+1}}.$$

Also, from Lemma 2.1, we see in particular that $J_N = o(x^{-\mu})$ $(x \to \infty)$ for $\mu = \alpha + \beta N + 1$, so that $R_N(x) = o\big(\frac{1}{x^{\alpha+\beta N+1}}\big)$ $(x \to \infty)$, as desired. $\square$

- Recall the Laplace transform considered in Eq. (1.1):

$$I(x) = \int_0^\infty \frac{e^{-xt}}{1+t}\, dt$$

Since

$$\frac{1}{1+t} \sim \sum_{n=0}^{\infty} (-1)^n t^n \quad (t \to 0^+).$$

we see that

$$I(x) \sim \sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^{n+1}} \quad (x \to \infty).$$

- (Modified Bessel function) Using the substitution $t = s - 1$ we can express

$$K_0(x) = \int_1^\infty e^{-xs}(s^2-1)^{-1/2}\,ds = e^{-x}\int_0^\infty e^{-xt}(t^2+2t)^{-1/2}\,dt.$$

From Prob. 2.2, we know the asymptotic expansion

$$\left(t^2+2t\right)^{-1/2} = (2t)^{-1/2}\left(1+\frac{t}{2}\right)^{-1/2} \sim \sqrt{\frac{\pi}{2t}}\sum_{n=0}^{\infty}\frac{t^n}{2^n n!\,\Gamma\left(\frac{1}{2}-n\right)} \qquad (t\to 0^+).$$

Hence

$$K_0(x) \sim e^{-x}\sqrt{\frac{\pi}{2}}\sum_{n=0}\frac{\Gamma\left(n+\frac{1}{2}\right)}{2^n n!\,\Gamma\left(\frac{1}{2}-n\right)x^{n+\frac{1}{2}}} \qquad (x\to\infty).$$

- If $a > 0$, we can use the substitution $u = t - a$ to find the asymptotic behaviour as $x \to \infty$ of

$$I(x) = \int_a^\infty e^{-xt}t^\lambda\,dt = e^{-ax}\int_0^\infty e^{-xu}(u+a)^\lambda\,du = e^{-ax}a^\lambda\int_0^\infty e^{-xu}\left(1+\frac{u}{a}\right)^\lambda du.$$

Since

$$\left(1+\frac{u}{a}\right)^\lambda \sim \sum_{n=0}^{\infty}\binom{\lambda}{n}\frac{u^n}{a^n} \qquad (u\to 0^+),$$

we have

$$\begin{aligned} I(x) &\sim e^{-ax}a^\lambda\sum_{n=0}^{\infty}\binom{\lambda}{n}\frac{\Gamma(n+1)}{a^n x^{n+1}} \\ &\sim e^{-ax}a^\lambda\sum_{n=0}^{\infty}\frac{\prod_{j=0}^{n-1}(\lambda-j)}{a^n x^{n+1}} \qquad (x\to\infty). \end{aligned}$$

Note that this is consistent with Lemma 2.1.

Remark: The next theorem extends Watson's lemma to the case of a bounded interval, since, as just observed, the contribution from the tail of the improper integral is asymptotically small.

Corollary 2.1.1 (Generalized Watson's Lemma): If $f(t) \sim t^\alpha\sum_{n=0}^{\infty}a_n t^{\beta n}$ $(t\to 0^+)$, where $\alpha > -1$ and $\beta > 0$, and $\int_0^\infty e^{-xt}f(t)\,dt$ converges for all sufficiently large x, then for any $a > 0$,

$$I_a(x) \doteq \int_0^a e^{-xt}f(t)\,dt \sim \sum_{n=0}^{\infty}\frac{a_n\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \qquad (x\to\infty).$$

Proof: From Watson's Lemma, we know that

$$I_\infty(x) \doteq \int_0^\infty e^{-xt} f(t)\,dt \sim \sum_{n=0}^\infty \frac{a_n \Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \qquad (x \to \infty).$$

Let

$$R_a(x) \doteq I_\infty(x) - I_a(x) = \int_a^\infty e^{-xt} f(t)\,dt = o\left(\frac{1}{x^{\alpha+\beta n+1}}\right) \qquad (x \to \infty),$$

by Lemma 2.1, for every $n \in \mathbb{N}_0$. Hence in view of Theorem 1.4, I_a and I_∞ have the same asymptotic expansion with respect to $1/x^{\alpha+\beta n+1}$ as $x \to \infty$. □

Remark: Watson's Lemmas can sometimes be used to find amptotic expansions for Laplace integrals even when $h(t) \neq -t$, using the substitution $\xi = -h(t)$.

- The asymptotic behaviour of

$$I(x) = \int_0^\infty e^{-x \sinh t}\,dt$$

as $x \to \infty$ can be found by first doing the substitution $\xi = \sinh t$:

$$I(x) = \int_0^\infty e^{-x\xi} \frac{d\xi}{\sqrt{1+\xi^2}}.$$

From Prob. 2.2, we know

$$\left(1+\xi^2\right)^{-1/2} \sim \sqrt{\pi} \sum_{n=0}^\infty \frac{\xi^{2n}}{n!\,\Gamma\left(\frac{1}{2}-n\right)} \qquad (\xi \to 0).$$

Hence

$$I(x) \sim \sqrt{\pi} \sum_{n=0}^\infty \frac{\Gamma(2n+1)}{n!\,\Gamma\left(\frac{1}{2}-n\right) x^{2n+1}} \qquad (x \to \infty).$$

- To find the asymptotic behaviour as $x \to \infty$ of

$$I(x) = \int_0^{\pi/2} e^{-x \sin^2 t}\,dt,$$

let $\xi = \sin^2 t$, so that $d\xi = 2 \sin t \cos t\,dt = 2\xi^{1/2}(1-\xi)^{1/2}\,dt$ and

$$I(x) = \frac{1}{2} \int_0^1 \frac{e^{-x\xi}}{\xi^{1/2}(1-\xi)^{1/2}}\,d\xi.$$

From the asymptotic expansion

$$\xi^{-1/2}(1-\xi)^{-1/2} \sim \sqrt{\pi}\xi^{-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^n}{n!\,\Gamma\big(\frac{1}{2}-n\big)} \qquad (\xi \to 0),$$

we then see that

$$I(x) \sim \frac{1}{2}\sqrt{\frac{\pi}{x}}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\frac{1}{2}+n)}{n!\,\Gamma\big(\frac{1}{2}-n\big)x^n} \qquad (x \to \infty).$$

We now turn to more general Laplace integrals. If we want to use Watson's Lemma to find the asymptotic behaviour of

$$I(x) = \int_a^b e^{xh(t)} f(t)\,dt$$

as $x \to \infty$, where h is a real-valued strictly decreasing continuous function on $[a,b]$, the substitution $\xi = H(t) \doteq h(a) - h(t)$ leads to

$$I(x) = e^{xh(a)}\int_0^{h(a)-h(b)} e^{-x\xi}\,F(\xi)\,d\xi,$$

where $F(\xi) \doteq f(H^{-1}(\xi))(H^{-1})'(\xi)$.

The following result extends our Generalized Watson's Lemma to the more general situation where the function h in the integrand is known to be decreasing in a neighbourhood of a global maximum at a.

Corollary 2.1.2 (Laplace's Method): Suppose f and h are real-valued functions on $[a,b]$, such that $f \in C$, $h \in C^1$, and $h' < 0$ on some subinterval (a,c). Suppose also that $h(t) \leqslant M < h(a)$ for $t \in (c,b)$, so that the maximum of h is approached only at a. Define $H(t) \doteq h(a) - h(t)$ for $t \in (a,c)$ and $F(\xi) \doteq f(H^{-1}(\xi))(H^{-1})'(\xi)$ and suppose

$$F(\xi) \sim \xi^{\alpha}\sum_{n=0}^{\infty}\gamma_n\xi^{\beta n} \qquad \xi \to 0^+,$$

with $\alpha > -1$ and $\beta > 0$. Then

$$I(x) = \int_a^b e^{xh(t)} f(t)\,dt \sim e^{xh(a)}\sum_{n=0}^{\infty}\frac{\gamma_n\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \quad (x \to \infty),$$

provided the integral converges absolutely for all $x \geqslant X$.

Proof: On making the substitution $\xi = H(t)$ for $t \in [a, c]$, we find for $x > X$ that

$$\begin{aligned}\left| I(x) - e^{xh(a)} \int_0^{h(a)-h(c)} e^{-x\xi}\, F(\xi)\, d\xi \right| &= \left| \int_c^b e^{xh(t)} f(t)\, dt \right| \\ &\leqslant \int_c^b e^{(x-X)h(t)} e^{Xh(t)}\, |f(t)|\, dt \\ &\leqslant e^{(x-X)M} \int_c^b e^{Xh(t)}\, |f(t)|\, dt \\ &= o\left(\frac{e^{xh(a)}}{x^{\alpha+\beta n+1}} \right) \qquad (x \to \infty)\end{aligned}$$

for any $n \in \mathbb{N}_0$. In view of Theorem 1.4, we then see from Corollary 2.1.1 that

$$I(x) \sim e^{xh(a)} \int_0^{h(a)-h(c)} e^{-x\xi}\, F(\xi)\, d\xi \sim e^{xh(a)} \sum_{n=0}^{\infty} \frac{\gamma_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}} \quad (x \to \infty). \quad \square$$

Remark: In order to use this result requires a knowledge of F, which depends on H^{-1}. Of course the difficulty here is in having to compute H^{-1}.

Remark: If h is holomorphic, then an asymptotic expansion for $H^{-1}(\xi)$ can be found by *series reversion*. Letting $\tau = t - a$ and $\xi = g(\tau) = H(a + \tau)$, so that $g(0) = 0$, we need to invert

$$\xi = g(\tau) = a_1\tau + a_2\tau^2 + a_3\tau^3 + \ldots + a_N\tau^N + R_N, \tag{2.10}$$

where $a_1 \neq 0$ and $R_N = o(\tau^N)$ as $\tau \to 0$, for the inverse function $\tau = g^{-1}(\xi)$. A (unique) series expansion

$$\tau = g^{-1}(\xi) = b_1\xi + b_2\xi^2 + b_3\xi^3 + \ldots + b_N\xi^N + S_N, \tag{2.11}$$

where $S_N = o(\xi^N)$ as $\xi \to 0$, for the inverse function can be obtained by substituting Eq. (2.11) into Eq. (2.10):

$$\begin{aligned}\xi = a_1b_1\xi &+ \left(a_2b_1^2 + a_1b_2\right)\xi^2 + \left(a_3b_1^3 + 2a_2b_1b_2 + a_1b_3\right)\xi^3 \\ &+ \left(3a_3b_1^2b_2 + a_2b_2^2 + a_2b_1b_3\right) + \ldots + o(\xi^N) \qquad (\xi \to 0),\end{aligned}$$

on noting as $\xi \to 0$ that $\tau \sim a_1^{-1}\xi$ and hence $\tau^N = \mathcal{O}(\xi^N)$, so that $R_N = o(\tau^N)$ implies $R_N = o(\xi^N)$. On equating like coefficients, we obtain

$$\begin{aligned} b_1 &= a_1^{-1}, \\ b_2 &= -a_1^{-3}a_2, \\ b_3 &= a_1^{-5}(2a_2^2 - a_1a_3), \end{aligned}$$

and so on. A general formula for the coefficients b_n is derived in Appendix A. The desired series for $H^{-1}(\xi)$ is then given by $a + g^{-1}(\xi)$.

- When $h(t) = -t^2$, we reproduce the (exact) result

$$\int_0^\infty e^{-xt^2}\,dt = \int_0^\infty e^{-x\xi}\frac{d\xi}{2\sqrt{\xi}} \sim \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{x}} = \frac{1}{2}\sqrt{\frac{\pi}{x}} \quad (x \to \infty).$$

- When $h(t) = -t^\lambda$ and $f(t) = t^\alpha$ we reproduce another exact result (cf. Eq. (2.3)):

$$\begin{aligned}\int_0^\infty e^{-xt^\lambda}t^\alpha\,dt &= \int_0^\infty e^{-x\xi}\xi^{\alpha/\lambda}\frac{d\xi}{\lambda\xi^{(\lambda-1)/\lambda}}\\ &= \frac{1}{\lambda}\int_0^\infty e^{-x\xi}\xi^{(\alpha+1)/\lambda-1}d\xi\\ &\sim \frac{1}{\lambda}\frac{\Gamma\left(\frac{\alpha+1}{\lambda}\right)}{x^{\frac{\alpha+1}{\lambda}}} \quad (x \to \infty).\end{aligned}$$

Remark: Because of the difficulty in computing H^{-1}, a widely used alternative to doing a change of variables in the integrand is to substitute an asymptotic series for the functions $h(t)$ and $f(t)$ directly into the integral. This technique relies on the following elementary property.

Lemma 2.2: *If $f \sim g$ and g is bounded as $x \to x_0$ then $e^f \sim e^g$ as $x \to x_0$.*

Proof: There exist numbers $\delta_0 > 0$ and $M > 0$ such that $|g| \leqslant M$ whenever $|x - x_0| < \delta_0$. Then given $\varepsilon > 0$, we can find a $\delta \in (0, \delta_0) > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f - g| \leqslant \frac{\varepsilon}{M}|g| \leqslant \varepsilon.$$

Thus $\lim_{x\to x_0}[f(x) - g(x)] = 0$. Since e^x is a continuous function, we then see that

$$\lim_{x\to x_0}\frac{e^{f(x)}}{e^{g(x)}} = \lim_{x\to x_0}e^{f(x)-g(x)} = e^{\lim_{x\to x_0}[f(x)-g(x)]} = e^0 = 1.$$

Hence $e^f \sim e^g$ as $x \to x_0$.

Remark: It is not necessarily true that $f \sim g$ implies $h(f) \sim h(g)$ for any continuous function h: consider $f(x) = x$, $g(x) = \pi$ and $h(x) = \sin x$ as $x \to \pi$.

Problem 2.3: Prove that if h is continuous everywhere and there exists a λ so that $|h(g(z))| \geqslant \lambda > 0$ and g is bounded for all z sufficiently near z_0, then $f \sim g \Rightarrow h(f) \sim h(g)$ as $z \to z_0$.

Remark: In particular we see from Problem 2.3 that a complex version of Lemma 2.2 also holds: if $f \sim g$ and g is bounded as $z \to z_0$ then $e^f \sim e^g$ as $z \to z_0$.

If h has a maximum at $t = a$, with $h'(a) < 0$ and $f(a) \neq 0$, then one obtains

$$I(x) \sim \int_a^b e^{x[h(a)+h'(a)(t-a)]} f(a)\, dt \sim e^{xh(a)} f(a) \int_0^\infty e^{xh'(a)\tau}\, d\tau = \frac{e^{xh(a)} f(a)}{-xh'(a)} \quad (x \to \infty).$$

- Laplace's method predicts the leading-order behaviour

$$\int_0^{\pi/2} e^{-x \sin t}\, dt \sim \frac{1}{x} \quad (x \to \infty).$$

- Laplace's method predicts the leading-order behaviour

$$\int_0^{\pi/2} e^{x \cos t} \sin t\, dt \sim \int_0^\infty e^{x(1-t^2/2)} t\, dt = e^x \left[\frac{e^{-xt^2/2}}{-x}\right]_0^\infty \sim \frac{e^x}{x} \quad (x \to \infty).$$

In this case we can actually compute the exact behaviour of the original integral:

$$\int_0^{\pi/2} e^{x \cos t} \sin t\, dt = \left[\frac{e^{x \cos t}}{-x}\right]_0^{\pi/2} = \frac{e^x - 1}{x}.$$

In the previous examples $h'(a) \neq 0$ What if $h'(a) = 0$? Corollary 2.1.2 can be applied to validate and generalize the heuristic leading-order asymptotic expansion when h has a maximum of arbitrary order:

Corollary 2.1.3 (Maximum with $N-1$ Zero Derivatives): Let f and h be infinitely differentiable real-valued functions on $[a, b]$. Suppose $f(a) \neq 0$ and h has an exterior maximum at a, with $h^{(n)}(a) = 0$ for $n = 1, 2, \ldots, N-1$, $h^{(N)}(a) < 0$, and $\sup_{[c,b]} h(t) < h(a)$ for all $c \in (a, b)$. Then the leading-order asymptotic expansion as $x \to \infty$ of $I(x) = \int_a^b e^{xh(t)} f(t)\, dt$ is

$$I(x) \sim \frac{1}{N}\Gamma\left(\frac{1}{N}\right) e^{xh(a)} f(a) \left(\frac{N!}{-h^{(N)}(a)x}\right)^{1/N} \quad (x \to \infty).$$

Proof: In terms of the positive constant $K \doteq -h^{(N)}(a)/N!$ we have

$$h(t) \sim h(a) - K(t-a)^N \qquad (t \to a^+).$$

Moreover, from Taylor's remainder theorem for h', we see that

$$h'(t) = -NK(t-a)^{N-1} + \frac{h^{(N+1)}(\xi)}{N!}(t-a)^N,$$

for $t \in (a, b]$ and some $\xi \in (a, t)$. Let $M > 0$ be an upper bound for the continuous function $h^{(N+1)}/N!$ on $[a, b]$. Then

$$h'(t) < (t-a)^{N-1}[-NK + M(t-a)] < 0$$

if $a < t < a + NK/M \doteq c$. We may thus apply Corollary 2.1.2 with

$$u = H(t) = h(a) - h(t) \sim K(t-a)^N \qquad (t \to a^+).$$

Note that H is invertible on (a, c) and

$$\xi = H(H^{-1}(\xi)) \sim K(H^{-1}(\xi) - a)^N \qquad (\xi \to 0^+),$$

so that

$$H^{-1}(\xi) \sim a + \left(\frac{\xi}{K}\right)^{1/N} \qquad (\xi \to 0^+).$$

We then see that $f(H^{-1}(\xi)) \sim f(a)$ $(\xi \to 0^+)$ and

$$(H^{-1})'(\xi) \sim \frac{1}{NK^{1/N}}\xi^{1/N-1} \qquad (\xi \to 0^+).$$

Hence

$$\begin{aligned} I(x) &\sim e^{xh(a)} f(a) \frac{\Gamma(1/N)}{N(Kx)^{1/N}} \\ &\sim \frac{1}{N}\Gamma\left(\frac{1}{N}\right) e^{xh(a)} f(a) \left(\frac{N!}{-h^{(N)}(a)x}\right)^{1/N} \quad (x \to \infty). \quad \square \end{aligned}$$

- In this example we have $h(t) = \cos t$ with $N = 2$. Laplace's method predicts the leading-order behaviour

$$\int_0^{\pi/2} e^{x\cos t}\, dt \sim e^x \sqrt{\frac{\pi}{2x}} \quad (x \to \infty).$$

In order to obtain more than the leading order behaviour of the integral, more terms of the asymptotic expansions of f and h must be used. The natural question which then arises is: How many terms from the asymptotic expansions of h and f must we retain in order to achieve some predetermined order of approximation for the integral? From our Generalized Watson's Lemma we find for $\alpha > \lambda$ that

$$\int_0^1 e^{-xt^\lambda} e^{xt^\alpha} t^\beta \, dt = \int_0^1 e^{-xt^\lambda} \sum_{k=0}^{\infty} \frac{x^k t^{\alpha k+\beta}}{k!} \, dt \sim \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{\alpha k+\beta+1}{\lambda}\right)}{k!\, \lambda x^{\frac{\alpha k+\beta+1}{\lambda}-k}} \qquad (x \to \infty),$$

so to expand to order $x^{-\mu}$ we require all terms that satisfy

$$\alpha k + \beta \leqslant \lambda k + \lambda \mu - 1. \tag{2.12}$$

- Revisiting the behaviour of (p. 39)

$$I(x) = \int_0^{\pi/2} e^{-x \sin^2 t} \, dt,$$

as $x \to \infty$, one could alternatively expand

$$\sin^2 t \sim \left(t - \frac{t^3}{3!} + \mathcal{O}(t^5)\right)^2 \sim t^2 - \frac{t^4}{3} + \mathcal{O}(t^6) \qquad (t \to 0).$$

On setting $\lambda = 2$, $\beta = 0$, and $\mu = 3/2$ in Eq. (2.12) we see that to expand to order $x^{-3/2}$ we need to keep terms with $4k \leqslant \alpha k \leqslant 2k + 2$ since $\alpha \geqslant 4$; this implies that $k = 0$ or $k = 1, \alpha = 4$:

$$I(x) \sim \int_0^1 e^{-xt^2} e^{xt^4/3} \, dt \sim \int_0^\infty e^{-xt^2} \left(1 + x\frac{t^4}{3}\right) dt \sim \frac{1}{2}\frac{\Gamma(\frac{1}{2})}{\sqrt{x}} + \frac{1}{3}\frac{x\Gamma(\frac{5}{2})}{2x^{5/2}}$$
$$\sim \frac{1}{2}\frac{\Gamma(\frac{1}{2})}{\sqrt{x}} + \frac{1}{3}\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma(\frac{1}{2})}{2x^{3/2}} \sim \frac{1}{2}\sqrt{\frac{\pi}{x}}\left(1 + \frac{1}{4x}\right) \quad (x \to \infty).$$

- For $a > -1$, let us find the behaviour of

$$I(x) = \int_0^1 e^{-x(t+at^2)} \, dt,$$

as $x \to \infty$ in three different ways:

(i) Letting $\xi = t + at^2$, we see for $t \in (0, 1/2)$ that $\frac{d\xi}{dt} = 1 + 2at > 1 - 2t > 0$, and $t = (-1+\sqrt{1+4a\xi})/2a$. On $[1/2, 1]$ we see for $a \geqslant 0$ that $\xi \geqslant \min(\xi(1/2), \xi(1)) > 0$ and for $-1 < a < 0$ that $\xi \geqslant \min(\xi(1/2), \xi(1), -1/(4a)) > 0$. We may thus apply Corollary 2.1.2:

$$\begin{aligned}
I(x) &\sim \int_0^{\xi(1/2)} e^{-x\xi} \frac{1}{\sqrt{1+4a\xi}}\, d\xi \qquad (x \to \infty) \\
&\sim \int_0^{\infty} e^{-x\xi} \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(4a\xi)^n}{n!\, \Gamma\left(\frac{1}{2} - n\right)}\, d\xi \qquad (x \to \infty) \\
&\sim \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(4a)^n}{\Gamma\left(\frac{1}{2} - n\right) x^{n+1}} \qquad (x \to \infty) \\
&\sim \frac{1}{x} - \frac{2a}{x^2} + \frac{12a^2}{x^3} \qquad (x \to \infty).
\end{aligned}$$

(ii) Alternatively, we could find the first few terms with series reversion. For

$$\xi = h(t) = t + at^2$$

we find

$$h^{-1}(\xi) = \xi - a\xi^2 + 2a^2\xi^3 + \ldots \quad (\xi \to 0)$$

so that $(h^{-1})'(\xi) = 1 - 2a\xi + 6a^2\xi^2 + \ldots$ and

$$\begin{aligned}
I(x) &\sim \int_0^{\infty} e^{-x\xi} \left(1 - 2a\xi + 6a^2\xi^2 + \ldots\right) du \\
&\sim \frac{1}{x} - \frac{2a}{x^2} + \frac{12a^2}{x^3} \qquad (x \to \infty).
\end{aligned}$$

(iii) As a final check, let us Taylor expand e^{-xat^2} about $t = 0$. On setting $\lambda = 1$, $\alpha = 2$, $\beta = 0$, and $\mu = 3$ in Eq. (2.12) we see that to expand to order x^{-3} we need to retain terms with $2k \leqslant k + 2$; this implies that $k \leqslant 2$:

$$I(x) \sim \int_0^1 e^{-xt} \left(1 - xat^2 + \frac{x^2a^2t^4}{2}\right) dt \sim \frac{1}{x} - \frac{2a}{x^2} + \frac{12a^2}{x^3} \qquad (x \to \infty).$$

Note that when using methods (ii) and (iii), one still needs to check that the conditions of Corollary 2.1.2 hold.

Problem 2.4: Find the leading asymptotic behaviour as $x \to \infty$ of

$$\int_0^1 \exp\left[x\left(t + \frac{t^3}{6} - \sinh t\right)\right] \cos t \, dt.$$

- (Movable maximum) Let us find the asymptotic behaviour of

$$I(x) = \int_0^\infty e^{-xt} e^{-1/t} \, dt,$$

as $x \to \infty$. If we let $h(t) = -t$, which has a maximum at 0, the other piece, $f(t) = e^{-1/t}$ is $o(t^m)$ for all $m \in \mathbb{R}$ $(t \to 0)$, so its asymptotic expansion with respect to $\phi_n(t) = t^n$ is

$$f(t) \sim 0 + 0t + 0t^2 + \dots \qquad (t \to 0).$$

Watson's Lemma then yields $I(x) \sim \frac{0}{x} + \frac{0}{x^2} + \dots$ $(x \to \infty)$, which, while correct, isn't very useful. To find the leading-order behaviour, we need to determine the maximum of the entire integrand. Letting $g(t) = -xt - 1/t$, we see that $e^{g(t)}$ has a maximum when $0 = g'(t) = -x + 1/t^2$, that is, when $t = 1/\sqrt{x}$ (noting $g''(t) = -2/t^3 < 0$). This point is called a *moveable maximum* since its location depends on x. Let us transform into this moving frame by letting $s = t\sqrt{x}$, so that the maximum occurs at the fixed value $s = 1$, independent of x:

$$I(x) = \frac{1}{\sqrt{x}} \int_0^\infty e^{-\sqrt{x}(s+1/s)} \, ds.$$

We can now apply Laplace's method. Since the function $h(s) \doteq -s - 1/s$ has an interior maximum at $s = 1$, we shift the integration variable to $\xi = s - 1$, so that the peak occurs at 0:

$$I(x) = \frac{1}{\sqrt{x}} \int_{-1}^\infty e^{\sqrt{x} h(1+\xi)} \, d\xi.$$

We then expand $h(1+\xi)$ in a Taylor's series about $\xi = 0$:

$$h(1+\xi) = -\xi - 1 - \frac{1}{1+\xi} \sim -\xi - 1 - (1 - \xi + \xi^2) \sim -2 - \xi^2 \qquad (\xi \to 0).$$

Hence

$$I(x) \sim \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \int_{-1}^\infty e^{-\sqrt{x}\xi^2} \, d\xi \sim \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \int_{-\infty}^\infty e^{-\sqrt{x}\xi^2} \, d\xi \sim \frac{\sqrt{\pi} e^{-2\sqrt{x}}}{x^{3/4}} \qquad (x \to \infty).$$

To check that we have retained enough terms, note that for $\lambda = 2$, $\mu = 1/2$, and $\beta = 0$, Eq. (2.12) implies that $\alpha k \leqslant 2k$, so that $k = 0$ or $\alpha \leqslant 2$.

- (Stirling's Formula) Let us now determine the asymptotic behaviour of $\Gamma(x)$ for $x \to \infty$:

$$\Gamma(x) = \int_{0^+}^{\infty} e^{-t} t^{x-1}\, dt = \int_{0^+}^{\infty} e^{(x-1)\log t - t}\, dt = \int_{0^+}^{\infty} e^{x \log t} \frac{e^{-t}}{t}\, dt.$$

However, $\log t$ has no maximum in $[0, \infty)$, so we need to find the maximum of the entire integrand. Letting $g(t) = (x-1)\log t - t$, we see that

$$0 = g'(t) = \frac{(x-1)}{t} - 1 \Rightarrow t = x - 1,$$

and $g''(x-1) = -1/(x-1) < 0$ for $x > 1$. For large x the maximum occurs at $t = x - 1 \sim x$. We can therefore transform to the frame of this moving maximum by introducing the variable $s = t/x$:

$$\Gamma(x) = \int_{0^+}^{\infty} e^{-sx}(sx)^{x-1} x\, ds = x^x \int_{0^+}^{\infty} \frac{e^{x(\log s - s)}}{s}\, ds.$$

Letting $h(s) = \log s - s$, we see that the maximum now occurs at $s = 1$. On shifting the integration variable to $\xi = s - 1$, we find

$$\Gamma(x) = x^x \int_{-1^+}^{\infty} \frac{e^{xh(1+\xi)}}{1+\xi}\, d\xi.$$

Now

$$h(1+\xi) = \log(1+\xi) - (1+\xi) \sim -1 - \frac{\xi^2}{2} + \frac{\xi^3}{3} - \frac{\xi^4}{4} \qquad (\xi \to 0)$$

and

$$f(\xi) = \frac{1}{1+\xi} \sim 1 - \xi + \xi^2 \qquad (\xi \to 0).$$

On setting $\lambda = 2$ and $\mu = 3/2$ in Eq. (2.12) we see that to expand to order $x^{-3/2}$ we need to keep all terms with **even** values of $\alpha k + \beta \leqslant 2k + 2$. Since $\alpha \geqslant 3$ then $k \leqslant 2$ and we need only consider even values of $\alpha k + \beta$ in $[3k, 2k+2]$: for $k = 0$ we need to retain the ξ^0 and ξ^2 terms, for $k = 1$ we need the ξ^4 terms and for $k = 2$

we need only the ξ^6 term:

$$\begin{aligned}
\Gamma(x) &\sim x^x e^{-x} \int_{-1}^{1} e^{-x\xi^2/2} e^{x(\xi^3/3-\xi^4/4)} (1-\xi+\xi^2)\,d\xi \\
&\sim x^x e^{-x} \int_{-\infty}^{\infty} e^{-x\xi^2/2} \left[1 + x\left(\frac{\xi^3}{3} - \frac{\xi^4}{4}\right) + x^2\frac{\xi^6}{18}\right] (1-\xi+\xi^2)\,d\xi \\
&\sim x^x e^{-x} \int_{-\infty}^{\infty} e^{-x\xi^2/2} \left[1 - \frac{x\xi^4}{4} + \frac{x^2\xi^6}{18} - \frac{x\xi^4}{3} + \xi^2\right] d\xi \\
&\sim x^x e^{-x} \int_{-\infty}^{\infty} e^{-x\xi^2/2} \left[1 + \xi^2 - \frac{7}{12}x\xi^4 + \frac{x^2\xi^6}{18}\right] d\xi \\
&\sim x^x e^{-x} \left(\frac{\Gamma(\frac{1}{2})2^{1/2}}{x^{1/2}} + \frac{\Gamma(\frac{3}{2})2^{3/2}}{x^{3/2}} - \frac{7x\Gamma(\frac{5}{2})2^{5/2}}{12x^{5/2}} + \frac{x^2\Gamma(\frac{7}{2})2^{7/2}}{18x^{7/2}}\right) \\
&\sim \sqrt{2\pi}x^x e^{-x} \left(\frac{1}{x^{1/2}} + \frac{1}{x^{3/2}} - \frac{7}{4x^{3/2}} + \frac{5}{6x^{3/2}}\right) \\
&\sim \sqrt{\frac{2\pi}{x}} x^x e^{-x} \left(1 + \frac{1}{12x}\right) \qquad (x \to \infty).
\end{aligned}$$

The first term leads to *Stirling's formula*:

$$\begin{aligned}
n! &\sim \sqrt{\frac{2\pi}{(n+1)}}\,(n+1)^{(n+1)} e^{-(n+1)} \\
&\sim \sqrt{2\pi(n+1)}\,n^n \left(1 + \frac{1}{n}\right)^n e^{-(n+1)} \\
&\sim \sqrt{2\pi n}\,\frac{n^n}{e^n} \qquad (n \to \infty).
\end{aligned}$$

2.E Fourier Integrals

Let us now determine the behaviour as $x \to \infty$ of the *Fourier integral*

$$I(x) = \int_a^b e^{ixt} f(t)\,dt,$$

where $x, t \in \mathbb{R}$. For Laplace integrals the primary result that allowed us to obtain asymptitic expansions was Watson's Lemma. For Fourier integrals, the corresponding result is the Riemann–Lebesgue Lemma.

Theorem 2.2 (Riemann–Lebesgue Lemma):

(i) If f is piecewise continuous on a bounded interval $[a, b]$ then

$$\int_a^b e^{ixt} f(t)\, dt = o(1) \qquad (x \to \infty).$$

(ii) If f is continuous on an unbounded interval (a, b), except perhaps at a finite number of points, then

$$\int_a^b e^{ixt} f(t)\, dt = o(1) \qquad (x \to \infty),$$

provided for sufficiently large x the integral converges uniformly.

Proof:

(i) Without loss of generality, we may suppose that f is continuous on $[a, b]$ (if not, we can subdivide $[a, b]$ into a finite number of subintervals on which it is continuous and then sum the results).

Let $M = \max_{[a,b]} |f|$. Since f continuous on $[a, b] \Rightarrow f$ is uniformly continuous on $[a, b]$, given $\varepsilon > 0$, there exists a sufficiently fine partition

$$\{a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\}$$

of $[a, b]$ such that for $j = 1, 2, \ldots n$,

$$t \in [t_{j-1}, t_j] \Rightarrow |f(t) - f(t_j)| < \frac{\varepsilon}{2(b-a)}.$$

We will use the fact that for $x > 0$,

$$\left| \int_a^b e^{ixt}\, dt \right| = \left| \frac{e^{ibx} - e^{iax}}{ix} \right| \leqslant \frac{\left|e^{ibx}\right| + \left|e^{iax}\right|}{x} = \frac{2}{x}.$$

We then find for $x > 4nM/\varepsilon$,

$$\begin{aligned}
\left| \int_a^b e^{ixt} f(t)\, dt \right| &= \left| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} e^{ixt} [f(t_j) + f(t) - f(t_j)]\, dt \right| \\
&\leqslant \sum_{j=1}^n |f(t_j)| \left| \int_{t_{j-1}}^{t_j} e^{ixt}\, dt \right| + \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left|e^{ixt}\right| |f(t) - f(t_j)|\, dt \\
&\leqslant \sum_{j=1}^n M \frac{2}{x} + \sum_{j=1}^n \frac{\varepsilon}{2(b-a)} (t_j - t_{j-1}) \\
&= nM \frac{2}{x} + \frac{\varepsilon}{2(b-a)} (b-a) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus

$$\lim_{x\to\infty}\int_a^b e^{ixt}f(t)\,dt=0,$$

as desired.

(ii) Let c_j, $j=1,2,\ldots,n$ be the points of discontinuity of f in (a,b) listed in ascending order. Then given $\varepsilon>0$ we know from the Cauchy criterion and (i) that there exists a number $\delta\in(0,\frac{1}{2}\min\limits_{1\leqslant j<n}[c_{j+1}-c_j])$ such that the integrals $\int_a^{c_1-\delta}e^{ixt}f(t)\,dt$, $\int_{c_j-\delta}^{c_j+\delta}e^{ixt}f(t)\,dt$ for $j=1,2,\ldots n$, and $\int_{c_n+\delta}^{b}e^{ixt}f(t)\,dt$ are each bounded by $\frac{1}{2}\varepsilon/(n+2)$ for all sufficiently large x. Since f is continuous on $[c_j+\delta,c_{j+1}-\delta]$ for each $j=1,2,\ldots,n-1$ we know from (i) that the contribution from each of these $n-1$ subintervals is less than $\frac{1}{2}\varepsilon/(n-1)$ for sufficiently large x. On summing up all $2n+1$ contributions we find for sufficiently large x that

$$\int_a^b e^{ixt}f(t)\,dt<\varepsilon.\quad\square$$

Remark: If $f\in C^1[a,b]$ then we may use the Riemann–Lebesgue Lemma to find the asymptotic behaviour as $x\to\infty$ of

$$I(x)=\int_a^b e^{ixt}f(t)\,dt.$$

On integrating by parts we find

$$\begin{aligned}I(x)&=\frac{e^{ixt}f(t)}{ix}\bigg|_a^b-\frac{1}{ix}\int_a^b e^{ixt}f'(t)\,dt\\&=\frac{i}{x}\big[e^{iax}f(a)-e^{ibx}f(b)\big]+R_1(x),\end{aligned}$$

where

$$R_1(x)=\frac{i}{x}\int_a^b e^{ixt}f'(t)\,dt=o\left(\frac{1}{x}\right)\qquad(x\to\infty)$$

since $f'\in C[a,b]$. Hence

$$I(x)\sim\frac{i}{x}\big[e^{iax}f(a)-e^{ibx}f(b)\big]\qquad(x\to\infty).$$

Furthermore, if $f\in C^N[a,b]$ then repeated integration by parts N times yields

$$I(x)=\sum_{n=1}^{N}\left(\frac{i}{x}\right)^n\big[e^{iax}f^{(n-1)}(a)-e^{ibx}f^{(n-1)}(b)\big]+R_N(x),$$

where

$$R_N(x) = \left(\frac{i}{x}\right)^N \int_a^b e^{ixt} f^{(N)}(t)\, dt = o\left(\frac{1}{x^N}\right) \qquad (x \to \infty).$$

If $f \in C^\infty[a, b]$ then

$$I(x) \sim \sum_{n=1}^{\infty} \left(\frac{i}{x}\right)^n \left[e^{iax} f^{(n-1)}(a) - e^{ibx} f^{(n-1)}(b)\right] \qquad (x \to \infty).$$

Note that the contribution to the asymptotic expansion comes only from the endpoints.

2.F Method of Stationary Phase

Let us now consider the generalization

$$I(x) = \int_a^b e^{ixh(t)} f(t)\, dt$$

of a Fourier integral, where $x, t \in \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ and f is differentiable on $[a, b]$. The heuristic reasoning underlying Laplace's method doesn't apply in this case since $\left|e^{ixh(t)}\right| = 1$ for all $x, t \in R$.

The idea behind the method of stationary phase is that for large x, contributions from the integrand oscillate rapidly and will typically cancel each other out, except near (i) the endpoints a and b due to lack of symmetry, and (ii) a zero of $h'(t)$, where h changes relatively slowly.

Definition: A point at which $h'(t)$ vanishes is called a *stationary point*.

Remark: In physical problems involving wave propagation, $h(t)$ has the interpretation of a *phase*, hence the name *method of stationary phase*.

- Let us first find the asymptotic behaviour of

$$I(x) = \int_a^b e^{ixh(t)} f(t)\, dt,$$

in the case where f is differentiable and h has no stationary points in $[a, b]$. Letting $\xi = h(t)$, we find

$$I(x) = \int_{h(a)}^{h(b)} e^{ix\xi} f(h^{-1}(\xi))\, \frac{d\xi}{h'(h^{-1}(\xi))} \sim \frac{i}{x}\left[\frac{e^{ixh(a)} f(a)}{h'(a)} - \frac{e^{ixh(b)} f(b)}{h'(b)}\right] \qquad (x \to \infty).$$

Thus, $I(x) = \mathcal{O}(1/x)$ as $x \to \infty$.

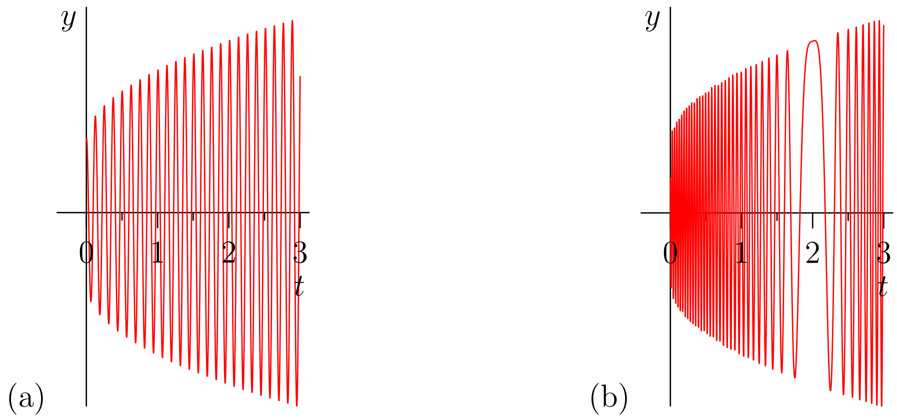


Figure 2.5: Comparison of $y = \operatorname{Re} e^{ixh(t)}(1+\sqrt{t})$ with $x = 50$ for (a) $h(t) = t$, which has no stationary points; (b) $h(t) = (2-t)^2$, which has a stationary point at $t = 2$.

Remark: To find the asymptotic behaviour of $I(x)$ when h has a stationary point, we will need the following lemma.

Definition: If $h^{(n)}(a) = 0$ for $n = 1, 2, \ldots, N-1$, $h^{(N)}(a) \neq 0$, and $h' \neq 0$ on $(a, b]$, we say that N is the *order* of the stationary point of h at a.

Lemma 2.3: *Let*

$$I(x) = \int_a^b e^{ixh(t)} f(t)\, dt,$$

where f is differentiable on $[a, b]$, $f(a) \neq 0$, h has a stationary point of order N at a, and $h' \neq 0$ on $(a, b]$. Then

$$I(x) \sim \frac{1}{N}\Gamma\left(\frac{1}{N}\right) f(a) e^{ixh(a)} \left(\frac{N!\, i}{h^{(N)}(a) x}\right)^{1/N} \qquad (x \to \infty).$$

Proof: As $t \to a^+$ we have $e^{ix[h(t)-h(a)]} f(t)/f(a) \sim F(x, t)$, where $F(x,t) \doteq e^{ixh^{(N)}(a)(t-a)^N/N!}$. That is,

$$e^{ix[h(t)-h(a)]} f(t) = [F(x,t) + R(x,t)] f(a),$$

where $R(x,t) = o(F(x,t))$ as $t \to a^+$. Given $x > 0$, choose δ sufficiently small so that

$$a \leqslant t < a + \delta \Rightarrow |R(x,t)| \leqslant \frac{1}{x} |F(x,t)|.$$

Decompose $I(x) = I_1(x) + I_2(x)$ where

$$\begin{aligned} I_1(x) &= \int_a^{a+\delta} e^{ixh(t)} f(t)\, dt = e^{ixh(a)} f(a) \int_a^{a+\delta} [F(x,t) + R(x,t)]\, dt \\ &\sim e^{ixh(a)} f(a) \int_a^{a+\delta} F(x,t)\, dt \qquad (x \to \infty) \end{aligned}$$

and

$$I_2(x) = \int_{a+\delta}^{b} e^{ixh(t)} f(t)\, dt = \mathcal{O}\left(\frac{1}{x}\right) \qquad (x \to \infty),$$

noting that h has no stationary points in $[a+\delta, b]$. On introducing the substitution $\xi = \left|h^{(N)}(a)\right| (t-a)^N / N!$, we find on using Eq. 2.4 (or its complex conjugate) that

$$\begin{aligned}\int_a^{a+\delta} F(x,t)\, dt &= \int_a^{\infty} F(x,t)\, dt - \int_{a+\delta}^{\infty} F(x,t)\, dt \\ &= \frac{1}{N}\left(\frac{N!}{|h^{(N)}(a)|}\right)^{1/N} \int_0^{\infty} e^{\operatorname{sgn}(h^{(N)}(a)) ix\xi} \xi^{1/N-1}\, d\xi + \mathcal{O}\left(\frac{1}{x}\right) \\ &\sim \frac{1}{N}\Gamma\left(\frac{1}{N}\right)\left(\frac{N!\, i}{h^{(N)}(a)x}\right)^{1/N} \qquad (x \to \infty)\end{aligned}$$

since $(t-a)^N$ has no stationary points in $[a+\delta, \infty)$. Here, the complex plane is cut along the negative real axis, so that $\operatorname{Arg} z \in (-\pi, \pi)$. It follows that

$$I(x) \sim \frac{1}{N}\Gamma\left(\frac{1}{N}\right) f(a) e^{ixh(a)} \left(\frac{N!\, i}{h^{(N)}(a)x}\right)^{1/N} \qquad (x \to \infty).$$

Remark: The previous result gives just the leading order expansion of $I(x)$. If more terms are required, then the neglected contributions of order $\mathcal{O}(1/x)$ must be taken into account.

Problem 2.5: By substituting $\tau = -t$, show that if h instead has a stationary point of order N at b, with $h' \neq 0$ on $[a, b)$ and $f(b) \neq 0$, then

$$I(x) \sim -\frac{1}{N}\Gamma\left(\frac{1}{N}\right) f(b) e^{ixh(b)} \left(\frac{(-1)^N N!\, i}{h^{(N)}(b)x}\right)^{1/N} \qquad (x \to \infty).$$

Remark: Slight modifications would be necessary if we wanted to extend the above analysis to the case where h has no stationary points ($N = 1$): both endpoints contribute to the leading-order behaviour, as seen previously.

Remark: If h has a stationary point at $c \in (a, b)$, we simply sum the contributions from the subintervals $[a, c]$ and $[c, b]$.

- Consider the behaviour as $x \to \infty$ of

$$I(x) = \int_0^{\pi/2} e^{ix\cos t}\, dt.$$

Here $f(t) = 1$ and $h(t) = \cos t$. Since $h'(0) = 0$ but $h''(0) = -1 \neq 0$ and $h'(t) = -\sin t \neq 0$ on $(0, \pi/2]$, we see that h has a single stationary point of order $N = 2$ at $t = 0$. Thus

$$I(x) \sim \sqrt{\frac{\pi}{2x}} e^{i(x-\pi/4)} \qquad (x \to \infty).$$

Remark: The leading asymptotic behaviour of $I(x) = \int_a^b e^{ixh(t)} f(t)\, dt$ is determined by the highest-order stationary point of h in $[a, b]$.

Remark: To get the full asymptotic series one must take into account contributions from all stationary points and end points. This is in contrast with Laplace's method for Laplace integrals, where the full asymptotic series is determined entirely from the immediate neighbourhood of the point at which h attains its global maximum.

2.G Method of Steepest Descent

Let $\lambda \in R$, $D \subset \mathbb{C}$ and the complex-valued functions h and f be holomorphic on D. Given a contour C in D, suppose we wish to find the asymptotic behaviour as $\lambda \to \infty$ of

$$I(\lambda) = \int_C e^{\lambda h(z)} f(z)\, dz.$$

In the method of *steepest descent*, one

(i) deforms the contour to a new contour $\widetilde{C}$ on which $\operatorname{Im} h(z) = \text{const}$;

(ii) parametrizes $\widetilde{C}$ as $z = \zeta(t)$ with $t \in [a, b]$;

(iii) uses Laplace's method on the resulting integral, expressing $h(\zeta(t)) = u(\zeta(t)) + iv$, where $u(z)$ and v are real valued:

$$I(\lambda) = e^{i\lambda v} \int_{\widetilde{C}} e^{\lambda u(z)} f(z)\, dz.$$

Remark: One could alternatively find a contour on which the real part of h is constant and use the method of stationary phase instead, but Laplace's method is typically easier.

Remark: The following lemma shows that the path of constant phase is typically a path of steepest descent (or ascent), hence the name "method of steepest descent." Given a surface $u = u(x, y)$, note that $\boldsymbol{\nabla} u = (u_x, u_y)$ is the direction in which u increases most rapidly.

Lemma 2.4 (Steepest Descent): *If*

(i) $h(x + iy) = u(x, y) + iv(x, y)$ *is holomorphic at* $z_0 \doteq x_0 + iy_0$*,*

(ii) $h'(z_0) \neq 0$*,*

(iii) C *is the curve through* z_0 *defined by* $v(x, y) = v_0$*,*

then $\boldsymbol{\nabla} u$ *is tangent to* C *at* z_0*.*

Proof: From the Cauchy–Riemann equations we know that

$$\begin{aligned} u_x &= v_y, \\ u_y &= -v_x. \end{aligned}$$

Hence $h'(z_0) \neq 0$ implies that

$$0 \neq u_x + iv_x = u_x - iu_y = v_y + iv_x,$$

so that the vectors $\boldsymbol{\nabla} u$ and $\boldsymbol{\nabla} v$ are both nonzero at z_0.

Parametrize C as $(\xi(t), \eta(t))$ with tangent $\boldsymbol{T} = (\xi'(0), \eta'(0))$ and normal $\boldsymbol{N} = (\eta'(0), -\xi'(0))$ at $z_0 = (\xi(0), \eta(0))$.

At the point z_0 we then see that

$$\boldsymbol{\nabla} u \cdot \boldsymbol{N} = (u_x, u_y) \cdot (\eta'(0), -\xi'(0)) = (v_y, -v_x) \cdot (\eta'(0), -\xi'(0)) = (v_x, v_y) \cdot (\xi'(0), \eta'(0)) = 0$$

since $v(\xi(t), \eta(t)) = v_0$. Hence $\boldsymbol{\nabla} u$ is perpendicular to $\boldsymbol{N}$ and therefore parallel to $\boldsymbol{T}$ at z_0.

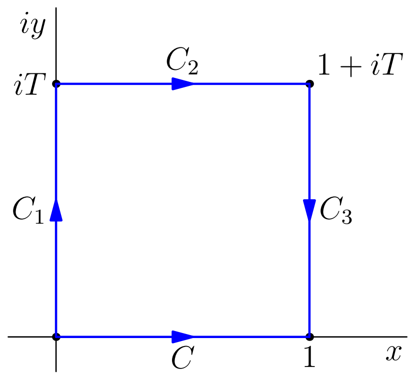
Remark: If $h'(z_0) \neq 0$, Lemma 2.4 guarantees that there is a unique path $z = (\xi(t), \eta(t))$ of constant phase through $z_0 = (x_0, y_0) = (\xi(0), \eta(0))$, in the direction of the gradient of u. Denoting $U(t) = u(\xi(t), \eta(t))$ we know that $U'(0) = u_x(x_0, y_0)\xi'(0) + u_y(x_0, y_0)\eta'(0) = \boldsymbol{\nabla} u \cdot \boldsymbol{T} \neq 0$, since $\boldsymbol{\nabla} u$ is nonzero and parallel to $\boldsymbol{T}$ at z_0.

Remark: From the Laplace method, we know that the dominant contribution to $\int_a^b e^{\lambda U(t)} f(\zeta(t))\, dt$ comes from points where U achieves its maximum. These are either the end points a and b or critical points of U (points where U' is zero or does not exist). In the case where $h'(z_0) \neq 0$, we have just seen that $U'(0) \neq 0$, so such points must be end points of the integration path. In the following example this dominant contribution comes from the endpoint $t = 0$.

- To find the asymptotic behaviour as $\lambda \to \infty$ of

$$I(\lambda) = \int_{0^+}^{1} e^{i\lambda t} \log t \, dt,$$

we cannot use integration by parts since $\log(0) = -\infty$ and $\log(1) = 0$. Let $h(z) = iz$ and $f(z) = \log z$. The contours on which $\operatorname{Im} h(z)$ are constant are the vertical lines $x =$ constant, such as the contours C_1 and C_3 in the following figure.



We thus deform the original contour C to $C_1 \cup C_2 \cup C_3$.
For $\lambda > 0$ we see as $T \to \infty$ that

$$\int_{C_2} e^{\lambda h(z)} \log z \, dz = \int_0^1 e^{i\lambda(t+iT)} \log(t+iT) \, dt = e^{-\lambda T} \int_0^1 e^{i\lambda t} \log(t+iT) \, dt = o(1)$$

since

$$\begin{aligned}
\log(t+iT) &= \frac{1}{2}\log(t^2+T^2) + i\tan^{-1}\frac{T}{t} \\
&= \frac{1}{2}\log(T^2) + \frac{1}{2}\log\left(1+\frac{t^2}{T^2}\right) + i\tan^{-1}\frac{T}{t} \\
&\sim \log(T) \qquad (T \to \infty).
\end{aligned}$$

Also,

$$\begin{aligned}
\lim_{T\to\infty} \int_{C_1} e^{\lambda h(z)} \log z \, dz &= \int_0^\infty e^{-\lambda t} \log(it) i \, dt \\
&= i \int_0^\infty e^{-\lambda t} \left(\log t + i\frac{\pi}{2}\right) dt \\
&= \frac{i}{\lambda} \int_0^\infty e^{-\xi} \left(\log \xi - \log \lambda + i\frac{\pi}{2}\right) d\xi \\
&= \frac{i}{\lambda}\left(-\gamma - \log\lambda + i\frac{\pi}{2}\right) \\
&= -\frac{i\log\lambda}{\lambda} - \frac{i\gamma + \pi/2}{\lambda},
\end{aligned}$$

on using the substitution $\xi = \lambda t$ and the fact that $\Gamma'(1) = \int_0^\infty e^{-\xi} \log \xi \, d\xi = -\gamma$.
Finally,

$$\begin{aligned}
\lim_{T\to\infty} \int_{C_3} e^{\lambda h(z)} \log z \, dz &= -\int_0^\infty e^{i\lambda(1+it)} \log(1+it) i \, dt \\
&= ie^{i\lambda} \int_0^\infty e^{-\lambda t} \sum_{n=1}^\infty \frac{(-it)^n}{n} \, dt \\
&\sim ie^{i\lambda} \sum_{n=1}^\infty \frac{(-i)^n (n-1)!}{\lambda^{n+1}} \quad (\lambda \to \infty).
\end{aligned}$$

Thus

$$I(\lambda) \sim -\frac{i \log \lambda}{\lambda} - \frac{i\gamma + \pi/2}{\lambda} + ie^{i\lambda} \sum_{n=1}^\infty \frac{(-i)^n (n-1)!}{\lambda^{n+1}} \quad (\lambda \to \infty).$$

Remark: When $h'(z_0) = 0$, then $U'(0) = 0$ no matter what path $z = (\xi(t), \eta(t))$ we choose through z_0. However, the following lemma shows that only certain directions through z_0 actually correspond to paths of steepest descent, while others correspond to paths of steepest ascent. For this reason, points where $h'(z_0) = 0$ are called *saddle points*. We will want to integrate along the steepest descent path, so that the integrand becomes sharply peaked at z_0 as $\lambda \to \infty$. Note that if z_0 is not an endpoint of the integration path, then U will have a local interior maximum at 0.

Lemma 2.5 (Saddle Points): *If*

(i) h is holomorphic at z_0,

(ii) $h^{(n)}(a) = 0$ for $n = 1, 2, \ldots, N-1$ and $h^{(N)}(a) = \rho e^{i\alpha}$, with $\rho > 0$,

then there are N paths of steepest descent (ascent) through z_0, with direction $\frac{(2n+1)\pi - \alpha}{N}$ $\left(\frac{2n\pi - \alpha}{N}\right)$ for $n = 0, 1, 2, \ldots, N-1$.

Proof:
Let $z - z_0 = re^{i\theta}$. Then

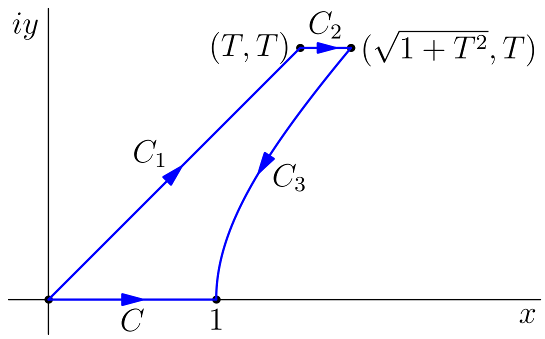
$$h(z) - h(z_0) \sim \frac{h^{(N)}(z_0)}{N!}(z - z_0)^N = \frac{\rho e^{i\alpha}}{N!} r^N e^{iN\theta} = \frac{\rho r^N}{N!}[\cos(\alpha + N\theta) + i \sin(\alpha + N\theta)].$$

The direction of steepest descent of $\text{Re}[h(z) - h(z_0)]$ is given by the value of θ for which $\text{Re}[h(z) - h(z_0)]$ is most negative, namely for $\alpha + N\theta = \cos^{-1}(-1) = (2n+1)\pi$. Likewise, the directions of steepest ascent satisfy $\alpha + N\theta = \cos^{-1}(1) = 2n\pi$. Notice that in each of these directions we have $\sin(\alpha + N\theta) = 0$, so that these are all directions of constant phase.

• We now use the method of steepest descent to find the asymptotic behaviour of

$$I(\lambda) = \int_0^1 e^{i\lambda t^2}\,dt$$

as $\lambda \to \infty$. Let $f(z) = 1$, $h(z) = iz^2 = i(x+iy)^2 = -2xy + i(x^2 - y^2)$. Since $h'(0) = 0$, but $h''(0) = 2i$, at the origin we have a saddle point of order $N = 2$, with $\alpha = \pi/2$. The steepest descent directions out of the origin are given by $\pi/4$ and $5\pi/4$, while the steepest ascent directions are given by $3\pi/4$ and $7\pi/4$. We choose C_1 to be the curve $x^2 - y^2 = 0$ coming out of the origin, at angle $\theta = \pi/4$. We denote the curve $x^2 - y^2 = 1$ as C_3 and deform the original contour C to $C_1 \cup C_2 \cup C_3$, as shown below:



Now

$$\left|\int_{C_2} e^{\lambda h(z)}\,dz\right| \leqslant \int_T^{\sqrt{1+T^2}} \left|e^{i\lambda(t+iT)^2}\right|\,dt = \int_T^{\sqrt{1+T^2}} e^{-2\lambda t T}\,dt \leqslant \sqrt{1+T^2} - T \underset{T\to\infty}{\to} 0.$$

Also

$$\lim_{T\to\infty} \int_{c_1} e^{\lambda h(z)}\,dz = (1+i)\int_0^\infty e^{-2\lambda t^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{2\lambda}}(1+i).$$

Let us parametrize C_3 by $t = y \in [0,T]$, so that $U(t) \doteq \operatorname{Re} h(\zeta(t)) = -2t\sqrt{1+t^2}$, which has a maximum in $[0,\infty)$ at $t = 0$. Letting $s = -U(t)$ we have $iz^2 = -s + i$ on C_3, so that $z = (1+is)^{1/2}$ and $dz = \frac{1}{2}i(1+is)^{-1/2}ds$. Then

$$\lim_{T\to\infty} \int_{C_3} e^{\lambda h(z)}\,dz = -\frac{i}{2}e^{i\lambda}\int_0^\infty e^{-\lambda s}\,(1+is)^{-1/2}\,ds.$$

Since

$$(1+is)^{-1/2} \sim \sqrt{\pi}\sum_{n=0}^\infty \frac{(is)^n}{n!\,\Gamma\left(\frac{1}{2}-n\right)} \qquad (s \to 0),$$

we then find using Laplace's method that

$$\lim_{T\to\infty} \int_{C_3} e^{\lambda h(z)}\,dz \sim -\frac{i}{2}e^{i\lambda}\sqrt{\pi}\sum_{n=0}^\infty \frac{i^n}{\Gamma\left(\frac{1}{2}-n\right)\lambda^{n+1}} \qquad (\lambda \to \infty).$$

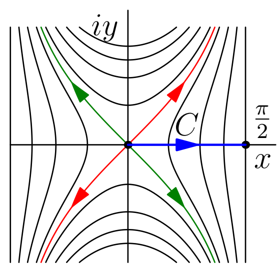
Thus

$$I(\lambda) \sim \frac{1}{2}\sqrt{\frac{\pi}{2\lambda}}(1+i) - \frac{i\sqrt{\pi}}{2}e^{i\lambda}\sum_{n=0}^{\infty}\frac{i^n}{\Gamma\left(\frac{1}{2}-n\right)\lambda^{n+1}} \qquad (\lambda \to \infty).$$

- (Bessel Function) Consider the asymptotic behaviour of

$$\begin{aligned} J_0(\lambda) &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(\lambda\cos\theta)\,d\theta \\ &= \frac{2}{\pi}\int_0^{\pi/2}\cos(\lambda\cos\theta)\,d\theta \\ &= \frac{2}{\pi}\,\mathrm{Re}\int_C e^{i\lambda\cos z}\,dz \end{aligned}$$

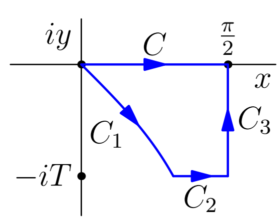
as $\lambda \to \infty$, where C is the contour shown below.



Letting

$$h(z) = i\cos z = i\cos(x+iy) = \sin x\sinh y + i\cos x\cosh y,$$

we see that there is a unique curve of constant phase $\cos x\cosh y = 0$ through $z = \pi/2$, since $h'(\pi/2) = i \neq 0$. However, since $h'(0) = 0$ and $h''(0) = -i$ we see that $N = 2$ and $\alpha = -\pi/2$, so that the two steepest descent curves of constant phase, shown above in green, leave the origin at $\theta = 3\pi/4$ and $7\pi/4$. The two steepest ascent curves are shown in red. All four curves satisfy the equation $\cos x\cosh y = 1$. We want to deform C to a path connecting the origin and $(\pi/2, 0)$ that leaves the origin on a path of steepest descent. We thus consider the deformed contour $C_1 \cup C_2 \cup C_3$:



We find for $T > 0$

$$\left|\mathrm{Re}\int_{C_2} e^{i\lambda\cos z}\,dz\right| \leqslant \int_{\cos^{-1}(\mathrm{sech}\,T)}^{\pi/2} e^{-\lambda\sin t\sinh T}\,dt \leqslant \int_{\cos^{-1}(\mathrm{sech}\,T)}^{\pi/2} 1\,dt = \pi/2 - \cos^{-1}(\mathrm{sech}\,T) \underset{T\to\infty}{\to} 0.$$

Also,

$$\mathrm{Re}\int_{C_3} e^{i\lambda\cos z}\,dz = \mathrm{Re}\int_{-T}^{0} e^{\lambda\sinh t} i\,dt = 0.$$

Finally, recall that the contour C_1 was chosen so that $h(\zeta(t)) = U(t)+i$, making it convenient to introduce the change of variables $s = -U(t)$, so that $s = i - i\cos(z)$, with

$$ds = i\sin z\,dz = i(1-\cos^2 z)^{1/2}\,dz = i(1-(1+is)^2)^{1/2}\,dz = i(s^2-2is)^{1/2}\,dz$$

on C_1, where we use the branch $\arg z \in [-\pi,\pi]$ to evaluate the square root. On noting that $U = 0$ at the origin and $s = -\sin x\sinh y \to \infty$ as $x \to \pi/2$ and $y \to -\infty$, we thus see that

$$\begin{aligned}
J_0(\lambda) &= \frac{2}{\pi}\,\mathrm{Re}\lim_{T\to\infty}\int_{C_1} e^{i\lambda\cos z}\,dz \\
&= -\frac{2}{\pi}\,\mathrm{Re}\,i\int_0^\infty e^{-\lambda s+\lambda i}(s^2-2is)^{-1/2}\,ds \\
&= -\frac{2}{\pi}\,\mathrm{Re}\,ie^{i\lambda}(-2i)^{-1/2}\int_0^\infty e^{-\lambda s}s^{-1/2}\left(1-\frac{s}{2i}\right)^{-1/2} ds \\
&\sim -\frac{2}{\pi}\,\mathrm{Re}\,ie^{i\lambda}(-2i)^{-1/2}\int_0^\infty e^{-\lambda s}s^{-1/2}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)}{n!\,\Gamma\left(\frac{1}{2}-n\right)}\left(\frac{is}{2}\right)^n ds \\
&\sim \mathrm{Re}\,e^{i(\lambda-\pi/4)}\sqrt{\frac{2}{\pi\lambda}}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)i^n}{2^n n!\,\Gamma\left(\frac{1}{2}-n\right)\lambda^n} \qquad (\lambda\to\infty).
\end{aligned}$$

Remark: The strategy of the steepest descent method is:

1. Identify possible stationary points of h.
2. Determine the paths of steepest descent from each stationary point.
3. Justify, via Cauchy's theorem, the deformation of the original contour onto one or more paths of steepest descent.
4. Determine the asymptotic expansion.

Chapter 3

Asymptotic Solution of Linear ODEs

3.A Classification of Singular Points

[Bender & Orszag 1999, pp. 62–63]

Exact solutions in closed form can only rarely be obtained for ordinary differential equations, either linear or nonlinear. Most of the time we must content ourselves with some sort of approximation to the solution. For linear equations one can usually predict the local behaviour of the solution near a point without knowing how to solve the equation explicitly. It suffices to examine the coefficient functions of the differential equation in a neighbourhood of the point.

The general homogeneous linear ordinary differential equation of order n is

$$u^{(n)}(z) + p_{n-1}(z)u^{(n-1)}(z) + \ldots + p_2(z)u''(z) + p_1(z)u'(z) + p_0(z)u(z) = 0. \quad (3.1)$$

Definition: The point $z_0 \neq \infty$ is an *ordinary point* of Eq. (3.1) if $p_k(z)$ is analytic at z_0 for $k = 0, 1, \ldots, n-1$.

- For the equation $zu' = u$ every point except $z = 0$ is an ordinary point.

- The equation $u' = |z|u$ has no ordinary points since $|z|$ is nowhere analytic.

Remark: All n linearly independent solutions of Eq. (3.1) are analytic at ordinary points. If any solution is expanded in a Taylor series about an ordinary point, then the radius of convergence will be at least as large as the distance to the nearest singularity of the coefficients $p_k(z)$.

Definition: The point $z_0 \neq \infty$ is a *regular singular point* of Eq. (3.1) if (i) z_0 is not an ordinary point of Eq. (3.1); and (ii) $(z-z_0)^{n-k}p_k(z)$ is analytic at z_0 for $k = 0, 1, \ldots, n-1$.

- The equations $zu' = u$, $z^2u'' = u$, and $z^2u' = u$ all have a singularity at $z = 0$. The point $z = 0$ is a regular singular point of the first and second equations but not the third.

Remark: If a solution of Eq. (3.1) is not analytic, its singularity will be either a pole or a branch point. There is always at least one solution of the form

$$u(z) = A(z)(z-z_0)^{\alpha},$$

where A is analytic at z_0, $A(z_0) \neq 0$, and α is called the *indicial exponent*. The Taylor series of A, expanded about z_0, has a radius of convergence at least as large as the distance to the next nearest singularity. If Eq. (3.1) is of order $n \geqslant 2$, then there is a second linearly independent solution of the form

$$\begin{aligned} u(z) &= B(z)(z-z_0)^{\beta}, \\ \text{or } u(z) &= A(z)(z-z_0)^{\alpha}\log(z-z_0) + B(z)(z-z_0)^{\beta}, \end{aligned}$$

where B is analytic at z_0. In general, the n^{th} solution is, at worst, of the form

$$u(z) = \sum_{j=0}^{n-1} A_j(z)(z-z_0)^{\gamma_j}[\log(z-z_0)]^j,$$

where A_j is analytic at z_0 for $j = 0, 1, \ldots, n-1$.

Definition: The point $z_0 \neq \infty$ is an *irregular singular point* of Eq. (3.1) if it is neither an ordinary point nor a regular singular point.

Remark: There is no comprehensive theory for irregular singular points. What can be said is the following:

(i) at least one solution is not of the form of those given previously for ordinary and regular singular points;

(ii) while it may happen that a solution is analytic, or has a branch point at an irregular point z_0, typically every solution has an essential singularity at z_0.

Remark: To classify the point $z_0 = \infty$, one considers the behaviour of

$$w(\zeta) = u(1/\zeta)$$

near $\zeta = 0$. From Eq. (3.1), one then obtains, upon repeated differentiation of $u(z) = w(1/z)$, an equation of the form

$$w^{(n)}(\zeta) + q_{n-1}(\zeta)w^{(n-1)}(\zeta) + \ldots + q_2(\zeta)w''(\zeta) + q_1(\zeta)w'(\zeta) + q_0(\zeta)w(\zeta) = 0. \tag{3.2}$$

Definition: The point $z_0 = \infty$ is:

(i) an ordinary point of Eq. (3.1) if $\zeta_0 = 0$ is an ordinary point of Eq. (3.2);

(ii) a regular singular point of Eq. (3.1) if $\zeta_0 = 0$ is a regular singular point of Eq. (3.2);

(iii) an irregular singular point of Eq. (3.1) if $\zeta_0 = 0$ is an irregular singular point of Eq. (3.2).

Problem 3.1: Classify all points of the *Airy equation* $u'' = zu$.

3.B Behaviour near Ordinary Points

Since all solutions of a linear ordinary differential equation are analytic at an ordinary point z_0, they can be expanded in a Taylor series about z_0:

$$u(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Substituting this series into Eq. (3.1) and equating like-degree terms yields a recursion relation for the a_ns.

- We want to determine the local behaviour of solutions to the Airy equation $u'' = zu$ near $z = 0$. The point $z = 0$ is an ordinary point, so all solutions are analytic at $z = 0$. Let us expand u in a Taylor series about $z = 0$:

$$u(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We insert this solution into the Airy equation to obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} = \sum_{n=0}^{\infty} a_n z^{n+1}$$

and then collect like-degree terms:

$$2a_2 + \sum_{n=0}^{\infty} [(n+2)(n+3)a_{n+3} - a_n]z^{n+1} = 0.$$

This implies that a_0, and a_1 are arbitrary, $a_2 = 0$, and

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)}, \quad n = 0, 1, 2, \ldots.$$

For $j = 1, 2, \ldots$, we deduce for

$$\begin{aligned}
n = 3j-3: &\quad a_{3j} = \frac{a_{3j-3}}{(3j-1)(3j)} \Rightarrow a_{3j} = \frac{a_0 \Gamma\left(\frac{2}{3}\right)}{9^j j!\; \Gamma\left(j+\frac{2}{3}\right)}, \\
n = 3j-2: &\quad a_{3j+1} = \frac{a_{3j-2}}{(3j)(3j+1)} \Rightarrow a_{3j+1} = \frac{a_1 \Gamma\left(\frac{4}{3}\right)}{9^j j!\; \Gamma\left(j+\frac{4}{3}\right)}, \\
n = 3j-1: &\quad a_{3j+2} = \frac{a_{3j-1}}{(3j+1)(3j+2)} \Rightarrow a_{3j+2} = 0.
\end{aligned}$$

If we define $c_0 = a_0\Gamma\left(\frac{2}{3}\right)$ and $c_1 = a_1\Gamma\left(\frac{4}{3}\right)$, then the general solution of the Airy equation can be written as

$$u(z) = c_0 \sum_{n=0}^{\infty} \frac{z^{3n}}{9^n n!\; \Gamma\left(n+\frac{2}{3}\right)} + c_1 \sum_{n=0}^{\infty} \frac{z^{3n+1}}{9^n n!\; \Gamma\left(n+\frac{4}{3}\right)}.$$

Since the coefficient function in the Airy equation is entire, we know that this series must have an infinite radius of convergence. It is conventional to define two special linearly independent solutions:

$$\mathrm{Ai}(z) \doteq 3^{-2/3} \sum_{n=0}^{\infty} \frac{z^{3n}}{9^n n!\; \Gamma\left(n+\frac{2}{3}\right)} - 3^{-4/3} \sum_{n=0}^{\infty} \frac{z^{3n+1}}{9^n n!\; \Gamma\left(n+\frac{4}{3}\right)}, \tag{3.3a}$$

$$\mathrm{Bi}(z) \doteq 3^{-1/6} \sum_{n=0}^{\infty} \frac{z^{3n}}{9^n n!\; \Gamma\left(n+\frac{2}{3}\right)} + 3^{-5/6} \sum_{n=0}^{\infty} \frac{z^{3n+1}}{9^n n!\; \Gamma\left(n+\frac{4}{3}\right)}. \tag{3.3b}$$

The functions Ai, Bi are called *Airy functions*. The constants are chosen so that:

(i) For $x \in \mathbb{R}$, $\mathrm{Ai}(x)$ decays exponentially as $x \to +\infty$;

(ii) For $x \in \mathbb{R}$, $\mathrm{Bi}(x)$ oscillates 90^o out of phase with $\mathrm{Ai}(x)$ as $x \to -\infty$.

The qualitative behaviour of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ for $x \in \mathbb{R}$ changes dramatically as x passes through the origin. This can explained, at least heuristically, by comparing the real variable version of the Airy equation $u'' = xu$ with the constant coefficient equations (for real λ)

$$u'' = \lambda^2 u \tag{3.4a}$$

and

$$u'' = -\lambda^2 u. \tag{3.4b}$$

The solution of Eq. (3.4a), $u(x) = Ae^{\lambda x} + Be^{-\lambda x}$, grows or decays exponentially, depending on the constants of integration, whereas the solution of Eq. (3.4b), $u(x) = A\cos(\lambda x) + B\sin(\lambda x)$, oscillates. For the Airy equation, the solutions grow or decay exponentially when $x > 0$ and oscillate when $x < 0$. Points like this where the qualitative nature of the solutions changes due to a coefficient function passing through zero are called *turning points*. We will encounter turning points again later.

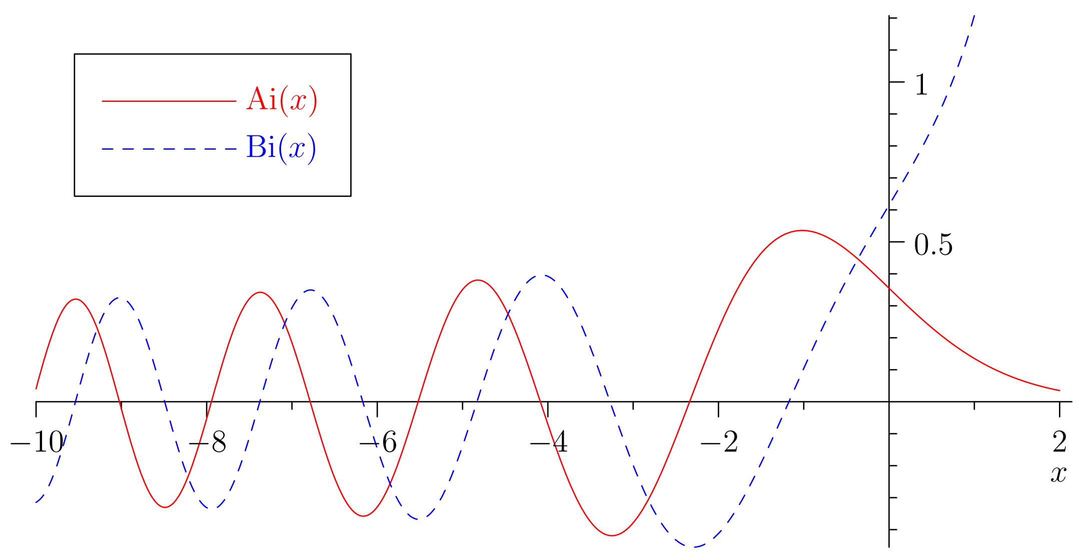


Figure 3.1: The Airy functions

3.C Behaviour near Regular Singular Points

The solution near a regular singular point can be found by the *Frobenius method*. This technique is discussed in introductory texts on differential equations. We will summarize the technique here.

In this section we shall restrict most of our analysis to second-order equations, although everything can be generalized to higher-order equations. Consider the differential equation

$$y'' + p(x)y' + q(x)y = 0. \tag{3.5}$$

If x_0 is an ordinary point of Eq. (3.5), then two linearly independent solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

can be found. What happens if x_0 is a singular point? We can get an idea as to what happens if we examine the Cauchy–Euler equation:

$$x^2 y'' + axy' + by = 0, \quad (a, b = \text{constant}). \tag{3.6}$$

If we look for a solution of the form $y = x^r$, this leads to

$$r^2 + (a-1)r + b = 0. \tag{3.7}$$

Thus, $y = x^r$ is a solution to Eq. (3.6) only if r is a root of the above quadratic equation.

- For the equation

  $$3x^2 y'' + 11xy' - 3y = 0,$$

  the quadratic (3.7) becomes $r^2 + \frac{8}{3}r - 1 = 0$, which leads to two linearly independent solutions: $y_1(x) = x^{1/3}$ and $y_2(x) = x^{-3}$.

Remark: The general solution to the Cauchy–Euler equation (3.6) can be found with the transformation $x = e^t$ (which converts it into an equation with constant coefficients):

$$y(x) = \begin{cases} c_1 x^{r_1} + c_2 x^{r_2}, & \text{if } r_1 \neq r_2, \\ c_1 x^r + c_2 x^r \log x, & \text{if } r_1 = r_2 = r, \\ x^\alpha [c_1 \cos(\beta \log x) + c_2 \sin(\beta \log x)], & \text{if } r_1, r_2 = \alpha \pm i\beta. \end{cases} \tag{3.8}$$

When written in the standard form

$$y'' + \frac{a}{x} y' + \frac{b}{x^2} y = 0,$$

it is clear that $x = 0$ is a regular singular point of the equation. The solutions will also usually be singular at $x = 0$, as in the previous example.

In general, if x_0 is a regular singular point of Eq. (3.5), then p and q can be written as

$$p(x) = \frac{A(x)}{x - x_0}, \quad q(x) = \frac{B(x)}{(x - x_0)^2},$$

where A and B are analytic at x_0. Equation (3.5) now becomes

$$y'' + \frac{A(x)}{x - x_0}y' + \frac{B(x)}{(x - x_0)^2}y = 0.$$

In terms of the operator

$$\mathcal{L} \doteq (x - x_0)^2 \frac{d^2}{dx^2} + A(x)(x - x_0)\frac{d}{dx} + B(x).$$

we may thus write Eq. (3.5) as $\mathcal{L}y = 0$.

This equation resembles the Cauchy–Euler equation, so it is reasonable to look for a solution of the form

$$y(x) = (x - x_0)^r Y(x),$$

where Y is analytic at x_0. This means that Y can be expanded in a Taylor series:

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}, \qquad a_0 \neq 0. \tag{3.9}$$

The series in Eq. (3.9) is called a *Frobenius* series. Since A and B are analytic at x_0, each can be expanded in a Taylor series:

$$A(x) = \sum_{n=0}^{\infty} A_n (x - x_0)^n, \qquad B(x) = \sum_{n=0}^{\infty} B_n (x - x_0)^n.$$

On substituting these expansions into the operator $\mathcal{L}$, we find that

$$\begin{aligned}
\mathcal{L}y &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-x_0)^{n+r} + \left(\sum_{j=0}^{\infty} A_j (x-x_0)^j\right)\left(\sum_{k=0}^{\infty}(k+r)a_k(x-x_0)^{k+r}\right) \\
&\quad + \left(\sum_{j=0}^{\infty} B_j (x-x_0)^j\right)\left(\sum_{k=0}^{\infty}(a_k(x-x_0)^{k+r}\right) \\
&= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-x_0)^{n+r} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(k+r)A_{n-k}a_k\right)(x-x_0)^{n+r} \\
&\qquad + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} B_{n-k}a_k\right)(x-x_0)^{n+r}.
\end{aligned}$$

If we define the coefficient of the term $a_0(x - x_0)^r$ by

$$P(r) \doteq r^2 + (A_0 - 1)r + B_0, \tag{3.10}$$

this result may be written

$$\mathcal{L}y = P(r)a_0(x - x_0)^r + \sum_{n=1}^{\infty}\left\{P(n+r)\, a_n + \sum_{k=0}^{n-1}[(k+r)A_{n-k} + B_{n-k}]a_k\right\}(x - x_0)^{n+r}. \tag{3.11}$$

Setting the individual coefficients of $(x - x_0)^{n+r}$ to zero then yields

$$n = 0: \quad P(r)\, a_0 = 0, \tag{3.12a}$$

$$n \geqslant 1: \quad P(n+r)\, a_n = -\sum_{k=0}^{n-1} [(k+r)A_{n-k} + B_{n-k}]a_k. \tag{3.12b}$$

If the nontrivial Frobenius series in Eq. (3.9) is to be a solution to $\mathcal{L}y = 0$, it follows from Eq. (3.12a) that $P(r) = 0$:

$$r^2 + (A_0 - 1)r + B_0 = 0. \tag{3.13}$$

Thus, r must be a root of the quadratic polynomial (3.10). Equation (3.13) is called the *indicial equation*. Using the solutions of the indicial equation, $r = r_1$ and $r = r_2$, the a_ns can then be determined recursively from Eq. (3.12b) in terms of an overall arbitrary constant a_0, provided that $P(n + r) \neq 0$ for every $n \in \mathbb{N}$. That is, at least one solution in Frobenius form can be found provided $r_1 - r_2$ does not equal a nonzero integer.

- Suppose we want to find a solution about $x = 0$ of the differential equation

$$(x + 2)x^2 y'' - xy' + (x + 1)y = 0. \tag{3.14}$$

The coefficient functions are given by

$$p(x) = -\frac{1}{x(x+2)}, \qquad q(x) = \frac{x+1}{x^2(x+2)}.$$

We see that $x = 0$ is a regular singular point since

$$xp(x) = -\frac{1}{x+2}, \qquad x^2 q(x) = \frac{x+1}{x+2},$$

are both analytic at $x = 0$.

Let us look for a Frobenius series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad a_0 \neq 0.$$

Instead of expressing $xp(x)$ and $x^2 q(x)$ as power series and using Eqs. (3.12), it is more convenient in this case to substitute the above expansion for $y(x)$ directly into Eq. (3.14):

$$\begin{aligned} 0 &= (x+2)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + (x+1)\sum_{n=0}^{\infty} a_n x^{n+r} \\ &= \sum_{n=1}^{\infty}(n+r-1)(n+r-2)a_{n-1}x^{n+r} + 2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} \\ &\quad - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=1}^{\infty} a_{n-1}x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r}. \end{aligned}$$

On letting $P(r) = 2r^2 - 3r + 1$, we then find that

$$n = 0: \quad P(r)a_0 = 0,$$
$$n \geqslant 1: \quad P(n+r)a_n = -[(n+r-1)(n+r-2)+1]a_{n-1}.$$

The indicial equation $P(r) = 0$ implies that $r = 1$ or $r = 1/2$. The recurrence relation gives for $r = 1$:

$$a_1 = -\frac{a_0}{3}, \quad a_2 = \frac{a_0}{10}, \quad a_3 = -\frac{a_0}{30}$$

and for $r = 1/2$:

$$a_1 = -\frac{3}{4}a_0, \quad a_2 = \frac{7}{32}a_0, \quad a_3 = -\frac{133}{1920}a_0.$$

Two linearly independent solutions are therefore (for $a_0 = 1$):

$$y_1(x) = x\left(1 - \frac{1}{3}x + \frac{1}{10}x^2 - \frac{1}{30}x^3 + \dots\right)$$

and

$$y_2(x) = x^{1/2}\left(1 - \frac{3}{4}x + \frac{7}{32}x^2 - \frac{133}{1920}x^3 + \dots\right),$$

so that the general solution is $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

Remark: What happens if the roots of the indicial equation are equal, that is, if $r_1 = r_2 = r$? We can get one Frobenius solution $y_1(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+r}$, but how do we determine a second linearly independent solution? A hint is given by the solution (3.8) to the Cauchy–Euler equation. In that case the linearly independent solutions are $y_1(x) = x^r$ and $y_2(x) = x^r \log x$. It therefore seems reasonable in the general case to look for a second solution of the form

$$y_2(x) = y_1(x)\log(x-x_0) + \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r}. \tag{3.15}$$

To show that a solution of the form (3.15) will always work in the case where $P(r)$ has a multiple real root, let us define $a_n(r)$ to be the solution of the recursion relation (3.12b) with $a_0(r) = 1$ and define

$$w(x,r) \doteq \sum_{n=0}^{\infty} a_n(r)(x-x_0)^{n+r}.$$

One solution to $\mathcal{L}w(x,r) = 0$ is given by $y_1(x) = w(x, r_1)$. When $P(r) = (r - r_1)^2$ and $a_0 = 1$, we see from Eq. (3.11) that

$$\mathcal{L}w(x,r) = P(r)(x - x_0)^r = (r - r_1)^2(x - x_0)^r. \tag{3.16}$$

Differentiation of Eq. (3.16) with respect to r then yields

$$\frac{\partial}{\partial r}\mathcal{L}w(x,r) = 2(r - r_1)(x - x_0)^r + (r - r_1)^2(x - x_0)^r \log(x - x_0).$$

On interchanging the order of differentiation and setting $r = r_1$, we deduce that $y_2(x) = \dfrac{\partial w}{\partial r}(x, r_1)$ is a second solution:

$$\mathcal{L}\left[\frac{\partial w}{\partial r}(x, r_1)\right] = \frac{\partial}{\partial r}\mathcal{L}w(x,r)\bigg|_{r=r_1} = 0.$$

From the definition of $w(x,r)$ we then see that

$$\begin{aligned}
y_2(x) &= \frac{\partial}{\partial r}\left\{(x - x_0)^r \sum_{n=0}^{\infty} a_n(r)(x - x_0)^n\right\}\bigg|_{r=r_1} \\
&= (x - x_0)^{r_1}\log(x - x_0)\sum_{n=0}^{\infty} a_n(r_1)(x - x_0)^n + (x - x_0)^{r_1}\sum_{n=0}^{\infty} a_n'(r_1)(x - x_0)^n \\
&= y_1(x)\log(x - x_0) + \sum_{n=0}^{\infty} a_n'(r_1)(x - x_0)^{n+r_1},
\end{aligned}$$

which is precisely the form suggested earlier, Eq. (3.15).

Remark: In the case where $r_1 - r_2 = N \in \mathbb{N}$, we note for $r = r_2$ that the coefficient a_N in Eq. (3.12b) is multiplied by $P(N + r_2) = P(r_1) = 0$. If the right-hand side of Eq. (3.12b) is zero, a_N is then arbitrary, yielding a second linearly independent solution in Frobenius form. However, if the right-hand side of Eq. (3.12b) is nonzero for $n = N$, then Eqs. (3.12) cannot be satisfied for $r = r_2$: there is only one equation in Frobenius form, corresponding to $r = r_1$. In this case differentiating

$$\mathcal{L}w(x,r) = (r - r_1)(r - r_2)(x - x_0)^r$$

with respect to r at $r = r_1$ shows only that $y_2(x) = \dfrac{\partial w}{\partial r}(x, r_1)$ is a *particular solution* of the *inhomogeneous equation*

$$\mathcal{L}\left[\frac{\partial w}{\partial r}(x, r_1)\right] = (r_1 - r_2)(x - x_0)^{r_1}.$$

Nevertheless, since $\mathcal{L}$ is linear, we only need to find a second (linearly independent) particular solution $\widetilde{y}(x)$ and then the difference of these two particular solutions will be a solution to the *homogeneous equation* $\mathcal{L}y = 0$. Fortunately, a second particular solution to

$$\mathcal{L}\widetilde{y}(x) = (r_1 - r_2)(x - x_0)^{r_1}.$$

in Frobenius form

$$\widetilde{y}(x) \doteq \sum_{n=0}^{\infty} \widetilde{a}_n (x - x_0)^{n+r_2}.$$

can always be found. For $n \neq N$ the coefficients $\widetilde{a}_n$ satisfy the same equations as the coefficients a_n, namely Eqs. (3.12) with $r = r_2$. For $n = N = r_1 - r_2$, equating the coefficients of $(x - x_0)^{r_1}$ yields a constraint that leaves $\widetilde{a}_N$ arbitrary but fixes the value of $\widetilde{a}_0$:

$$\frac{1}{N} \sum_{k=0}^{N-1} [(k + r_2) A_{N-k} + B_{N-k}] \widetilde{a}_k = 1.$$

The above constraint can always be satisfied by scaling $\widetilde{a}_0$, recalling for this case that the right-hand side of Eq. (3.12b), with a_k replaced by $\widetilde{a}_k$, is nonzero for $n = N$. The difference $y_2(x) - \widetilde{y}(x)$, containing the arbitrary constant $\widetilde{a}_N$ is the desired second solution to the inhomogeneous equation $\mathcal{L}y = 0$.

The following theorem summarizes these results.

Theorem 3.1: *Suppose that Eq. (3.5) has a regular singular point at x_0 and that the corresponding indicial equation $P(r) = 0$ has roots at r_1 and r_2.*

1. *If $r_1, r_2 \in \mathbb{R}$ with $r_1 - r_2 \notin \mathbb{Z}$, then there exist two linearly independent solutions of the form*

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \qquad a_0 \neq 0,$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \qquad b_0 \neq 0.$$

2. *If $r_1, r_2 \in \mathbb{R}$ with $r_1 = r_2$, then there exist two linearly independent solutions of the form*

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \qquad a_0 \neq 0,$$

$$y_2(x) = y_1(x) \log(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_1},$$

where the constants b_n may be zero.

3. *If $r_1, r_2 \in \mathbb{R}$ with $r_1 - r_2 \in \mathbb{N}$, then there exist two linearly independent solutions of the form*

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \qquad a_0 \neq 0,$$

$$y_2(x) = A y_1(x) \log(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \qquad b_0 \neq 0,$$

   where the constant A may be zero.

4. *If $r_1, r_2 = \alpha \pm i\beta$, then there exist two linearly independent solutions of the form*

$$y_1(x) = \cos(\beta \log(x - x_0)) \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\alpha}, \qquad a_0 \neq 0,$$

$$y_2(x) = \sin(\beta \log(x - x_0)) \sum_{n=0}^{\infty} b_n (x - x_0)^{n+\alpha}, \qquad b_0 \neq 0.$$

Problem 3.2: Find two linearly independent solutions about $x = 0$ to

$$x^2 y'' - x y' + (1 - x) y = 0.$$

$\boxed{4}$

3.D Behaviour near Irregular Singular Points

As was stated previously, there is no comprehensive theory for irregular singular points. However, we can get some insight by examining a first-order differential equation:

$$u' = p(z)u. \tag{3.17}$$

The solution is easily obtained and is

$$u(z) = ce^{\int p(z)\,dz}, \quad c = \text{const}.$$

Now suppose p has a pole of order N at $z = z_0$. Then

- z_0 is an ordinary point if $N = 0$;

- z_0 is a regular singular point if $N = 1$;
- z_0 is an irregular singular point if $N \geqslant 2$.

We can expand p in a Laurent series:

$$p(z) = \sum_{n=1}^{N} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

This leads to

$$\int p(z)\,dz = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1} & \text{if } N = 0, \\ \displaystyle b_1 \log(z-z_0) + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1} & \text{if } N = 1, \\ \displaystyle\sum_{n=2}^{N} \frac{b_n}{(1-n)(z-z_0)^{n-1}} + b_1 \log(z-z_0) + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1} & \text{if } N \geqslant 2. \end{cases}$$

Define

$$v(z) \doteq \exp\left(\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1} \right), \qquad \tilde{S}(z) \doteq \sum_{n=2}^{N} \frac{b_n}{(1-n)(z-z_0)^{n-1}}.$$

Then v is analytic at $z = z_0$, with $v(z_0) = 1$. The solution may now be written as

$$u(z) = c \exp\left(\int p(z)\,dz \right) = \begin{cases} c\, v(z) & \text{if } N = 0, \\ c\,(z-z_0)^{b_1}\, v(z) & \text{if } N = 1, \\ c\, e^{\tilde{S}(z)}\,(z-z_0)^{b_1}\, v(z) & \text{if } N \geqslant 2. \end{cases}$$

It is clear that if

1. $N = 0$, the solution is analytic at z_0;
2. $N = 1$, the solution has, at worst, a pole or a branch point at z_0;
3. $N \geqslant 2$, the solution has an essential singularity at z_0.

Now consider a second-order *linear* differential equation

$$u'' + p(z)u' + q(z)u = 0, \tag{3.18}$$

with an irregular singular point at $z = z_0$. The previous example suggests making a change of dependent variable as follows:

$$u(z) = e^{S(z)}.$$

Then

$$u'(z) = S'(z)u(z), \qquad u''(z) = [S''(z) + S'^2(z)]u(z),$$

so that Eq. (3.18) becomes

$$S''(z) + S'^2(z) + p(z)S'(z) + q(z) = 0. \tag{3.19}$$

Note that this is a second-order *nonlinear* differential equation and so may seem worse than Eq. (3.18) for u. This is where we make a crucial assumption:

$$\boxed{S'' = o(S'^2) \qquad (z \to z_0)}.$$

Most of the time this is a reasonable assumption, but it must be verified after the fact in each case. To see why this may be reasonable, let us re-examine the solution to the first-order equation (3.17). We have

$$S(z) \sim \frac{b_N}{(1-N)(z-z_0)^{N-1}} \qquad (z \to z_0) \qquad \text{if } N \geqslant 2.$$

On differentiating this asymptotic expansion, we find

$$S'(z) \sim \frac{b_N}{(z-z_0)^N}, \qquad S''(z) \sim \frac{-Nb_N}{(z-z_0)^{N+1}} \qquad (z \to z_0),$$

so that

$$\lim_{z\to z_0} \frac{S''(z)}{S'^2(z)} = \lim_{z\to z_0} \frac{-N}{b_N}(z-z_0)^{N-1} = 0 \qquad \text{for } N \geqslant 2.$$

Hence $S'' = o(S'^2) \quad (z \to z_0)$.

Equation (3.19) may now be approximated by

$$S'^2 \sim -p(z)S' - q(z).$$

This equation, while still nonlinear, is at least of first order. Moreover, it can readily be solved with the substitution $y = S'$, from which we can then determine the solution u to Eq. (3.18).

- Let us determine the behaviour of $x^3u'' = u$ as $x \to 0^+$. Noting that $x = 0$ is an irregular singular point, we let $u(x) = e^{S(x)}$:

$$S'' + S'^2 = \frac{1}{x^3} \tag{3.20}$$

  Assuming that $S'' = o(S'^2)$ $(x \to 0)$, we find

$$S' \sim \frac{\sigma}{x^{3/2}},$$

where σ denotes the sign ± 1. Then $S \sim \frac{-2\sigma}{x^{1/2}}$ $(x \to 0)$. Since

$$S'' \sim -\frac{3\sigma}{2x^{5/2}},$$

we can then check the consistency of our assumption:

$$\frac{S''}{S'^2} \sim -\frac{3\sigma x^3}{2x^{5/2}} = -\frac{3\sigma}{2}x^{1/2} \to 0 \qquad (x \to 0).$$

To find more terms in the asymptotic expansion of S, let us try to express

$$S(x) = -\frac{2\sigma}{x^{1/2}} + C(x),$$

with $C(x) = o(x^{-1/2})$. This implies that

$$S'(x) = \frac{\sigma}{x^{3/2}} + C'(x) \quad \text{with} \quad C' = o(x^{-3/2})$$

and

$$S''(x) = \frac{-3\sigma}{2x^{5/2}} + C''(x) \quad \text{with} \quad C'' = o(x^{-5/2}).$$

If we substitute these expressions into Eq. (3.20) we find:

$$\frac{-3\sigma}{2x^{5/2}} + C'' + \left(\frac{\sigma}{x^{3/2}} + C'\right)^2 = \frac{1}{x^3},$$

so that

$$C'' + \frac{2\sigma}{x^{3/2}}C' + C'^2 = \frac{3\sigma}{2x^{5/2}}. \tag{3.21}$$

Since $C'(x) = o(x^{-3/2})$, we see that $C'^2 = o(x^{-3/2}C')$; the dominant balance of terms is thus given by

$$\frac{2\sigma}{x^{3/2}}C' \sim \frac{3\sigma}{2x^{5/2}} \qquad (x \to 0).$$

Then

$$C' \sim \frac{3}{4x} \qquad (x \to 0)$$

and hence $C(x) \sim \frac{3}{4}\log x = o(x^{-1/2})$ as $x \to 0$. Now express $C(x) = \frac{3}{4}\log x + D(x)$, where we anticipate that $D(x) = o(\log x)$ $(x \to 0)$.

To go further, we find from Eq. (3.21) that

$$-\frac{3}{4x^2} + D'' + \frac{2\sigma}{x^{3/2}}\left(\frac{3}{4x} + D'\right) + \left(\frac{3}{4x} + D'\right)^2 = \frac{3\sigma}{2x^{5/2}}.$$

which simplifies to

$$D'' + \frac{2\sigma}{x^{3/2}}D' + \frac{3}{2x}D' + D'^2 = \frac{3}{16x^2}.$$

On differentiating $C(x) = \frac{3}{4}\log x + D(x)$, we see that $D'(x) = o(x^{-1})$ and $D''(x) = o(x^{-2})$ as $x \to 0$. The dominant balance is then given by

$$\frac{2\sigma}{x^{3/2}} D' \sim \frac{3}{16x^2} \qquad (x \to 0),$$

so that

$$D' \sim \frac{3\sigma}{32} x^{-1/2} \qquad (x \to 0)$$

and

$$D \sim k + \frac{3\sigma}{16} x^{1/2} \qquad (x \to 0),$$

where k is a constant. As desired, we verify that $D(x) = o(\log x)$ as $x \to 0$. Since D approaches a constant as $x \to 0$, we now have enough terms to ascertain the leading-order asymptotic behaviour of the solutions $u(x)$ to the original differential equation:

$$u(x) \sim K x^{3/4} e^{\pm 2x^{-1/2}} \qquad (x \to 0),$$

where K is a constant.

Chapter 4

Perturbation Theory

4.A Introduction

In perturbation problems one considers functions

$$u : \Omega \times I \to \mathbb{R}^m,$$

where $u(x;\varepsilon) = (u_1(x;\varepsilon), u_2(x;\varepsilon), \ldots, u_m(x,\varepsilon))$, $x = (x_1, x_2, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$ and $\varepsilon \in I \subset \mathbb{R}$. Usually the function u is defined implicitly as the solution of a system of differential equations with boundary and/or initial conditions. We may represent such problems as:

$$P : \begin{cases} M(u,\varepsilon) = 0, & M \text{ is a differential operator (possibly nonlinear)}, \\ B(u,\varepsilon) = 0, & B \text{ is a boundary condition operator.} \end{cases}$$

The objective is to solve the problem P for $u(x;\varepsilon)$. Usually, this is too difficult, so we settle for some sort of approximation $\varphi(x;\varepsilon)$ to $u(x;\varepsilon)$. Typically, the parameter ε will be a small positive constant (i.e. $\varepsilon \in I = (0,a)$) and we will seek an approximation φ which is asymptotic to u:

$$u(x;\varepsilon) \sim \varphi(x;\varepsilon) \qquad (\varepsilon \to 0^+) \quad \forall x \in \Omega.$$

Definition: A function $\delta : (0,a) \to \mathbb{R}^+$ is called a *gauge function* if δ is continuous and strictly monotonic.

- The functions $1/\varepsilon$, ε^n for $n \neq 0$, $\sqrt{\varepsilon}$, and $-\varepsilon \log \varepsilon$ are all gauge functions on $(0,a)$ for sufficiently small a, whereas $\varepsilon \sin(1/\varepsilon)$ is not a gauge function on $(0,a)$ for any a.

Suppose that

(i) $\varphi : \Omega \times (0,\varepsilon_0) \to \mathbb{R}, \quad \Omega \subset \mathbb{R}^n$;

(ii) $\delta : (0,a) \to \mathbb{R}^+$ is a gauge function;

(iii) $\Omega_0 \subset \Omega$.

Definition: We write $\varphi = \mathcal{O}(\delta)$ $(\varepsilon \to 0^+)$ uniformly in Ω_0 if there exist positive numbers a and M such that for all $x \in \Omega_0$,

$$0 < \varepsilon < a \Rightarrow |\varphi(x, \varepsilon)| \leqslant M\delta(\varepsilon).$$

This is equivalent to $\left|\dfrac{\varphi(x,\varepsilon)}{\delta(\varepsilon)}\right|$ being bounded on $\Omega_0 \times I$.

Definition: We write $\varphi = o(\delta)$ $(\varepsilon \to 0^+)$ uniformly in Ω_0 if for all $\tilde{\varepsilon} > 0$ there exists $a > 0$ such that for all $x \in \Omega_0$,

$$0 < \varepsilon < a \Rightarrow |\varphi(x, \varepsilon)| \leqslant \tilde{\varepsilon}\delta(\varepsilon).$$

Equivalently, $\lim\limits_{\varepsilon \to 0^+} \dfrac{\varphi(x,\varepsilon)}{\delta(\varepsilon)} = 0$ uniformly in Ω_0.

- Let $x_0 \in (0, 1)$, $\Omega_0 = (x_0, 1)$, $\Omega = (0, 1)$, $\varphi(x, \varepsilon) = \varepsilon/x$, and $\delta(\varepsilon) = \varepsilon$. Then

  $$|\varphi(x, \varepsilon)| = \frac{\varepsilon}{x} \leqslant \frac{\varepsilon}{x_0} = M\delta(\varepsilon)$$

  in Ω_0, where $M = 1/x_0$ is independent of x. That is, $\varphi = \mathcal{O}(\delta)$ $(\varepsilon \to 0^+)$ uniformly in Ω_0, but not uniformly in Ω.

- Let $L > 0$, $\Omega_0 = (0, L)$, $\Omega = (0, \infty)$, $\varphi(x, \varepsilon) = \varepsilon x$, $\psi(x, \varepsilon) = \varepsilon \sin x$, and $\delta(\varepsilon) = \varepsilon$. Then $|\varphi(x, \varepsilon)| = \varepsilon x \leqslant L\varepsilon$ for all $x \in \Omega_0$. That is, $\varphi = \mathcal{O}(\delta)$ $(\varepsilon \to 0^+)$ uniformly in Ω_0, but not uniformly in Ω. On the other hand $|\psi(x, \varepsilon)| = \varepsilon|\sin x| \leqslant \varepsilon$ for all $x \in \Omega$. Therefore $\psi = \mathcal{O}(\delta)$ $(\varepsilon \to 0^+)$ uniformly in Ω.

- Let $x_0 \in (0, 1)$, $\Omega_0 = (x_0, 1)$, $\Omega = (0, 1)$, $\varphi(x, \varepsilon) = e^{-x/\varepsilon}$, and $\delta(\varepsilon) = \varepsilon^n$, where $n \in \mathbb{N}$. Then $|\varphi(x, \varepsilon)| = e^{-x/\varepsilon} \leqslant e^{-x_0/\varepsilon}$ for all $x \in \Omega_0$. Hence in Ω_0,

  $$0 \leqslant \lim_{\varepsilon \to 0^+} \left|\frac{\varphi(x,\varepsilon)}{\delta(\varepsilon)}\right| \leqslant \lim_{\varepsilon \to 0^+} \varepsilon^{-n} e^{-x_0/\varepsilon} = 0.$$

  Therefore $\varphi = o(\delta)$ $(\varepsilon \to 0^+)$ uniformly in Ω_0, but not uniformly in Ω.

Definition: The function φ is a *uniform asymptotic approximation* to u in Ω_0 valid to order δ if $u - \varphi = o(\delta)$ $(\varepsilon \to 0^+)$ uniformly in Ω_0. We write $u \sim \varphi$ $(\varepsilon \to 0^+)$ uniformly in Ω_0.

- Let $x_0 \in (0,1)$. Suppose that

$$u(x,\varepsilon) = \frac{1}{x+\varepsilon}, \quad \varphi(x,\varepsilon) = \frac{1}{x}\left(1-\frac{\varepsilon}{x}\right), \quad \delta(\varepsilon) = \varepsilon.$$

Then

$$|u(x,\varepsilon) - \varphi(x,\varepsilon)| = \frac{\varepsilon^2}{x^2(x+\varepsilon)} = \mathcal{O}(\varepsilon) \qquad (\varepsilon \to 0).$$

uniformly in $(x_0, 1)$, so that φ is a uniform asymptotic approximation to u in $(x_0,1)$, but not in $(0,1)$:

$$\lim_{x\to 0^+}\lim_{\varepsilon\to 0^+} \frac{\varepsilon^2}{x^2(x+\varepsilon)} = 0 \neq \infty = \lim_{\varepsilon\to 0^+}\lim_{x\to 0^+} \frac{\varepsilon^2}{x^2(x+\varepsilon)}.$$

- Let $x_0 \in (0,1)$ and

$$u(x,\varepsilon) = x+\varepsilon+e^{-x/\varepsilon}, \quad \varphi(x,\varepsilon) = x+\varepsilon, \quad \delta(\varepsilon) = \varepsilon^n,$$

where $n \in \mathbb{N}$. Then

$$|u(x,\varepsilon) - \varphi(x,\varepsilon)| = e^{-x/\varepsilon} = \mathcal{O}(\varepsilon^n)$$

uniformly in $(x_0,1)$ but not in $(0,1)$:

$$\lim_{x\to 0^+}\lim_{\varepsilon\to 0^+} e^{-x/\varepsilon} = 0 \neq 1 = \lim_{\varepsilon\to 0^+}\lim_{x\to 0^+} e^{-x/\varepsilon}.$$

Definition: A sequence of functions $\{\varphi_n\}_{n=0}^N$, where $\varphi_n : \Omega \times (0,a) \to \mathbb{R}$ is a *uniform asymptotic sequence* in Ω_0 if $\varphi_{n+1} = \mathcal{O}(\varphi_n)$ $(\varepsilon \to 0)$ uniformly in Ω_0.

Definition: The series $\displaystyle\sum_{n=0}^N \varphi_n(x,\varepsilon)$ is a *uniform asymptotic series* in Ω_0 if $\{\varphi_n\}_{n=0}^N$ is a *uniform asymptotic sequence* in Ω_0.

- For example, $\varphi_n(x,\varepsilon) = u_n(x)\varepsilon^n$ is a uniform asymptotic sequence if $u_{n+1}(x)/u_n(x)$ is bounded for each $n \in \mathbb{N}$.

4.B Regular Perturbations

Let $u : \Omega \times (0,a) \to \mathbb{R}^m$. The most naive way to try to solve the problem

$$P : \begin{cases} M(u,\varepsilon) = 0, \\ B(u,\varepsilon) = 0 \end{cases} \tag{4.1}$$

is by assuming an asymptotic expansion of the *Poincaré type*:

$$u(x, \varepsilon) \sim \sum_{n=0}^{\infty} u_n(x)\varepsilon^n. \tag{4.2}$$

On substituting this expansion into P, one obtains a sequence of linear problems:

$$P_0 : \begin{cases} M(u_0, 0) = 0, \\ B(u_0, 0) = 0, \end{cases} \qquad P_n : \begin{cases} L(u_n) = R_n(u_0, \dots, u_{n-1}), \\ \widetilde{B}(u_n) = \widetilde{R}_n(u_0, \dots, u_{n-1}) \end{cases} \qquad n \geqslant 1.$$

- Consider the problem

$$P : \begin{cases} u' + (1 + \varepsilon x)u = 0, \quad x \in (0, \infty), \\ u(0) = 1, \end{cases}$$

which has the exact solution

$$u(x, \varepsilon) = e^{-x - \varepsilon x^2/2}. \tag{4.3}$$

On introducing the expansion (4.2), the problem P becomes

$$\begin{cases} 0 = \displaystyle\sum_{n=0}^{\infty} u_n'(x)\varepsilon^n + \sum_{n=0}^{\infty} u_n(x)\varepsilon^n + \sum_{n=0}^{\infty} x u_n(x)\varepsilon^{n+1} = u_0' + u_0 + \sum_{n=1}^{\infty}(u_n' + u_n + x u_{n-1})\varepsilon^n, \\ \displaystyle\sum_{n=0}^{\infty} u_n(0)\varepsilon^n = 1, \end{cases}$$

which reduces to the sequence of problems

$$P_0 : \begin{cases} u_0' + u_0 = 0, \\ u_0(0) = 1, \end{cases} \qquad P_n : \begin{cases} u_n' + u_n = -x u_{n-1}, \\ u_n(0) = 0 \end{cases} \qquad n \geqslant 1.$$

We can easily solve these first-order linear ordinary differential equations to find

$$u_0 = e^{-x},$$
$$u_n(x) = -e^{-x} \int_0^x \xi e^{\xi} u_{n-1}(\xi)\, d\xi, \qquad n = 1, 2, \dots.$$

That is,

$$\begin{aligned} u_1(x) &= -e^{-x} \int_0^x \xi\, d\xi = -\frac{x^2}{2} e^{-x}, \\ u_2(x) &= -e^{-x} \int_0^x \xi \left[-\frac{\xi^2}{2}\right] d\xi = \frac{x^4}{8} e^{-x}, \\ &\vdots \\ u_n(x) &= -e^{-x} \int_0^x \xi \left[\frac{(-1)^{n-1}\xi^{2(n-1)}}{2^{n-1}(n-1)!}\right] d\xi = \frac{(-1)^n x^{2n}}{2^n n!} e^{-x}. \end{aligned}$$

Since

$$\left|\frac{u_{n+1}(x)\varepsilon^{n+1}}{u_n(x)\varepsilon^n}\right| = \frac{x^2\varepsilon}{2(n+1)} = o(1) \quad (\varepsilon \to 0) \text{ uniformly in } (0, L) \text{ but not in } (0, \infty),$$

we see that

$$u(x, \varepsilon) \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} e^{-x}\varepsilon^n = e^{-x-\varepsilon x^2/2} \qquad (\varepsilon \to 0)$$

is a uniform asymptotic series in $(0, L)$ but not on $(0, \infty)$.

Definition: The problem P is a *regular perturbation* problem in Ω if

$$\sum_{n=0}^{\infty} u_n(x)\varepsilon^n$$

is a uniform asymptotic approximation to $u(x, \varepsilon)$ in Ω.

Definition: The problem P is a *singular perturbation* problem if P is not a regular perturbation problem.

Remark: The previous example is thus a regular perturbation problem in $\Omega_0 = (0, L)$ and a singular perturbation problem in $\Omega = (0, \infty)$.

Remark: Typically, a straightforward expansion will give rise to a uniform asymptotic expansion only in some proper subset Ω_0 of the domain Ω.

- (Boundary Layer) The problem

$$P : \begin{cases} \varepsilon u'' - u' = 0, & x \in (0, 1), \\ u(0) = 0, \quad u(1) = 1 \end{cases} \tag{4.4}$$

has the exact solution

$$u(x, \varepsilon) = \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}.$$

The expansion (4.2), reduces P to

$$P_0 : \begin{cases} u_0' = 0, \\ u_0(0) = 0, \quad u_0(1) = 1, \end{cases} \qquad P_n : \begin{cases} u_n' = u_{n-1}'', \\ u_n(0) = u_n(1) = 0 \end{cases} \qquad n \geqslant 1.$$

Unfortunately, problem P_0 has no solution; this means that the original problem P has no regular perturbation expansion.

Since ε multiplies the highest derivative, setting ε to zero (e.g. to obtain the problem P_0) drastically changes the nature of the differential equation. As seen in Fig. 4.1, for small but nonzero ε the solution is constant everywhere except in a narrow region of thickness $\mathcal{O}(\varepsilon)$ near $x = 1$, where the solution varies rapidly. This is known as a *boundary layer*. The large derivative in the boundary layer compensates the smallness of ε, so that the highest-order term cannot be neglected in the boundary layer.

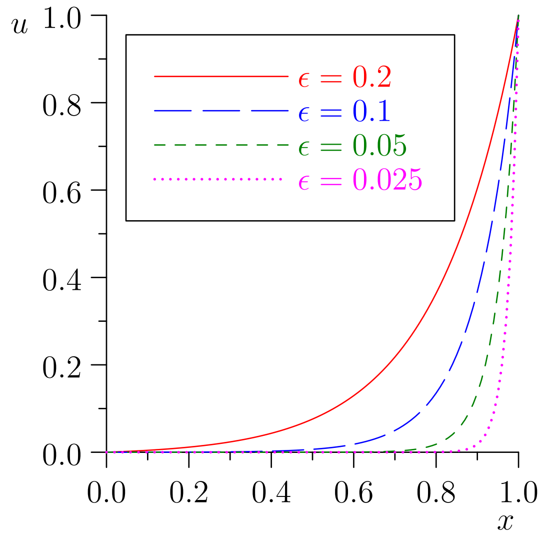


Figure 4.1: Formation of a boundary layer at $x = 1$.

- (Rapid Oscillation) The problem

$$P : \begin{cases} \varepsilon u'' + u = 0, & x \in (0, 1), \\ u(0) = 0, \quad u(1) = 1 \end{cases} \tag{4.5}$$

has the exact solution

$$u(x, \varepsilon) = \frac{\sin(x/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})}.$$

The expansion (4.2), reduces P to

$$P_0 : \begin{cases} u_0 = 0, \\ u_0(0) = 0, \quad u_0(1) = 1, \end{cases} \qquad P_n : \begin{cases} u_n = -u''_{n-1}, \\ u_n(0) = u_n(1) = 0 \end{cases} \quad n \geqslant 1.$$

Again, problem P_0 has no solution. As illustrated in Fig. 4.2, for small ε the exact solution varies rapidly over the entire interval.

- (Multiple Scales) The problem

$$P : \begin{cases} u'' + 2\varepsilon u' + u = 0, & x \in (0, L), \\ u(0) = 1, \quad u'(0) = -\varepsilon \end{cases}$$

has the exact solution

$$u(x, \varepsilon) = e^{-\varepsilon x} \cos(\sqrt{1 - \varepsilon^2} x).$$

The expansion (4.2), reduces P to

$$P_0 : \begin{cases} u_0'' + u_0 = 0, \\ u_0(0) = 1, \quad u_0'(0) = 0, \end{cases}$$

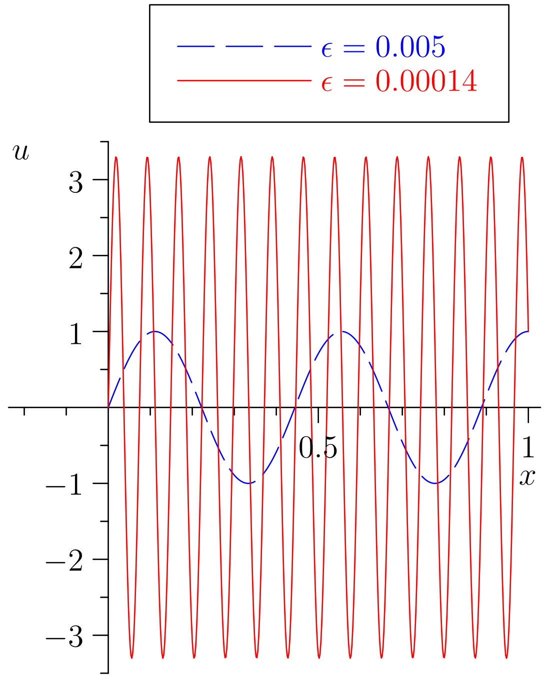


Figure 4.2: Rapid oscillations in a singular perturbation problem.

$$P_1: \begin{cases} u_1'' + u_1 = -2u_0', \\ u_1(0) = 0, \quad u_1'(0) = -1, \end{cases} \qquad P_n: \begin{cases} u_n'' + u_n = -2u_{n-1}', \\ u_n(0) = u_n'(0) = 0 \end{cases} \quad n \geqslant 2.$$

We find $u_0 = \cos x$, $u_1 = -x\cos x$, and that u_n contains a term proportional to $x^n \cos x$. We see that $u_1(x)\varepsilon/u_0(x) = -\varepsilon x = o(\varepsilon)$ $(\varepsilon \to 0)$ uniformly in $(0, L)$. The asymptotic expansion then contains terms of the form $(\varepsilon x)^n \cos x$. This problem thus has two scales, namely x and εx, as illustrated in Fig. 4.3.

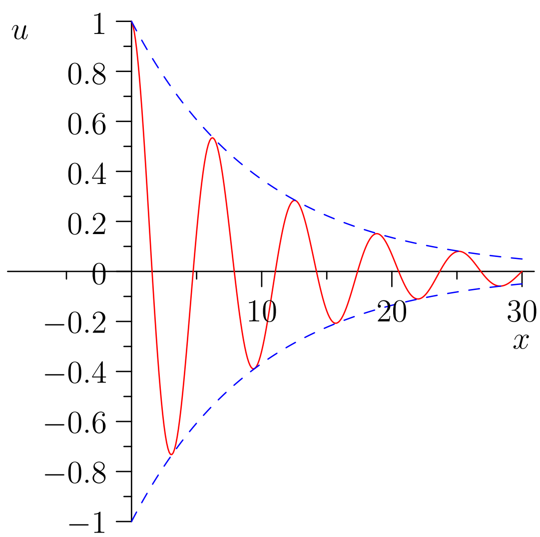


Figure 4.3: Multiple-scale behaviour for $\varepsilon = 0.1$ of $e^{-\varepsilon x}\cos(\sqrt{1-\varepsilon^2}x)$ (solid red) vs. $\pm e^{-\varepsilon x}$ (dashed blue).

Chapter 5

Matched Asymptotic Expansions

5.A A Simple Example

The expansion (4.2) can be used to reduce the problem

$$P : \begin{cases} \varepsilon u'' + (1+\varepsilon)u' + u = 0, & x \in (0,1), \\ u(0) = \alpha, \quad u(1) = \beta \end{cases}$$

to

$$P_0 : \begin{cases} u_0' + u_0 = 0, \\ u_0(0) = \alpha, \quad u_0(1) = \beta, \end{cases} \qquad P_n : \begin{cases} u_n' + u_n = -u_{n-1}'' - u_{n-1}', \\ u_n(0) = u_n(1) = 0 \end{cases} \qquad n \geqslant 1.$$

However, we see that problem P_0 has no solution. The solution to the differential equation in P_0 is

$$u_0 = Ae^{-x}.$$

Clearly, one boundary cannot be satisfied and must be dropped. If we apply only the left boundary condition, then

$$u_0(0) = \alpha \Rightarrow A = \alpha,$$

whereas if we apply only the right boundary condition then

$$u_0(1) = \beta \Rightarrow A = \beta e.$$

The exact solution to P is

$$u(x,\varepsilon) = \frac{(\beta - \alpha e^{-1/\varepsilon})e^{1-x} + (\alpha - \beta e)e^{-x/\varepsilon}}{1 - e^{1-1/\varepsilon}}. \tag{5.1}$$

For small ε one finds

$$u(x,\varepsilon) \sim \beta e^{1-x} + (\alpha - \beta e)e^{-x/\varepsilon} \tag{5.2}$$

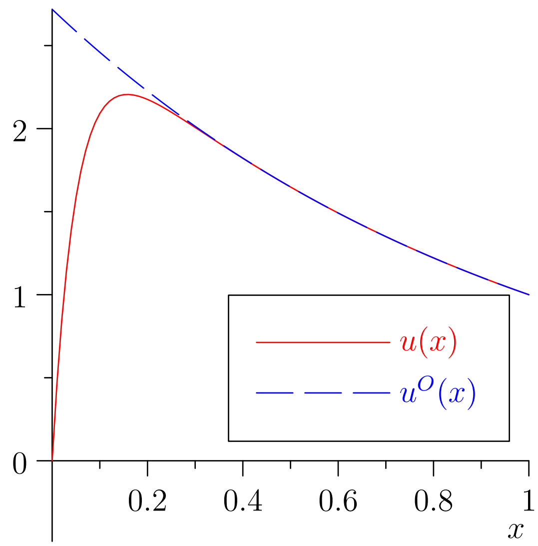


Figure 5.1: Outer solution of Eq. (5.1) for $\varepsilon = 0.05$, $\alpha = 0$, and $\beta = 1$.

and for $x > 0$,

$$u(x, \varepsilon) \sim \beta e^{1-x} \doteq u^O. \tag{5.3}$$

We denote the latter solution by u^O since it is the leading term of the *outer solution*. Equation (5.2) implies that

$$u(0, \varepsilon) \sim \alpha.$$

In contrast, since the $e^{-x/\varepsilon}$ term is not negligible for $x = 0$, Eq. (5.3) incorrectly predicts

$$u^O(0, \varepsilon) \sim \beta e.$$

The underlying problem here is that the order in which one takes the limits $x \to 0$ and $\varepsilon \to 0$ is crucial:

$$\lim_{\varepsilon\to 0^+} \lim_{x\to 0^+} u(x, \varepsilon) = \alpha \neq \beta e = \lim_{x\to 0^+} \lim_{\varepsilon\to 0^+} u(x, \varepsilon).$$

To address this problem, it is helpful to introduce a magnified scale $\xi = x/\varepsilon$, which allows us to zoom in on the boundary layer. Let

$$w(\xi, \varepsilon) \doteq u(\varepsilon\xi, \varepsilon) = \frac{(\beta - \alpha e^{-1/\varepsilon})e^{1-\varepsilon\xi} + (\alpha - \beta e)e^{-\xi}}{1 - e^{1-1/\varepsilon}}.$$

Now expand w for small ε (holding ξ fixed):

$$w(\xi, \varepsilon) \sim \beta e + (\alpha - \beta e)e^{-\xi}.$$

In terms of the original variable, we can write this as the leading term of the *inner solution* u^I:

$$w\left(\frac{x}{\varepsilon}, \varepsilon\right) \sim \beta e + (\alpha - \beta e)e^{-x/\varepsilon} \doteq u_0^I(x, \varepsilon). \tag{5.4}$$

We then recover the correct left boundary condition $u_0^I(0,\varepsilon) = \alpha$ but the incorrect right boundary condition $u_0^I(1,\varepsilon) = \beta e + (\alpha - \beta e)e^{-1/\varepsilon} \sim \beta e$ as $\varepsilon \to 0$.

If we retain an extra term in the inner solution we obtain, as $\varepsilon \to 0$,

$$w(\xi,\varepsilon) \sim \beta e^{1-\varepsilon\xi} + (\alpha - \beta e)e^{-\xi} \sim \beta e(1-\varepsilon\xi) + (\alpha - \beta e)e^{-\xi},$$

so that

$$w\left(\frac{x}{\varepsilon},\varepsilon\right) \sim \beta e(1-x) + (\alpha - \beta e)e^{-x/\varepsilon} \doteq u_1^I(x,\varepsilon). \tag{5.5}$$

Now we still obtain the correct left boundary condition $u_1^I(0,\varepsilon) = \alpha$ but the incorrect boundary condition $u_1^I(1,\varepsilon) = (\alpha - \beta e)e^{-1/\varepsilon} \to 0$ as $\varepsilon \to 0$, as shown in Fig. 5.2.

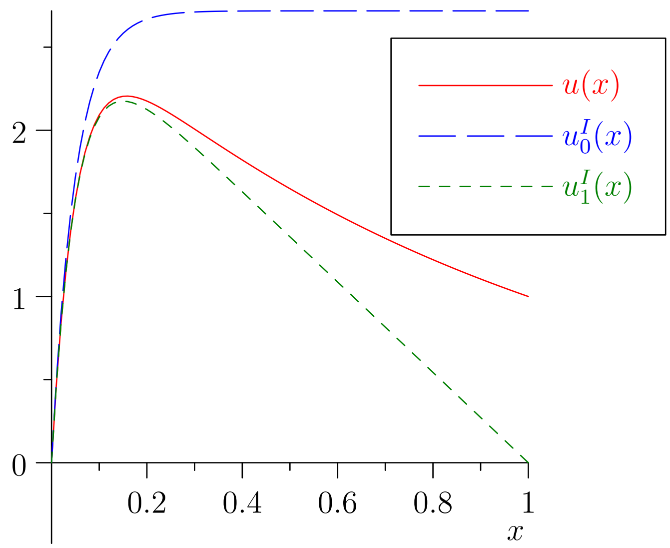


Figure 5.2: Inner solutions given by Eqs. (5.4) and (5.5) for $\varepsilon = 0.05$, $\alpha = 0$, and $\beta = 1$.

Remark: We call $\xi = x/\varepsilon$ the *inner variable*. However, the question naturally arises: what is the best choice for the inner variable? Suppose we instead chose the inner variable to be the magnified scale $\xi = x/\varepsilon^2$. We would then find that

$$\begin{aligned} w(\xi,\varepsilon) \doteq u(\varepsilon^2\xi,\varepsilon) &= \frac{(\beta - \alpha e^{-1/\varepsilon})e^{1-\varepsilon^2\xi} + (\alpha - \beta e)e^{-\varepsilon\xi}}{1 - e^{1-1/\varepsilon}} \\ &\sim \beta e(1-\varepsilon^2\xi) + (\alpha - \beta e)(1-\varepsilon\xi) \qquad (\varepsilon \to 0). \end{aligned}$$

Then

$$w\left(\frac{x}{\varepsilon^2},\varepsilon\right) \sim \beta e(1-x) + (\alpha - \beta e)\left(1 - \frac{x}{\varepsilon}\right) \doteq u_1^I(x,\varepsilon). \tag{5.6}$$

As shown in Fig. 5.3, for the case where $\alpha < \beta e$, this inner solution satisfies $u_1^I(0,\varepsilon) = \alpha$ and $u_1^I(1,\varepsilon) = (\alpha - \beta e)(1 - 1/\varepsilon) \to \infty$ as $\varepsilon \to 0$.

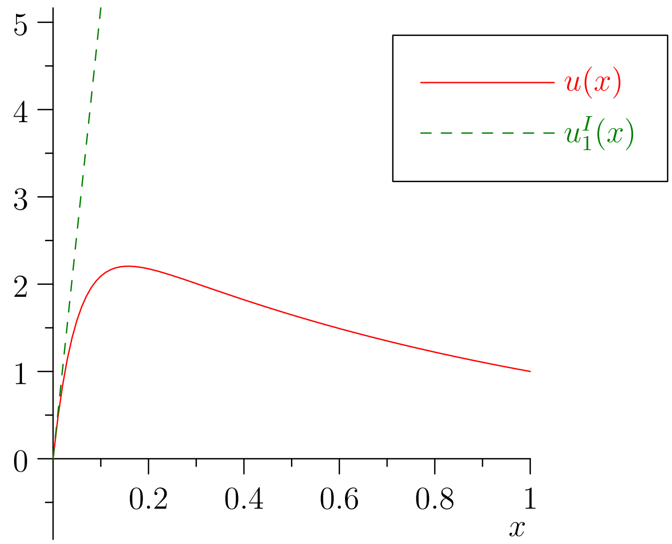


Figure 5.3: Inner solution given by Eq. (5.6) for $\varepsilon = 0.05$, $\alpha = 0$, and $\beta = 1$.

Remark: If instead we choose the inner variable to be the magnified scale $\xi = x/\sqrt{\varepsilon}$, we find

$$\begin{aligned} w(\xi, \varepsilon) &\doteq u(\sqrt{\varepsilon}\xi, \varepsilon) \\ &= \frac{(\beta - \alpha e^{-1/\varepsilon})e^{1-\sqrt{\varepsilon}\xi} + (\alpha - \beta e)e^{-\xi/\sqrt{\varepsilon}}}{1 - e^{1-1/\varepsilon}} \\ &\sim \beta e(1 - \sqrt{\varepsilon}\xi) \qquad (\varepsilon \to 0). \end{aligned}$$

Then

$$w\left(\frac{x}{\sqrt{\varepsilon}}, \varepsilon\right) \sim \beta e(1 - x) \doteq u_1^I. \tag{5.7}$$

As shown in Fig. 5.4, this choice leads to incorrect boundary conditions at both endpoints: $u_1^I(0, \varepsilon) = \beta e$ and $u_1^I(1, \varepsilon) = 0$.

Remark: The basic idea underlying the method of matched asymptotic expansions is to find asymptotic expressions valid over different intervals and then match them on their overlapping domains. Specifically, we can write u^O from Eq. (5.3) in terms of the inner variable $\xi = x/\varepsilon$ and expand it for small ε, holding ξ fixed:

$$u^O = \beta e^{1-\varepsilon\xi} \sim \beta e(1 - \varepsilon\xi) \doteq (u^O)^I \qquad (\varepsilon \to 0).$$

Likewise, we can expand u^I from Eq. (5.5) for small ε, holding x fixed:

$$u^I(x, \varepsilon) = \beta e(1 - x) + (\alpha - \beta e)e^{-x/\varepsilon} \sim \beta e(1 - x) \doteq (u^I)^O \qquad (\varepsilon \to 0).$$

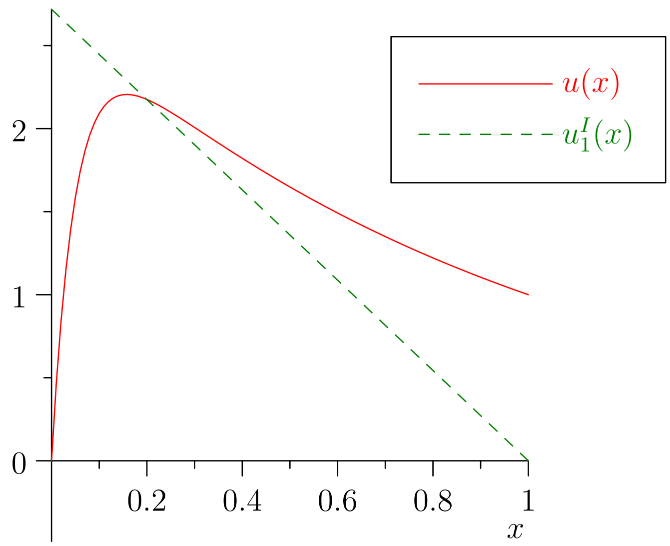


Figure 5.4: Inner solution given by Eq. (5.7) for $\varepsilon = 0.05$.

Notice that the inner expansion of the outer solution equals the outer expansion of the inner solution:

$$(u^O)^I = (u^I)^O.$$

This is called the *matching principle*. When satisfied, the asymptotic matching procedure has been successful.

Problem 5.1: If we instead choose the inner variable $\xi = x/\varepsilon^2$, show that the matching principle is not satisfied when $\alpha \neq \beta e$.

5.B Expansion Operators

The heuristic arguments of the previous section were based on finding approximations to u valid on overlapping subsets of Ω and matching these expressions on this overlap region.

Consider the function

$$u(x, \varepsilon) = x + \varepsilon + e^{-x/\varepsilon}$$

for $x \in [0, 1]$.

The natural choice for the outer and inner variables are the variables that appear in the function, respectively x and $\xi = x/\varepsilon$.

For small ε, the outer solution ($x > 0$) is

$$u^O(x, \varepsilon) = u_0(x) + \varepsilon u_1(x) = x + \varepsilon,$$

where $u_0(x) = x$ and $u_1(x) = 1$.

For small ε, the inner solution

$$u^I(x, \varepsilon) = w(\xi, \varepsilon) = w_0(\xi) + \varepsilon w_1(\xi) = e^{-\xi} + \varepsilon(\xi + 1),$$

where $w_0(\xi) = e^{-\xi}$ and $w_1(\xi) = \xi + 1$.

In the outer region,

$$u(x, \varepsilon) - u_0(x) = \varepsilon + e^{-x/\varepsilon} = \mathcal{O}(1) \quad (\varepsilon \to 0)$$
$$u(x, \varepsilon) - u_0(x) - \varepsilon u_1(x) = e^{-x/\varepsilon} = \mathcal{O}(\varepsilon) \quad (\varepsilon \to 0)$$

uniformly for $x \in [x_0, 1]$ for any $x_0 > 0$.

In the inner region,

$$w(\xi, \varepsilon) - w_0(\xi) = \varepsilon(\xi + 1) = x + \varepsilon = \mathcal{O}(1) \quad (\varepsilon \to 0)$$

uniformly for $x \in [0, \varepsilon]$.

As $\varepsilon \to 0$, we see there is unfortunately no overlap between the regions $[x_0, 1]$ and $[0, \varepsilon]$. However, we can actually allow x_0 to depend on ε, say $x_0 = \eta(\varepsilon)$ and still get a valid asymptotic expansion for the outer solution, provided $\lim\limits_{\varepsilon \to 0} \eta(\varepsilon)/\varepsilon = \infty$:

$$u(x, \varepsilon) - u_0(x) = \varepsilon + e^{-x/\varepsilon} \leqslant \varepsilon + e^{-\eta(\varepsilon)/\varepsilon} = o(1).$$

For example, we can choose $\eta(\varepsilon) = \sqrt{\varepsilon}$. Thus, the outer solution is actually uniformly valid on the enlarged interval $[\sqrt{\varepsilon}, 1]$.

Similarly, the inner expansion is valid in a region larger than $[0, \zeta(\varepsilon)]$, provided $\lim\limits_{\varepsilon \to 0} \zeta(\varepsilon) = 0$:

$$w(\xi, \varepsilon) - w_0(\xi) = x + \varepsilon \leqslant \zeta(\varepsilon) + \varepsilon = \mathcal{O}(1) \quad (\varepsilon \to 0).$$

For example, we can choose $\zeta(\varepsilon) = \varepsilon^{1/3}$, thereby leaving an overlap region of validity $[\varepsilon^{1/2}, \varepsilon^{1/3}]$ for the inner and outer solutions.

We now introduce some formal notation. Suppose

(i) $u : \Omega \times (0, \varepsilon_0) \to \mathbb{R}, \quad \Omega \subset \mathbb{R}^m$;

(ii) $\{\alpha_n(\varepsilon)\}_{n=0}^{\infty}$ is an asymptotic sequence of gauge functions;

(iii) $\Omega_0 \subset \Omega$.

Definition: The *expansion operator* $E_x^n u : \Omega_0 \times (0, a) \to \mathbb{R}$ relative to the asymptotic sequence $\{\alpha_n(\varepsilon)\}_{n=0}^{\infty}$ is defined by $(E_x^n u)(x, \varepsilon) = \sum_{j=0}^{n} u_j(x)\alpha_j(\varepsilon)$, where

$$u_0(x) = \lim_{\varepsilon \to 0^+} \frac{u(x, \varepsilon)}{\alpha_0(\varepsilon)},$$

$$u_j(x) = \lim_{\varepsilon \to 0^+} \frac{u(x, \varepsilon) - (E_x^{j-1} u)(x, \varepsilon)}{\alpha_j(\varepsilon)} \qquad j = 1, 2, \ldots, n,$$

provided the limits exist in Ω_0.

Remark: As in the proof of Theorem 1.3, we see that

$$0 = \lim_{\varepsilon \to 0} \frac{u(x,\varepsilon) - (E_x^{n-1}u)(x,\varepsilon) - u_n(x)\alpha_n(\varepsilon)}{\alpha_n(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{u(x,\varepsilon) - (E_x^n u)(x,\varepsilon)}{\alpha_n(\varepsilon)}.$$

For $x \in \Omega_0$ we thus have

$$u(x,\varepsilon) = (E_x^n u)(x,\varepsilon) + o(\alpha_n(\varepsilon)) \qquad (\varepsilon \to 0^+).$$

- Consider again the function $u(x,\varepsilon) = x + \varepsilon + e^{-x/\varepsilon}$ with $\alpha_n(\varepsilon) = \varepsilon^n$, with $\Omega_0 = (0,1]$. We find

$$\begin{aligned} u_0(x) &= \lim_{\varepsilon \to 0^+} \frac{u(x,\varepsilon)}{1} = x, \\ u_1(x) &= \lim_{\varepsilon \to 0^+} \frac{u(x,\varepsilon) - u_0(x)}{\varepsilon} = 1, \\ u_n(x) &= \lim_{\varepsilon \to 0^+} \frac{e^{-x/\varepsilon}}{\varepsilon^n} = 0 \qquad \text{for } n \geqslant 2. \end{aligned}$$

  Thus

$$\begin{aligned} (E_x^0 u)(x,\varepsilon) &= x, \\ (E_x^n u)(x,\varepsilon) &= x + \varepsilon \qquad \text{for } n \geqslant 1. \end{aligned}$$

Definition: Given the change of variable $\xi = x/\delta(\varepsilon)$, denote $w(\xi,\varepsilon) \doteq u(x,\varepsilon) = u(\delta(\varepsilon)\xi,\varepsilon)$, and suppose $\{\beta_n(\varepsilon)\}_{n=0}^{\infty}$ is an asymptotic sequence of gauge functions. The *expansion operator* H_ξ^n relative to the asymptotic sequence $\{\beta_n(\varepsilon)\}_{n=0}^{\infty}$ is defined by $(H_\xi^n u)(\xi,\varepsilon) = \sum_{j=0}^{n} w_j(\xi)\beta_j(\varepsilon)$, where

$$\begin{aligned} w_0(\xi) &= \lim_{\varepsilon \to 0^+} \frac{w(\xi,\varepsilon)}{\beta_0(\varepsilon)}, \\ w_j(\xi) &= \lim_{\varepsilon \to 0^+} \frac{w(\xi,\varepsilon) - (H_\xi^{j-1}u)(\xi,\varepsilon)}{\beta_j(\varepsilon)} \qquad j = 1, 2, \ldots, n. \end{aligned}$$

- For $u(x,\varepsilon) = x + \varepsilon + e^{-x/\varepsilon}$ with $\beta_n(\varepsilon) = \varepsilon^n$, we have $w(\xi,\varepsilon) = u(\varepsilon\xi,\varepsilon) = e^{-\xi} + \varepsilon(\xi + 1)$, since

$$\begin{aligned} w_0(\xi) &= \lim_{\varepsilon \to 0^+} \frac{w(\xi,\varepsilon)}{1} = e^{-\xi}, \\ w_1(\xi) &= \lim_{\varepsilon \to 0^+} \frac{w(\xi,\varepsilon) - w_0(x)}{\varepsilon} = \xi + 1, \\ w_n(\xi) &= 0 \qquad \text{for } n \geqslant 2. \end{aligned}$$

  Thus

$$\begin{aligned} (H_\xi^0 u)(\xi,\varepsilon) &= e^{-\xi}, \\ (H_\xi^n u)(\xi,\varepsilon) &= e^{-\xi} + \varepsilon(\xi + 1) \qquad \text{for } n \geqslant 1. \end{aligned}$$

Remark: From the previous example we have

$$E_x^0 u = x, \qquad H_\xi^0 u = e^{-\xi}.$$

Thus

$$E_x^0 H_\xi^0 u = E_x^0 e^{-\xi} = E_x^0(e^{-x/\varepsilon}) = \lim_{\varepsilon\to 0^+} \frac{e^{-x/\varepsilon}}{1} = 0,$$

and

$$H_\xi^0 E_x^0 u = H_\xi^0(x) = H_\xi^0(\varepsilon\xi) = \lim_{\varepsilon\to 0^+} \frac{\varepsilon\xi}{1} = 0.$$

That is,

$$E_x^0 H_\xi^0 u = H_\xi^0 E_x^0 u.$$

We also found

$$E_x^1 u = x + \varepsilon, \qquad H_\xi^1 u = e^{-\xi} + \varepsilon(\xi + 1).$$

Thus

$$E_x^1 H_\xi^1 u = E_x^1(e^{-\xi} + \varepsilon(\xi+1)) = E_x^1(e^{-x/\varepsilon} + x + \varepsilon) = E_x^1 u = x + \varepsilon,$$

and

$$H_\xi^1 E_x^1 u = H_\xi^1(x+\varepsilon) = H_\xi^1(\varepsilon\xi + \varepsilon) = 0 + \varepsilon \lim_{\varepsilon\to 0^+} \frac{\varepsilon\xi + \varepsilon}{\varepsilon} = \varepsilon(\xi+1) = x + \varepsilon.$$

That is,

$$E_x^1 H_\xi^1 u = H_\xi^1 E_x^1 u.$$

The statement that

$$E_x^n H_\xi^n u = H_\xi^n E_x^n u \qquad \forall n \in \mathbb{N}_0 \tag{5.8}$$

is a way of expressing the matching principle.

Remark: A generalized version of the matching principle,

$$E_x^n H_\xi^m u = H_\xi^m E_x^n u,$$

is sometimes used in situations where the case $n = m$ does not work.

Definition: When the matching principle is successful, the *composite expansion operator* C_x^n defined by

$$C_x^n u = E_x^n u + H_\xi^n u - E_x^n H_\xi^n u$$

can be used to generate global approximations to singular perturbation problems.

- For $u(x, \varepsilon) = x + \varepsilon + e^{-x/\varepsilon}$ we find

$$C_x^0 u = E_x^0 u + H_\xi^0 u - E_x^0 H_\xi^0 u = x + e^{-\xi} - 0 = x + e^{-x/\varepsilon}.$$
$$C_x^1 u = E_x^1 u + H_\xi^1 u - E_x^1 H_\xi^1 u = x + \varepsilon + e^{-\xi} + \varepsilon(\xi + 1) - (x + \varepsilon) = x + \varepsilon + e^{-x/\varepsilon}.$$

5.C The Method of Matched Asymptotic Expansions

The results of the previous section were obtained with the benefit of the exact solution. We now attempt to ascertain these properties directly from the differential equation. First, we must determine the location of the boundary layer. If physical considerations do not indicate the location of the boundary layer, one may have to resort to trial and error. If the wrong location is used, the matching will not be successful.

- Let us revisit the problem

$$P : \begin{cases} \varepsilon u'' + (1+\varepsilon)u' + u = 0, & x \in [0,1], \\ u(0,\varepsilon) = \alpha, \quad u(1,\varepsilon) = \beta \end{cases}.$$

Suppose we determine that the boundary layer occurs at $x = 0$. On substituting the expansion (4.2) and discarding the boundary condition at 0, the *outer problem* becomes

$$P_0 : \begin{cases} u_0' + u_0 = 0, \\ u_0(1) = \beta, \end{cases} \qquad P_n : \begin{cases} u_n' + u_n = -u_{n-1}' - u_{n-1}'', \\ u_n(1) = 0 \end{cases} \qquad n \geqslant 1.$$

The solutions to these first-order differential equations are given by

$$u_0(x) = \beta e^{1-x},$$

$$u_n(x) = \int_x^1 e^{\lambda - x}(u_{n-1}'(\lambda) + u_{n-1}''(\lambda))\, d\lambda \qquad n \geqslant 1.$$

We recursively find $u_n(x) = 0$ for all $n \geqslant 1$.

The outer solution is thus given by

$$u^O(x,\varepsilon) = \beta e^{1-x} + o(\varepsilon^n) \qquad (\varepsilon \to 0)$$

for any $n \in \mathbb{N}$.

To determine the correct inner variable, try $\xi = x/\delta(\varepsilon)$, where $\delta(\varepsilon) = \mathcal{O}(1)$ as $\varepsilon \to 0$. Then

$$u(x,\varepsilon) = w\Big(\frac{x}{\delta},\varepsilon\Big) = w(\xi,\varepsilon),$$
$$u'(x,\varepsilon) = \frac{1}{\delta}w'\Big(\frac{x}{\delta},\varepsilon\Big) = \frac{1}{\delta}w'(\xi,\varepsilon),$$
$$u''(x,\varepsilon) = \frac{1}{\delta^2}w''\Big(\frac{x}{\delta},\varepsilon\Big) = \frac{1}{\delta^2}w''(\xi,\varepsilon),$$

so that the *inner problem* becomes

$$P^I : \begin{cases} \dfrac{\varepsilon}{\delta^2}w'' + (1+\varepsilon)\dfrac{1}{\delta}w' + w = 0, \quad x \in \Omega = [0,1], \\ w(0,\varepsilon) = \alpha. \end{cases}$$

We have discarded the right-hand boundary condition since the right boundary lies outside the boundary layer. The resulting system, being a second-order ordinary differential equation with only one boundary condition, is underdetermined. The extra constant of integration will be determined by matching to the outer solution.

There are three cases of interest:

Case (i): $\lim_{\varepsilon\to 0}\delta(\varepsilon)/\varepsilon = \infty$

The inner problem may be written

$$P^I : \begin{cases} \dfrac{\varepsilon}{\delta}w'' + (1+\varepsilon)w' + \delta w = 0, \\ w(0,\varepsilon) = \alpha. \end{cases}$$

On expressing $w(\xi,\varepsilon) = w_0(\xi) + \mathcal{O}(1)$, the lowest-order problem becomes, noting that $\varepsilon/\delta = \mathcal{O}(1)$ and $\delta = \mathcal{O}(1)$,

$$P_0^I : \begin{cases} w_0' = 0 \\ w_0(0) = \alpha, \end{cases}$$

which has the solution $w_0(\xi) = \alpha$. The leading-order behaviours of the inner and outer solutions are then respectively given by

$$E_x^0 u = u_0 = \beta e^{1-x}, \qquad H_\xi^0 u = w_0 = \alpha.$$

Since $\delta = o(1)$, we thus find that

$$E_x^0 H_\xi^0 u = E_x^0\left(w_0\Big(\frac{x}{\delta}\Big)\right) = w_0(\infty) = \alpha,$$
$$H_\xi^0 E_x^0 u = H_\xi^0(u_0(\delta\xi)) = u_0(0) = \beta e.$$

Since $\alpha \neq \beta e$ in general, we see that the matching principle, Eq. (5.8), is not satisfied.

Case (ii): $\lim_{\varepsilon\to 0} \delta(\varepsilon)/\varepsilon = 0$

Express the inner problem as

$$P^I : \begin{cases} w'' + \dfrac{\delta}{\varepsilon}(1+\varepsilon)w' + \dfrac{\delta^2}{\varepsilon}w = 0, \\ w(0,\varepsilon) = \alpha. \end{cases}$$

Since $\delta = \mathcal{O}(1)$, the lowest-order problem becomes, expressing $w(\xi,\varepsilon) = w_0(\xi) + \mathcal{O}(1)$,

$$P_0^I : \begin{cases} w_0'' = 0, \\ w_0(0) = \alpha. \end{cases}$$

which has the solution $w_0(\xi) = \alpha + A\xi$ where A is a constant. The leading-order behaviours of the inner and outer solutions are then respectively given by

$$E_x^0 u = u_0 = \beta e^{1-x}, \qquad H_\xi^0 u = w_0 = \alpha + A\xi.$$

Since $\delta = o(1)$, we thus find that

$$E_x^0 H_\xi^0 u = E_x^0\left(w_0\left(\frac{x}{\delta}\right)\right) = w_0(\infty) = \begin{cases} \alpha & \text{if } A = 0, \\ \infty & \text{if } A > 0, \\ -\infty & \text{if } A < 0. \end{cases}$$

$$H_\xi^0 E_x^0 u = H_\xi^0(u_0(\delta\xi)) = u_0(0) = \beta e.$$

Again, the matching condition is not satisfied.

Case (iii): $\delta(\varepsilon) = \varepsilon$

The inner problem reduces to

$$P^I : \begin{cases} w'' + (1+\varepsilon)w' + \varepsilon w = 0, \\ w(0,\varepsilon) = \alpha, \end{cases}$$

On expressing $w(\xi,\varepsilon) = \sum_{n=0}^{\infty} w_n(\xi)\varepsilon^n$, the inner problem becomes

$$P_0^I : \begin{cases} w_0'' + w_0' = 0, \\ w_0(0) = \alpha. \end{cases}$$

$$P_1^I : \begin{cases} w_1'' + w_1' = -w_0' - w_0, \\ w_1(0) = 0, \end{cases}$$

from which we deduce that $w_0(\xi) = \alpha - A_0(1 - e^{-\xi})$ and $w_1(\xi) = (A_0 - \alpha)\xi + A_1(1 - e^{-\xi})$, where A_0 and A_1 are constants.

Thus

$$E_x^0 H_\xi^0 u = E_x^0 \left(w_0\left(\frac{x}{\varepsilon}\right)\right) = w_0(\infty) = \alpha - A_0,$$

$$H_\xi^0 E_x^0 u = H_\xi^0(u_0(\varepsilon\xi)) = u_0(0) = \beta e.$$

The lowest-order matching condition $E_x^0 H_\xi^0 u = H_\xi^0 E_x^0 u$ can thus be satisfied if we choose $A_0 = \alpha - \beta e$, so that

$$w_0(\xi) = \beta e + (\alpha - \beta e)e^{-\xi}$$

$$w_1(\xi) = -\beta e\xi + A_1(1 - e^{-\xi}).$$

At the next order, we find, noting that $u_1 = 0$,

$$E_x^1 H_\xi^1 u = E_x^1 \left(w_0\left(\frac{x}{\varepsilon}\right) + \varepsilon w_1\left(\frac{x}{\varepsilon}\right)\right) = f_0(x) + \varepsilon f_1(x),$$

$$H_\xi^1 E_x^1 u = H_\xi^1(u_0(\varepsilon\xi)) = g_0(\xi) + \varepsilon g_1(\xi),$$

where

$$\begin{aligned}
f_0(x) &= \lim_{\varepsilon\to 0}\left[w_0\left(\frac{x}{\varepsilon}\right) + \varepsilon w_1\left(\frac{x}{\varepsilon}\right)\right]\\
&= \lim_{\varepsilon\to 0}\left[\beta e + (\alpha - \beta e)e^{-x/\varepsilon} - \beta e x + \varepsilon A_1(1 - e^{-x/\varepsilon})\right] = \beta e(1 - x),\\
f_1(x) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[w_0\left(\frac{x}{\varepsilon}\right) + \varepsilon w_1\left(\frac{x}{\varepsilon}\right) - f_0(x)\right]\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\beta e + (\alpha - \beta e)e^{-x/\varepsilon} - \beta e x + \varepsilon A_1(1 - e^{-x/\varepsilon}) - \beta e(1 - x)\right]\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[(\alpha - \beta e)e^{-x/\varepsilon} + \varepsilon A_1(1 - e^{-x/\varepsilon})\right] = A_1,\\
g_0(\xi) &= \lim_{\varepsilon\to 0} u_0(\varepsilon\xi) = \lim_{\varepsilon\to 0}\beta e^{1-\varepsilon\xi} = \beta e,\\
g_1(\xi) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\beta e^{1-\varepsilon\xi} - \beta e\right] = \beta e\lim_{\varepsilon\to 0}\frac{e^{-\varepsilon\xi} - 1}{\varepsilon} = -\beta e\xi.
\end{aligned}$$

We can thus make $E_x^1 H_\xi^1 u = \beta e(1 - x) + \varepsilon A_1$ equal $H_\xi^1 E_x^1 u = \beta e(1 - \varepsilon\xi) = \beta e(1 - x)$ by choosing $A_1 = 0$.

We have thus recovered the inner solution, Eq. (5.5), that we previously obtained with foreknowledge of the exact solution:

$$w(\xi, \varepsilon) \sim w_0(\xi) + \varepsilon w_1(\xi) = \beta e(1 - \varepsilon\xi) + (\alpha - \beta e)e^{-\xi}.$$

The choice $\delta \sim \varepsilon$ thus yields an inner variable that allows the corresponding inner solution to be matched to the outer solution $u(x, \varepsilon) \sim \beta e^{1-x}$. Note that $w(0, \varepsilon) = \alpha$ and $u(1, \varepsilon) = \beta$, as desired.

Remark: With $\xi = x/\delta(\varepsilon)$, we say that the *boundary layer thickness* is $\mathcal{O}(\delta(\varepsilon))$. The full solution can be expressed as

$$u(x,\varepsilon) \sim \begin{cases} u^I(x,\varepsilon) & \text{near the origin,} \\ u^O(x,\varepsilon) & \text{away from the origin.} \end{cases}$$

While u^I and u^O are valid on overlapping domains, the place to switch from one domain to the other is not so easily determined. This difficulty can be circumvented with the composite expansion operator:

$$\begin{aligned} C_x^0 u &= E_x^0 u + H_\xi^0 u - E_x^0 H_\xi^0 u. \\ &= u_0(x) + w_0\left(\frac{x}{\varepsilon}\right) - \beta e \\ &= \beta e^{1-x} + \beta e + (\alpha - \beta e)e^{-x/\varepsilon} - \beta e. \\ &= \beta e^{1-x} + (\alpha - \beta e)e^{-x/\varepsilon}. \end{aligned}$$

Going to higher order, we obtain the same result:

$$\begin{aligned} C_x^1 u &= E_x^1 u + H_\xi^1 u - E_x^1 H_\xi^1 u. \\ &= u_0(x) + w_0\left(\frac{x}{\varepsilon}\right) + \varepsilon w_1\left(\frac{x}{\varepsilon}\right) - \beta e(1-x) \\ &= \beta e^{1-x} + \beta e(1-x) + (\alpha - \beta e)e^{-x/\varepsilon} - \beta e(1-x) \\ &= \beta e^{1-x} + (\alpha - \beta e)e^{-x/\varepsilon}. \end{aligned} \tag{5.9}$$

In the limit $\varepsilon \to 0$, we see that this composite solution obeys the desired boundary conditions. It is instructive to compare this result with the exact solution:

$$\begin{aligned} u(x,\varepsilon) &= \frac{(\beta - \alpha e^{-1/\varepsilon})e^{1-x} + (\alpha - \beta e)e^{-x/\varepsilon}}{1 - e^{1-1/\varepsilon}} \\ &\sim \beta e^{1-x} + (\alpha - \beta e)e^{-x/\varepsilon} \qquad (\varepsilon \to 0). \end{aligned}$$

As shown in Fig. 5.5, the matched asymptotic and exact solutions are indistinguishable to graphical accuracy, even at relatively large values of the perturbation parameter, such as $\varepsilon = 0.1$.

Remark: The correct choice of $\delta(\varepsilon)$ must be such that the terms neglected in the outer equation are retained in P^I. In the previous example, since u'' was neglected, along with a u' term, in the lowest-order outer equation, both w'' and w' must be retained in the lowest-order approximation to P^I.

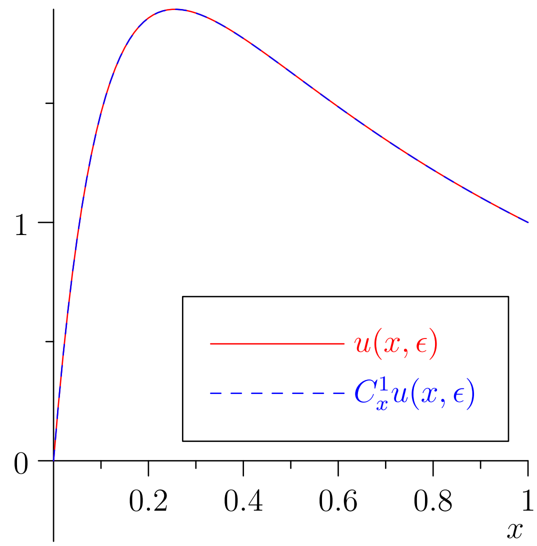


Figure 5.5: Matched asymptotic solution and exact solution given by Eq. (5.9) for $\varepsilon = 0.1$.

Remark: Typically one wants to choose $\delta(\varepsilon)$ so as to obtain the "least degenerate" form of the inner equation in the sense that the coefficients of the differential equation are of comparable size, to the extent possible. In our example,

$$\frac{\varepsilon}{\delta^2}w'' + \frac{1}{\delta}(1+\varepsilon)w' + w = 0,$$

which can be rewritten as

$$w'' + \frac{\delta}{\varepsilon}(1+\varepsilon)w' + \frac{\delta^2}{\varepsilon}w = 0,$$

we know that $\delta = o(1)$, so $\dfrac{\delta^2}{\varepsilon} = o\left(\dfrac{\delta}{\varepsilon}\right)$. Of the three cases

(i) $\dfrac{\varepsilon}{\delta} \prec 1$,

(ii) $\dfrac{\delta^2}{\varepsilon} \prec \dfrac{\delta}{\varepsilon} \prec 1$,

(iii) $\dfrac{\delta^2}{\varepsilon} \prec \dfrac{\delta}{\varepsilon} \sim 1$,

Case (iii) is the "least degenerate".

- (Nonlinear ODE) [Bender & Orszag 1999, p. 463]

  Consider the problem

$$P : \begin{cases} \varepsilon u'' + 2u' + e^u = 0, & x \in [0,1], \\ u(0,\varepsilon) = u(1,\varepsilon) = 0. \end{cases} \tag{5.10}$$

Let us assume that the boundary layer is at $x = 0$ and look for an outer solution of the form

$$u(x,\varepsilon) \sim \sum_{n=0}^{\infty} u_n(x)\varepsilon^n.$$

On discarding the boundary condition at $x = 0$, we find

$$P_0 : \begin{cases} 2u_0' + e^{u_0} = 0, \\ u_0(1) = 0, \end{cases}$$

so that $u_0(x) = \log \dfrac{2}{1+x}$.

Let the inner variable be $\xi = x/\delta(\varepsilon)$ and $w(\xi,\varepsilon) = u(x,\varepsilon)$. The inner problem is

$$P^I : \begin{cases} w'' + 2\dfrac{\delta}{\varepsilon} w' + \dfrac{\delta^2}{\varepsilon} e^w = 0, \\ w(0,\varepsilon) = 0. \end{cases}$$

We choose the least degenerate case $\dfrac{\delta^2}{\varepsilon} \prec \dfrac{\delta}{\varepsilon} \sim 1$, so that

$$P^I : \begin{cases} w'' + 2w' + \varepsilon e^w = 0, \\ w(0,\varepsilon) = 0, \end{cases}$$

Let us look for an inner solution of the form

$$w(\xi,\varepsilon) \sim \sum_{n=0}^{\infty} w_n(\xi)\varepsilon^n.$$

The lowest-order problem is

$$P_0^I : \begin{cases} w_0'' + 2w_0' = 0, \\ w_0(0) = 0, \end{cases}$$

which has the solution $w_0(\xi) = A(1 - e^{-2\xi})$.

Thus

$$E_x^0 H_\xi^0 u = E_x^0 \left(w_0\left(\frac{x}{\varepsilon}\right)\right) = w_0(\infty) = A,$$

$$H_\xi^0 E_x^0 u = H_\xi^0(u_0(\varepsilon\xi)) = u_0(0) = \log 2.$$

The lowest-order matching condition $E_x^0 H_\xi^0 u = H_\xi^0 E_x^0 u$ can thus be satisfied if we choose $A = \log 2$, so that the inner solution is

$$w_0(\xi) = \log 2(1 - e^{-2\xi}).$$

The composite solution is then given by

$$\begin{aligned} C_x^0 u &= E_x^0 u + H_\xi^0 u - E_x^0 H_\xi^0 u \\ &= u_0(x) + w_0\left(\frac{x}{\varepsilon}\right) - \log 2 \\ &= \log \frac{2}{1+x} - e^{-2x/\varepsilon} \log 2, \end{aligned} \tag{5.11}$$

as shown in Figure 5.6.

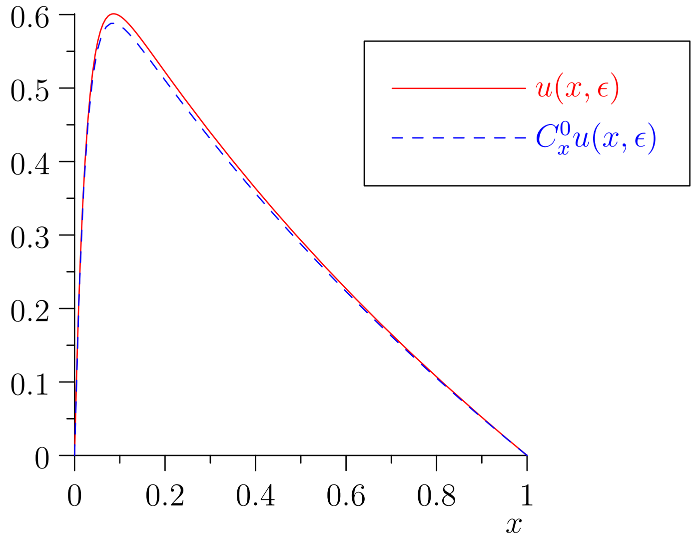


Figure 5.6: Matched asymptotic solution in Eq. (5.11) for $\varepsilon = 0.05$ compared to the exact solution of Eq. (5.10) computed numerically with a shooting method.

- (Two boundary layers) [Bender & Orszag 1999, p. 437]
  For the problem

$$P : \begin{cases} \varepsilon u'' - x^2 u' - u = 0, & x \in [0, 1], \\ u(0, \varepsilon) = u(1, \varepsilon) = 1. \end{cases} \tag{5.12}$$

let us look for an outer solution of the form

$$u(x, \varepsilon) \sim \sum_{n=0}^{\infty} u_n(x) \varepsilon^n.$$

The outer problem

$$P_0 : \begin{cases} x^2 u_0' + u_0 = 0, \\ u_0(1) = 1, \end{cases}$$

has the solution $u_0(x) = e^{1/x-1}$. We see that the left boundary condition cannot be satisfied and therefore expect a boundary layer at $x = 0$.

Let the inner variable be $\xi = x/\delta(\varepsilon)$ and $w(\xi, \varepsilon) = u(x, \varepsilon)$. The inner problem is

$$P^I : \begin{cases} \dfrac{\varepsilon}{\delta^2} w'' - \delta \xi^2 w' - w = 0, \\ w(0, \varepsilon) = 1, \end{cases}$$

or

$$P^I : \begin{cases} w'' - \dfrac{\delta^3}{\varepsilon} \xi^2 w' - \dfrac{\delta^2}{\varepsilon} w = 0, \\ w(0, \varepsilon) = 1. \end{cases}$$

We choose the least degenerate case $\dfrac{\delta^3}{\varepsilon} \prec \dfrac{\delta^2}{\varepsilon} \sim 1$, taking $\delta = \sqrt{\varepsilon}$:

$$P^I : \begin{cases} w'' - \sqrt{\varepsilon} \xi^2 w' - w = 0, \\ w(0, \varepsilon) = 1. \end{cases}$$

Let us look for an inner solution of the form

$$w(\xi, \varepsilon) \sim \sum_{n=0}^{\infty} w_n(\xi) \varepsilon^n.$$

The lowest-order problem is

$$P_0^I : \begin{cases} w_0'' - w_0 = 0, \\ w_0(0) = 1, \end{cases}$$

which has the solution $w_0(\xi) = B e^{\xi} + (1 - B) e^{-\xi}$, where B is a constant. Then

$$E_x^0 H_\xi^0 u = E_x^0 \left(w_0 \left(\frac{x}{\sqrt{\varepsilon}} \right) \right) = w_0(\infty) = \begin{cases} 0 & \text{if } B = 0, \\ \operatorname{sgn}(B) \cdot \infty & \text{if } B \neq 0, \end{cases}$$

$$H_\xi^0 E_x^0 u = H_\xi^0 (u_0(\varepsilon \xi)) = u_0(0) = \infty.$$

Since the outer solution diverges, no match is possible.

What went wrong here is that the outer solution u_0 is actually incorrect: there is another boundary layer at $x = 1$. That is, we need to drop both boundary conditions; the outer solution then becomes $u_0(x) = A e^{1/x}$, where the constant A will be determined by asymptotic matching.

Let us try the inner variable $\zeta = (1-x)/\mu(\varepsilon)$, with $\mu(\varepsilon) = o(1)$ and express $u(x,\varepsilon) = v(\zeta,\varepsilon) = v((1-x)/\mu,\varepsilon)$. Then $u' = -v'/\mu$ and $u'' = v''/\mu^2$, so that the inner problem at the right boundary is

$$P_R^I : \begin{cases} \dfrac{\varepsilon}{\mu^2}v'' + \dfrac{(1-\mu\zeta)^2}{\mu}v' - v = 0, \\ v(0,\varepsilon) = 1, \end{cases}$$

or

$$P_R^I : \begin{cases} v'' + \dfrac{\mu(1-\mu\zeta)^2}{\varepsilon}v' - \dfrac{\mu^2}{\varepsilon}v = 0, \\ v(0,\varepsilon) = 1. \end{cases}$$

We choose the least degenerate case $\dfrac{\mu^2}{\varepsilon} \prec \dfrac{\mu}{\varepsilon} \sim 1$, taking $\mu = \varepsilon$. The inner problem at the right boundary becomes

$$P_R^I : \begin{cases} v'' + (1-\varepsilon\zeta)^2 v' - \varepsilon v = 0, \\ v(0,\varepsilon) = 1. \end{cases}$$

We express

$$v(\zeta,\varepsilon) \sim \sum_{n=0}^{\infty} v_n(\zeta)\varepsilon^n,$$

and find

$$P_R^I : \begin{cases} v_0'' + v_0' = 0, \\ v_0(0) = 1, \end{cases}$$

which has the solution $v_0(\zeta) = C + (1-C)e^{-\zeta}$, where C is a constant. On matching the outer solution $u_0(x) = Ae^{1/x}$ to the left boundary layer, we find

$$E_x^0 H_\xi^0 u = E_x^0\left(w_0\left(\frac{x}{\sqrt{\varepsilon}}\right)\right) = w_0(\infty) = \begin{cases} 0 & \text{if } B = 0, \\ \operatorname{sgn}(B)\cdot\infty & \text{if } B \neq 0, \end{cases}$$

$$H_\xi^0 E_x^0 u = H_\xi^0(u_0(\varepsilon\xi)) = u_0(0) = \begin{cases} 0 & \text{if } A = 0, \\ \operatorname{sgn}(A)\cdot\infty & \text{if } A \neq 0. \end{cases}$$

The matching condition thus requires $A = B = 0$. That is, $u(x) = 0$ and $w_0 = e^{-\xi}$. On matching the outer solution to the right boundary layer, we find

$$E_x^0 H_\zeta^0 u = E_x^0\left(v_0\left(\frac{1-x}{\varepsilon}\right)\right) = v_0(\infty) = C.$$

$$H_\zeta^0 E_x^0 u = H_\zeta^0(u_0(1-\varepsilon\xi)) = u_0(1) = Ae = 0.$$

The matching condition requires $C = 0$. That is, $v_0 = e^{-\zeta}$.

Finally, the composite solution is

$$
\begin{aligned}
C_x^0 u &= E_x^0 u + H_\xi^0 u + H_\zeta^0 u - E_x^0 H_\xi^0 u - E_x^0 H_\zeta^0 u. \\
&= u_0(x) + w_0\left(\frac{x}{\sqrt{\varepsilon}}\right) + v_0\left(\frac{1-x}{\varepsilon}\right) - 0 - 0 \\
&= e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\varepsilon}.
\end{aligned} \tag{5.13}
$$

In Figure 5.7, we see that the boundary layer thickness at the left is wide, $\mathcal{O}(\sqrt{\varepsilon})$, and at the right is narrow. $\mathcal{O}(\varepsilon)$.

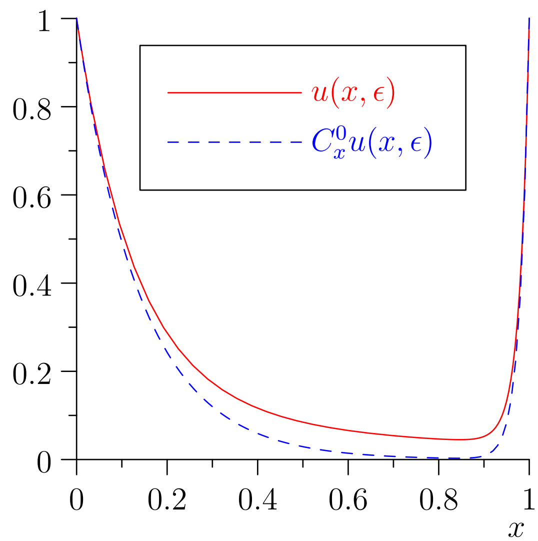


Figure 5.7: Matched asymptotic solution in Eq. (5.13) for $\varepsilon = 0.02$ compared to the exact solution of Eq. (5.12) computed numerically with a shooting method.

5.D Application to PDEs

Consider the following first-order linear partial differential equation in two independent variables:

$$
\begin{cases}
\varepsilon[a(x,t)u_t + b(x,t)u_x] + c(x,t)u = d(x,t) & x \in \mathbb{R},\ t \in [0,\infty), \\
u(x,0,\varepsilon) = f(x).
\end{cases}
$$

Depending on the coefficients a, b, c, and d, such a problem can lead to boundary layers on the boundary $t = 0$ (initial layer) or even internal boundary layers.

Typically, if $(x,t) \in \Omega$, where Ω is an n-dimensional set, boundary layers will be $(n-1)$ dimensional.

• (Burgers' equation) [Holmes 1995, Example 2, p. 91]

For the problem

$$P : \begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \varepsilon\dfrac{\partial^2 u}{\partial x^2} & x \in \mathbb{R},\ t \in [0, \infty), \\ u(x, 0, \varepsilon) = \varphi(x). \end{cases}$$

Here the initial condition $\varphi(x)$ satisfies $\varphi'(x) \leqslant 0$ for $x \neq 0$. Of particular interest is the case where φ has a jump discontinuity at $x = 0$: $\varphi(0^+) < \varphi(0^-)$. For example,

$$\varphi(x) = \begin{cases} c_L & x < 0, \\ c_R & x > 0, \end{cases}$$

where $c_L > c_R \geqslant 0$.

Let us look for an outer solution of the form

$$u(x, \varepsilon) \sim \sum_{n=0}^{\infty} u_n(x)\varepsilon^n.$$

The outer problem

$$P_0 : \begin{cases} \dfrac{\partial u_0}{\partial t} + u_0\dfrac{\partial u_0}{\partial x} = 0, \\ u_0(x, 0, \varepsilon) = \varphi(x), \end{cases}$$

$$P_1 : \begin{cases} \dfrac{\partial u_1}{\partial t} + u_0\dfrac{\partial u_1}{\partial x} + u_1\dfrac{\partial u_0}{\partial x} = \dfrac{\partial^2 u_0}{\partial x^2}, \\ u_1(x, 0, \varepsilon) = 0, \end{cases}$$

can be solved by the method of characteristics. Letting λ parametrize the initial curve $u_0(x, 0)$, s be the characteristic variable, and $z = u_0(x, t)$, we find

$$P_0 : \begin{cases} \dfrac{dt}{ds} = 1 & t|_{s=0} = 0, \\ \dfrac{dx}{ds} = u_0 = z & x|_{s=0} = \lambda, \\ \dfrac{dz}{ds} = 0 & z|_{s=0} = \varphi(\lambda). \end{cases}$$

Thus

$$P_0 : \begin{cases} t = s, \\ x = sz + \lambda = s\varphi(\lambda) + \lambda, \\ z = \varphi(\lambda). \end{cases}$$

We thus see that z satisfies the implicit equation $z = \varphi(\lambda) = \varphi(x - sz) = \varphi(x - tz)$, that is, $u_0 = \varphi(x - u_0 t)$.

Since $c_L > c_R$, this implies that the solution is multiple-valued in the region $c_R t < x < c_L t$ of an (x, t) diagram. Let us assume that there is a single smooth curve

$$x = \gamma(t), \quad \gamma(0) = 0$$

along which the outer solution has a jump discontinuity, often called a *shock*. That is, we look for a *shock layer*:

$$u_0(x,t) = \begin{cases} c_L & x < \gamma(t), \\ c_R & x > \gamma(t). \end{cases}$$

Accordingly, we choose an inner variable $\xi = \dfrac{x - \gamma(t)}{\delta(\varepsilon)}$ and set $w(\xi, t, \varepsilon) = u(x, t, \varepsilon)$, so that $u(x,t,\varepsilon) = w\left(\dfrac{x - \gamma(t)}{\delta}, t, \varepsilon\right)$. We see that

$$\frac{\partial u}{\partial t} = -\frac{\gamma'}{\delta}\frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial t}$$

and

$$\frac{\partial u}{\partial x} = \frac{1}{\delta}\frac{\partial w}{\partial \xi}.$$

The inner equation is then

$$-\frac{\gamma'}{\delta}\frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial t} + \frac{w}{\delta}\frac{\partial w}{\partial \xi} = \frac{\varepsilon}{\delta^2}\frac{\partial^2 w}{\partial \xi^2},$$

or

$$\frac{\partial^2 w}{\partial \xi^2} = \frac{\delta^2}{\varepsilon}\frac{\partial w}{\partial t} + \frac{\delta}{\varepsilon}(w - \gamma')\frac{\partial w}{\partial \xi}.$$

We choose the least degenerate case $\dfrac{\delta^2}{\varepsilon} \prec \dfrac{\delta}{\varepsilon} \sim 1$ and expand

$$w(\xi, t, \varepsilon) \sim \sum_{n=0}^{\infty} w_n(\xi, t)\varepsilon^n$$

to find

$$P_0^I : \frac{\partial^2 w_0}{\partial \xi^2} = (w_0 - \gamma')\frac{\partial w_0}{\partial \xi}.$$

$$P_1^I : \frac{\partial^2 w_1}{\partial \xi^2} = \frac{\partial w_0}{\partial t} + (w_0 - \gamma')\frac{\partial w_1}{\partial \xi} + w_1\frac{\partial w_0}{\partial \xi}.$$

We integrate once with respect to ξ to find

$$\frac{\partial w_0}{\partial \xi} = \frac{1}{2}w_0^2 - \gamma' w_0 + A(t),$$

so that

$$\frac{dw_0}{\frac{1}{2}w_0^2 - \gamma' w_0 + A(t)} = d\xi.$$

Let us now match to the outer solution:

$$(u^I)^O \sim w_0\left(\frac{x-\gamma(t)}{\varepsilon}, t\right) \sim \begin{cases} w_0(-\infty, t) & x < \gamma(t), \\ w_0(\infty, t) & x > \gamma(t) \end{cases} \qquad (\varepsilon \to 0),$$

$$(u^O)^I \sim u_0(\gamma(t) + \varepsilon\xi, t) \sim \begin{cases} c_L & x < \gamma(t), \\ c_R & x > \gamma(t) \end{cases} \qquad (\varepsilon \to 0).$$

Thus

$$\lim_{\xi \to -\infty} w_0 = c_L, \qquad \lim_{\xi \to \infty} w_0 = c_R.$$

In particular this means that $\lim_{\xi \to \pm\infty} \dfrac{\partial w_0}{\partial \xi} = 0$. Hence, as $\xi \to \pm\infty$ we find

$$0 = \frac{1}{2}c_R^2 - \gamma' c_R + A(t),$$

$$0 = \frac{1}{2}c_L^2 - \gamma' c_L + A(t).$$

On subtracting the second equation from the first, we find

$$0 = \frac{1}{2}c_R^2 - c_L^2 - \gamma'(c_R - c_L),$$

so that

$$\gamma' = \frac{1}{2}(c_L + c_R), \qquad A(t) = \frac{1}{2}c_L c_R.$$

Thus the shock takes the form of a straight line, $\gamma(t) = \frac{1}{2}(c_L + c_R)t$, that separates the trajectories $c_L t$ and $c_R t$. We can then determine the inner solution by partial fraction decomposition:

$$\begin{aligned} \frac{1}{2}\int d\xi &= \int \frac{dw_0}{(w_0 - c_L)(w_0 - c_R)} \\ &= \frac{1}{c_L - c_R}\int \left[\frac{1}{w_0 - c_L} - \frac{1}{w_0 - c_R}\right] dw_0 \\ &= \frac{1}{c_L - c_R}\left[\log \frac{w_0 - c_L}{w_0 - c_R} + \log B(t)\right]. \end{aligned}$$

Thus

$$w_0(\xi, t) = \frac{c_L - c_R E(\xi, t)}{1 - E(\xi, t)},$$

where $E(\xi, t) = B(t)e^{\frac{1}{2}\xi(c_L - c_R)}$. To determine the constant of integration, $B(t)$, one must perform the matching to higher order.

Chapter 6

WKB Theory

Recall the earlier examples (4.4):

$$P : \begin{cases} \varepsilon u'' - u' = 0, \\ u(0,\varepsilon) = 0, \quad u(1,\varepsilon) = 1 \end{cases} \qquad u(x,\varepsilon) = \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1},$$

where the solution varies rapidly over a narrow region (the boundary layer), and (4.5):

$$P : \begin{cases} \varepsilon u'' + u = 0, \\ u(0,\varepsilon) = 0, \quad u(1,\varepsilon) = 1 \end{cases} \qquad u(x,\varepsilon) = \frac{\sin(x/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})},$$

where the solution oscillates rapidly over the entire domain. Both solutions exhibit exponential behaviour, with a real exponent in the first case and an imaginary exponent in the second.

It is natural to seek an approximate solution of the form

$$u(x,\varepsilon) \sim A(x) e^{S(x)/\delta(\varepsilon)}, \qquad \delta(\varepsilon) = o(1).$$

Here $S(x)$ is called the *phase*. If S is real, there is a boundary layer of thickness $\delta(\varepsilon)$, while if S is imaginary there are rapid oscillations of wavelength $\sim \delta(\varepsilon)$.

The above expansion can be readily generalized to higher order in δ:

$$u(x,\varepsilon) \sim \exp\left(\frac{1}{\delta(\varepsilon)} \sum_{n=0}^{\infty} S_n(x) \delta^n(\varepsilon) \right). \tag{6.1}$$

This is known as either the *WKB* approximation, after Wentzel, Kramers, and Brillouin, who popularized the technique, or the *WKBJ* approximation, to acknowledge earlier work by Jeffreys.

Consider the second-order homogeneous linear differential equation

$$\begin{cases} \varepsilon^2 u'' = Q(x) u & x \in [0,1], \\ u(0,\varepsilon) = A, \qquad u(1,\varepsilon) = B, \end{cases}$$

where $Q(x) \neq 0$ on $[0,1]$. Substituting Eq. (6.1) leads to

$$\varepsilon^2 \left[\left(\frac{1}{\delta} \sum_{n=0}^{\infty} S_n'(x) \delta^n \right)^2 + \frac{1}{\delta} \sum_{n=0}^{\infty} S_n''(x) \delta^n \right] = Q(x).$$

To leading order as $\varepsilon \to 0$ we find

$$\frac{\varepsilon^2}{\delta^2} S_0'^{\,2} + \frac{2\varepsilon^2}{\delta} S_0' S_1' + \frac{\varepsilon^2}{\delta} S_0'' \sim Q(x)$$

Since $\dfrac{\varepsilon^2}{\delta} \prec \dfrac{\varepsilon^2}{\delta^2} \sim 1$, we choose the least degenerate case $\delta(\varepsilon) = \varepsilon$:

$$S_0'^{\,2} + \varepsilon(2S_0'S_1' + S_0'') = Q(x).$$

The dominant balance is called the *eikonal equation*

$$S_0'^{\,2} = Q(x),$$

which has the two solutions

$$S_0(x) = \pm F(x), \qquad \text{where } F(x) \doteq \int_0^x \sqrt{Q(t)}\, dt.$$

The higher-order balances are called *transport equations*:

$$2S_0'S_1' = -S_0'', \tag{6.2a}$$

$$\sum_{j=0}^{n} S_j' S_{n-j}' = -S_{n-1}'' \qquad (n \geqslant 1). \tag{6.2b}$$

The solution to Eq. (6.2a) is immediately seen to be

$$S_1(x) = -\frac{1}{4} \log Q(x) + k,$$

where k is arbitrary constant. The first-order WKB-approximation is then a linear combination of the two solutions:

$$\begin{aligned} u(x, \varepsilon) &\sim e^{S_0/\varepsilon + S_1} \\ &\sim \frac{c_1 e^{F(x)/\varepsilon} + c_2 e^{-F(x)/\varepsilon}}{\sqrt[4]{Q(x)}}. \end{aligned}$$

We now apply the boundary conditions to express these linear combinations as

$$u(x, \varepsilon) \sim \frac{A\sqrt{F'(0)} \sinh\left(\dfrac{F(1) - F(x)}{\varepsilon}\right) + B\sqrt{F'(1)} \sinh\left(\dfrac{F(x)}{\varepsilon}\right)}{\sqrt{F'(x)} \sinh\left(\dfrac{F(1)}{\varepsilon}\right)},$$

noting that $F(0) = 0$.

- In the special case where $A = 0$, $B = 1$, and $Q(x) = -1$, we see that $F(x) = ix$ and
$$u(x, \varepsilon) \sim \frac{\sin x/\varepsilon}{\sin 1/\varepsilon} \qquad (\varepsilon \to 0),$$
which is in fact the exact solution.

- In the special case where $A = 0$, $B = 1$, and $Q(x) = 1$, we see that $F(x) = x$ and one again recovers the exact solution:
$$u(x, \varepsilon) \sim \frac{\sinh x/\varepsilon}{\sinh 1/\varepsilon}, \qquad (\varepsilon \to 0).$$

- Let us look for a WKB solution for the problem
$$\begin{cases} \varepsilon u'' + p(x)u' + q(x)u = 0 & x \in [0, 1], \\ u(0, \varepsilon) = A, \qquad u(1, \varepsilon) = B, \end{cases}$$
where $p(x) \neq 0$ on $[0, 1]$. We find up to first order in ε that
$$\frac{\varepsilon}{\delta^2} S_0'^{\,2} + 2\frac{\varepsilon}{\delta} S_0' S_1' + \frac{\varepsilon}{\delta} S_0'' + \frac{1}{\delta} S_0' p + S_1' p + q = 0.$$
We choose the least degenerate case $\delta(\varepsilon) = \varepsilon$ so that $1 \sim \frac{\varepsilon}{\delta} \prec \frac{\varepsilon}{\delta^2}$. Thus
$$\frac{1}{\varepsilon}(S_0'^{\,2} + S_0' p) + 2S_0' S_1' + S_0'' + S_1' p + q = 0.$$
The eikonal equation is thus
$$S_0'^{\,2} + S_0' p = 0,$$
which has roots $S_0' = 0$ and $S_0' = -p(x)$, so that without loss of generality
$$S_0(x) = 0 \qquad \text{or} \qquad S_0(x) = -\int_0^x p(t)\, dt \doteq -G(x).$$
Thus
$$S_1(x) = -\int_0^x \frac{S_0''(t) + q(t)}{2S_0'(t) + p(t)}\, dt.$$
so that
$$S_0' = 0 \Rightarrow S_1(x) = -\int_0^x \frac{q(t)}{p(t)}\, dt \doteq -F(x),$$
and
$$S_0' = -p(x) \Rightarrow S_1(x) = -\log p(x) + F(x).$$

The first-order WKB-approximation is then a linear combination of the two solutions:

$$\begin{aligned} u(x,\varepsilon) &\sim e^{S_0/\varepsilon + S_1} \\ &\sim c_1(\varepsilon)e^{-F(x)} + \frac{c_2(\varepsilon)}{p(x)} e^{-G(x)/\varepsilon + F(x)} \qquad (\varepsilon \to 0). \end{aligned}$$

We now apply the boundary conditions to express these linear combinations as

$$\begin{aligned} u(x,\varepsilon) \sim& \frac{\lambda(\varepsilon)p(0)Ae^{F(1)} - Bp(1)}{\lambda(\varepsilon)p(0)e^{F(1)} - p(1)e^{-F(1)}} e^{-F(x)} \\ &+ \frac{p(0)p(1)(B - Ae^{-F(1)})}{p(x)[\lambda(\varepsilon)p(0)e^{F(1)} - p(1)e^{-F(1)}]} e^{-G(x)/\varepsilon + F(x)} \qquad (\varepsilon \to 0), \end{aligned}$$

where $\lambda(\varepsilon) \doteq e^{-G(1)/\varepsilon}$.

Remark: For the special case $A = 0$, $B = 1$, $p(x) = -1$, $q(x) = 0$ we have $G(x) = -x$, $F(x) = 0$, $\lambda(\varepsilon) = e^{1/\varepsilon}$, reproducing the exact solution in Eq. (4.4):

$$u(x,\varepsilon) = \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}.$$

Remark: The popularity of the WKB method stems from its ease of use and the fact that it works for problems with rapid variation in regions larger than thin boundary layers. However, a major drawback with the method is that it works only for linear ordinary differential equations.

6.A Turning Points

[Bender & Orszag 1999, p. 505]

Consider the WKB approximation for the problem

$$P : \begin{cases} \varepsilon^2 u'' = Q(x)u, \\ u(0,\varepsilon) = 1, \quad u(\infty,\varepsilon) = 0 \end{cases}$$

on the domain $\Omega = \mathbb{R}$, where $Q(x) \sim ax$ as $x \to 0$, $xQ(x) > 0$ for $x \neq 0$, and $\frac{1}{x^2} = o(Q(x))$ as $x \to \infty$. The point $x = 0$ is called a turning point since the sign of Q changes from negative ($\Rightarrow$ sinusoidal WKB solution) to positive ($\Rightarrow$ exponential WKB solution) as x passes from negative to positive values. The WKB approximation for $x > 0$ is

$$u^R(x,\varepsilon) \sim \frac{c_1(\varepsilon)e^{F(x)/\varepsilon} + c_2(\varepsilon)e^{-F(x)/\varepsilon}}{\sqrt[4]{Q(x)}} \qquad (\varepsilon \to 0),$$

where $F(x) \doteq \int_0^x \sqrt{Q(t)}\,dt$ and the boundary condition $u(\infty,\varepsilon) = 0$ implies that $c_1 = 0$. However, this approximation is clearly invalid near $x = 0$ since $Q \to 0$ there.

Since we cannot apply the boundary condition at $x = 0$, we expect a boundary layer near $x = 0$ which matches to WKB outer solutions u^L on the left and u^R on the right of the boundary layer.

For $x < 0$ we have

$$u^L(x,\varepsilon) \sim \frac{A(\varepsilon)\cos(G(x)/\varepsilon) + B(\varepsilon)\sin(G(x)/\varepsilon)}{\sqrt[4]{|Q(x)|}} \qquad (\varepsilon \to 0),$$

where $G(x) \doteq \int_0^x \sqrt{|Q(t)|}\,dt$.

Introduce the inner variable $\xi = x/\delta(\varepsilon)$ and $w(\xi,\varepsilon) = u(x,\varepsilon)$, so that the inner equation reads

$$\frac{\varepsilon^2}{\delta^2} w'' = Q(\delta\xi)w \sim a\delta\xi w \qquad (\delta \to 0).$$

Since the least degenerate case is $\dfrac{\varepsilon^2}{\delta^2} \sim a\delta$, we take $\delta(\varepsilon) = \dfrac{\varepsilon^{2/3}}{a^{1/3}}$. Thus $\xi = a^{1/3}x/\varepsilon^{2/3}$ and

$$w'' \sim \xi w \qquad (\varepsilon \to 0).$$

We recognize the latter as Airy's equation, with general solution

$$\begin{aligned} u^I(x,\varepsilon) = w(\xi,\varepsilon) &\sim \alpha(\varepsilon)\,\mathrm{Ai}(\xi) + \beta(\varepsilon)\,\mathrm{Bi}(\xi) \\ &= \alpha(\varepsilon)\,\mathrm{Ai}(a^{1/3}x/\varepsilon^{2/3}) + \beta(\varepsilon)\,\mathrm{Bi}(a^{1/3}x/\varepsilon^{2/3}) \qquad (\varepsilon \to 0), \end{aligned}$$

noting that the constants α and β that yield a match to the outer solution may depend on the parameter ε.

Problem 6.1: Show that

$$\mathrm{Ai}(\xi) \sim \frac{1}{2\sqrt{\pi}}\,\xi^{-1/4}e^{-\frac{2}{3}\xi^{3/2}} \qquad (\xi \to \infty),$$

$$\mathrm{Bi}(\xi) \sim \frac{1}{\sqrt{\pi}}\,\xi^{-1/4}e^{\frac{2}{3}\xi^{3/2}} \qquad (\xi \to \infty).$$

On matching u^I to u^R we find as $\varepsilon \to 0$ that

$$\begin{aligned} (u^I)^O &\sim \alpha(\varepsilon)\,\mathrm{Ai}(a^{1/3}x/\varepsilon^{2/3}) + \beta(\varepsilon)\,\mathrm{Bi}(a^{1/3}x/\varepsilon^{2/3}) \\ &\sim \frac{\varepsilon^{1/6}}{\sqrt{\pi}a^{1/12}}\,x^{-1/4}\left[\frac{\alpha(\varepsilon)}{2}e^{-\frac{2}{3}a^{1/2}x^{3/2}/\varepsilon} + \beta(\varepsilon)e^{\frac{2}{3}a^{1/2}x^{3/2}/\varepsilon}\right] \qquad \left(\varepsilon^{2/3} \prec x\right), \end{aligned}$$

$$(u^R)^I \sim c_2(\varepsilon)\frac{e^{-F(x)}/\varepsilon}{\sqrt[4]{Q(x)}} \sim c_2(\varepsilon)\frac{e^{-\frac{2}{3}a^{1/2}x^{3/2}/\varepsilon}}{a^{1/4}x^{1/4}} \qquad \left(\varepsilon^{2/3} \prec x \prec \varepsilon^{2/5}\right)$$

since $Q(x) \sim ax$ as $x \to 0$ and hence $F(x) \sim \int_0^x \sqrt{at}\,dt \sim \frac{2}{3}a^{1/2}x^{3/2} + \mathcal{O}(x^{5/2}/\varepsilon)$ as $x \to 0$. A match is thus possible if $\beta(\varepsilon) = 0$ and $\alpha(\varepsilon) = 2\sqrt{\pi}c_2(\varepsilon)/(a\varepsilon)^{1/6}$. That is,

$$w(\xi,\varepsilon) \sim \frac{2\sqrt{\pi}c_2(\varepsilon)}{(a\varepsilon)^{1/6}}\,\mathrm{Ai}(\xi) \qquad (\varepsilon \to 0).$$

On noting from Eq. (3.3a) that $\mathrm{Ai}(0) = \frac{1}{3^{2/3}\Gamma(2/3)}$ we see that the boundary condition $w(0,\varepsilon) = 1$ implies that

$$c_2(\varepsilon) = \frac{3^{2/3}\Gamma(2/3)(\alpha\varepsilon)^{1/6}}{2\sqrt{\pi}},$$

so that

$$w(\xi,\varepsilon) \sim 3^{2/3}\Gamma\left(\frac{2}{3}\right)\mathrm{Ai}(\xi) \qquad (\varepsilon \to 0).$$

Similarly, the match $(u^I)^O = (u^L)^I$ is possible if we choose

$$A(\varepsilon) = -B(\varepsilon) = \frac{3^{2/3}\Gamma\left(\frac{2}{3}\right)}{\sqrt{2\pi}}(a\varepsilon)^{1/6}.$$

The resulting matched WKB approximation to the one-turning point problem as $\varepsilon \to 0$ is then

$$u(x,\varepsilon) \sim \begin{cases} \dfrac{3^{2/3}\Gamma\left(\frac{2}{3}\right)(a\varepsilon)^{1/6}}{\sqrt{2\pi}}\left[\dfrac{\cos(G(x)/\varepsilon) - \sin(G(x)/\varepsilon)}{\sqrt[4]{|Q(x)|}}\right] & \varepsilon^{2/3} \prec x < 0, \\ 3^{2/3}\Gamma\left(\dfrac{2}{3}\right)\mathrm{Ai}\left(\dfrac{a^{1/3}x}{\varepsilon^{2/3}}\right) & x \prec \varepsilon^{2/5}, \\ \dfrac{3^{2/3}\Gamma\left(\frac{2}{3}\right)(a\varepsilon)^{1/6}}{2\sqrt{\pi}}\left[\dfrac{e^{-F(x)}/\varepsilon}{\sqrt[4]{Q(x)}}\right] & \varepsilon^{2/3} \prec x > 0. \end{cases}$$

We note as $\varepsilon \to 0$ that there is a nontrivial matching interval, $[\varepsilon^{2/3}, \varepsilon^{2/5}]$. The WKB approximation can be thought of as a special case of the multiple-scale analysis introduced in the next chapter.

Chapter 7

Multiple-Scale Analysis

7.A Secular Terms

Consider the problem for $u(t, \varepsilon)$:

$$\begin{cases} u'' + 2\varepsilon u' + u = 0 & t \in [0, \infty), \\ u(0, \varepsilon) = 1, \qquad u'(0, \varepsilon) = 0, \end{cases} \tag{7.1}$$

where the primes now denote differentiation with respect to t. The exact solution is

$$u(t, \varepsilon) = e^{-\varepsilon t}\left[\cos\left(\sqrt{1-\varepsilon^2}\, t\right) + \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \sin\left(\sqrt{1-\varepsilon^2}\, t\right)\right], \tag{7.2}$$

so that $|u(t, \varepsilon)| \leqslant 1 + \varepsilon/\sqrt{1-\varepsilon^2}$ for all $t \geqslant 0$.

If we try a straightforward perturbation expansion

$$u(t, \varepsilon) \sim \sum_{n=0}^{\infty} u_n(t) \varepsilon^n,$$

this yields

$$P_0 : \begin{cases} u_0'' + u_0 = 0, \\ u_0(0) = 1, \quad u_0'(0) = 0, \end{cases}$$

$$P_n : \begin{cases} u_n'' + u_n = -2u_{n-1}', \\ u_n(0) = u_n'(0) = 0 \end{cases} \qquad (n \geqslant 1).$$

The solutions for the first two contributions are

$$u_0(t) = \cos t, \quad u_1(t) = -t\cos t + \sin t,$$

so that

$$u(t, \varepsilon) \sim \cos t + \varepsilon(\sin t - t\cos t) + \mathcal{O}(\varepsilon).$$

However, the amplitude of the term $t\cos t$ grows without bound, despite the fact that the exact solution is bounded. This is known as a *secularity*. The perturbation expansion is invalid since it attempts to separate the true dependence of u on t and ε into a series containing products of functions of t and functions of ε; the exact solution evidently cannot be written in this form. Instead, we see that for small ε there are really two time scales, t and εt, as evident in Fig. 4.3. The *method of multiple scales* provides a means of dealing with such problems.

7.B Derivative Expansion Method

The *derivative expansion method* is probably the most common of the various multiple scale methods. One introduces several time (or length) scales and treats them as independent variables:

If t is the original variable and ε is the small parameter, we introduce the auxiliary time scales

$$\tau_1 = \varepsilon t,\ \tau_2 = \varepsilon^2 t,\ \ldots,\ \tau_N = \varepsilon^N t$$

and express $u(t,\varepsilon) = w(t,\tau_1,\ldots,\tau_N,\varepsilon)$. Then

$$u' = \frac{dw}{dt} = \left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau_1} + \varepsilon^2\frac{\partial}{\partial \tau_2} + \ldots + \varepsilon^N\frac{\partial}{\partial \tau_N}\right)w.$$

Remark: The original problem is thus transformed from an ordinary differential equation in u to a partial differential equation in w.

- Let us apply the derivative expansion method to the problem in Eq. (7.1):

$$\begin{cases} u'' + 2\varepsilon u' + u = 0 & t \in [0,\infty), \\ u(0,\varepsilon) = 1, \qquad u'(0,\varepsilon) = 0, & \end{cases}$$

We introduce new time scales $\tau_1 = \varepsilon t$, $\tau_2 = \varepsilon^2 t$ and let $u(t,\varepsilon) = w(t,\tau_1,\tau_2,\varepsilon)$. Then

$$\begin{aligned} u' &= \frac{\partial w}{\partial t} + \varepsilon\frac{\partial w}{\partial \tau_1} + \varepsilon^2\frac{\partial w}{\partial \tau_2}. \\ u'' &= \left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau_1} + \varepsilon^2\frac{\partial}{\partial \tau_2}\right)^2 w \\ &= \frac{\partial^2 w}{\partial t^2} + 2\varepsilon\frac{\partial^2 w}{\partial t\partial \tau_1} + \varepsilon^2\left(2\frac{\partial^2 w}{\partial t\partial \tau_2} + \frac{\partial^2 w}{\partial \tau_1^2}\right) + o(\varepsilon^2). \end{aligned}$$

The problem then becomes:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} + 2\varepsilon\left(\dfrac{\partial^2 w}{\partial t\partial \tau_1} + \dfrac{\partial w}{\partial t}\right) + \varepsilon^2\left(2\dfrac{\partial^2 w}{\partial t\partial \tau_2} + \dfrac{\partial^2 w}{\partial \tau_1^2} + 2\dfrac{\partial w}{\partial \tau_1}\right) + w + o(\varepsilon^2) = 0, \\ w(0,0,0,\varepsilon) = 1, \\ \dfrac{\partial w}{\partial t}(0,0,0,\varepsilon) + \varepsilon\dfrac{\partial w}{\partial \tau_1}(0,0,0,\varepsilon) + \varepsilon^2\dfrac{\partial w}{\partial \tau_2}(0,0,0,\varepsilon) = 0. \end{cases}$$

We look for a solution to this partial differential equation of the form

$$w(t, \tau_1, \tau_2, \varepsilon) = w_0(t, \tau_1, \tau_2) + \varepsilon w_1(t, \tau_1) + \varepsilon^2 w_2(t) + o(\varepsilon^2).$$

We find

$$\varepsilon^0 : \begin{cases} \dfrac{\partial^2 w_0}{\partial t^2} + w_0 = 0, \\ w_0(0,0,0) = 1, \\ \dfrac{\partial w_0}{\partial t}(0,0,0) = 0, \end{cases}$$

$$\varepsilon^1 : \begin{cases} \dfrac{\partial^2 w_1}{\partial t^2} + w_1 = -2\dfrac{\partial^2 w_0}{\partial t \partial \tau_1} - 2\dfrac{\partial w_0}{\partial t}, \\ w_1(0,0) = 0, \\ \dfrac{\partial w_1}{\partial t}(0,0) = -\dfrac{\partial w_0}{\partial \tau_1}(0,0,0), \end{cases}$$

$$\varepsilon^2 : \begin{cases} \dfrac{\partial^2 w_2}{\partial t^2} + w_2 = -2\dfrac{\partial^2 w_0}{\partial t \partial \tau_2} - \dfrac{\partial^2 w_0}{\partial \tau_1^2} - 2\dfrac{\partial w_0}{\partial \tau_1} - 2\dfrac{\partial^2 w_1}{\partial t \partial \tau_1} - 2\dfrac{\partial w_1}{\partial t}, \\ w_2(0) = 0, \\ \dfrac{\partial w_2}{\partial t}(0) = -\dfrac{\partial w_0}{\partial \tau_2}(0,0,0) - \dfrac{\partial w_1}{\partial \tau_1}(0,0). \end{cases}$$

The solution to the ε^0 problem is

$$w_0(t, \tau_1, \tau_2) = A(\tau_1, \tau_2)\cos t + B(\tau_1, \tau_2)\sin t,$$

where the initial conditions on w_0 imply that $A(0,0) = 1$ and $B(0,0) = 0$.

The ε^1 problem is then

$$\frac{\partial^2 w_1}{\partial t^2} + w_1 = 2\left(\frac{\partial A}{\partial \tau_1} + A\right)\sin t - 2\left(\frac{\partial B}{\partial \tau_1} + B\right)\cos t, \tag{7.3}$$

The only way to avoid secularities in solutions to this sinusoidal differential equation, which is being driven at its natural frequency, is to use our freedom in choosing the functions A and B to insist that

$$\frac{\partial A}{\partial \tau_1} + A = 0,$$
$$\frac{\partial B}{\partial \tau_1} + B = 0,$$

We thus find

$$
\begin{aligned}
A(\tau_1, \tau_2) &= \alpha(\tau_2)e^{-\tau_1},\\
B(\tau_1, \tau_2) &= \beta(\tau_2)e^{-\tau_1},
\end{aligned}
$$

where the initial conditions $A(0,0) = 1$ and $B(0,0) = 0$ imply that $\alpha(0) = 1$ and $\beta(0) = 0$. With this choice of A and B, the secular terms in Eq. (7.3) are thereby removed and we can now solve for both w_0 and w_1:

$$
\begin{aligned}
w_0(t, \tau_1, \tau_2) &= e^{-\tau_1}[\alpha(\tau_2)\cos t + \beta(\tau_2)\sin t],\\
w_1(t, \tau_1) &= C(\tau_1)\cos t + D(\tau_1)\sin t.
\end{aligned}
$$

The initial conditions on w_1 then imply that $C(0) = 0$ and $D(0) = \alpha(0) = 1$.

The ε^2 equation thus becomes

$$
\frac{\partial^2 w_2}{\partial t^2} + w_2 = \big[(2\alpha' + \beta)e^{-\tau_1} + 2(C' + C)\big]\sin t + \big[(-2\beta' + \alpha)e^{-\tau_1} - 2(D' + D)\big]\cos t.
$$

Again, we need to remove the secular terms:

$$
\begin{aligned}
(2\alpha' + \beta) + 2e^{\tau_1}(C' + C) &= 0,\\
(-2\beta' + \alpha) - 2e^{\tau_1}(D' + D) &= 0.
\end{aligned}
$$

We note that the terms involving α and β are functions of τ_2 only, while the terms involving C and D are functions of τ_1 only. Hence $2\alpha' + \beta = -2e^{\tau_1}(C' + C)$ and $(-2\beta' + \alpha) = 2e^{\tau_1}(D' + D)$ must be constants. For simplicity we choose these constants to be zero. The system

$$
\begin{aligned}
2\alpha' + \beta &= 0,\\
-2\beta' + \alpha &= 0,
\end{aligned}
$$

and the initial conditions $\alpha(0) = 1$, $\beta(0) = 0$ then imply that $\alpha(\tau_2) = \cos\frac{\tau_2}{2}$ and $\beta(\tau_2) = \sin\frac{\tau_2}{2}$.

The system

$$
\begin{aligned}
C' + C &= 0,\\
D' + D &= 0,
\end{aligned}
$$

and the initial conditions $C(0) = 0$, $D(0) = 1$ then imply that $C(\tau_1) = 0$ and $D(\tau_1) = e^{-\tau_1}$.

Thus

$$
\begin{aligned}
w_0(t, \tau_1, \tau_2) &= e^{-\tau_1}\left[\cos\left(\frac{\tau_2}{2}\right)\cos t + \sin\left(\frac{\tau_2}{2}\right)\sin t\right]\\
&= e^{-\tau_1}\cos\left(t - \frac{\tau_2}{2}\right),\\
w_1(t, \tau_1) &= e^{-\tau_1}\sin t.
\end{aligned}
$$

Finally, the solution to the ε^2 equation, given the initial conditions $w_2(0) = 0$ and

$$\frac{\partial w_2}{\partial t}(0) = -\frac{\partial w_0}{\partial \tau_2}(0,0,0) - \frac{\partial w_1}{\partial \tau_1}(0,0) = 0,$$

is simply $w_2(t) = 0$.

The resulting multiple scale solution,

$$u(t,\varepsilon) = e^{-\varepsilon t}\left[\cos\left(1 - \frac{\varepsilon^2}{2}\right)t + \varepsilon \sin t\right] + o(\varepsilon^2),$$

is compared with the exact solution in Fig. 7.1.

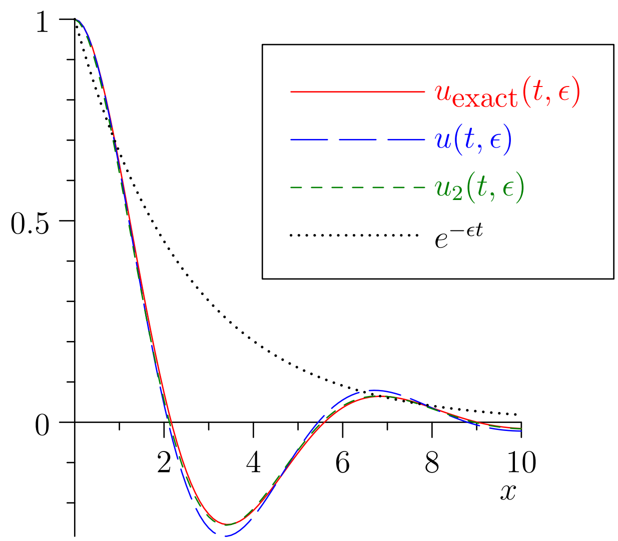


Figure 7.1: Multiple-scale solution *via* the derivative expansion (u) and two-variable expansion (u_2) versus exact solution u_{exact} of Eq. (7.1) for $\varepsilon = 0.4$.

7.C Two-variable expansion

Instead of introducing many slow variables $\tau_n = \varepsilon^n t$ for $n = 1, 2, \ldots, N$, it is often more convenient to consider only two time variables: the *slow variable* $\tau = \varepsilon t$ and the *modified fast variable*

$$T = (1 + \varepsilon^2\nu_2 + \varepsilon^3\nu_3 + \ldots + \varepsilon^N\nu_N)t$$

for some constants ν_j. We can then express

$$\frac{d}{dt} = \left(1 + \varepsilon^2\nu_2 + \varepsilon^3\nu_3 + \ldots + \varepsilon^N\nu_N\right)\frac{\partial}{\partial T} + \varepsilon\frac{\partial}{\partial t}.$$

- Let us revisit Eq. (7.1):

$$\begin{cases} u'' + 2\varepsilon u' + u = 0 & t \in [0, \infty), \\ u(0, \varepsilon) = 1, \qquad u'(0, \varepsilon) = 0 \end{cases}$$

and introduce the time scales $\tau = \varepsilon t$ and $T = (1 + \varepsilon^2 \nu)t$. On letting $u(t, \varepsilon) = w(T, \tau, \varepsilon)$ we find

$$\begin{aligned} u' &= (1 + \varepsilon^2 \nu)\frac{\partial w}{\partial T} + \varepsilon \frac{\partial w}{\partial \tau}. \\ u'' &= \left[(1 + \varepsilon^2 \nu)\frac{\partial}{\partial T} + \varepsilon \frac{\partial}{\partial \tau}\right]^2 w \\ &= \frac{\partial^2 w}{\partial T^2} + 2\varepsilon \frac{\partial^2 w}{\partial T \partial \tau} + \varepsilon^2 \left(2\nu \frac{\partial^2 w}{\partial T^2} + \frac{\partial^2 w}{\partial \tau^2}\right) + o(\varepsilon^2). \end{aligned}$$

The problem then becomes:

$$\begin{cases} \dfrac{\partial^2 w}{\partial T^2} + w + 2\varepsilon \left(\dfrac{\partial^2 w}{\partial T \partial \tau} + \dfrac{\partial w}{\partial T}\right) + \varepsilon^2 \left(2\nu \dfrac{\partial^2 w}{\partial T^2} + \dfrac{\partial^2 w}{\partial \tau^2} + 2\dfrac{\partial w}{\partial \tau}\right) + o(\varepsilon^2) = 0, \\ w(0, 0, \varepsilon) = 1, \\ \dfrac{\partial w}{\partial T}(0, 0, \varepsilon) + \varepsilon \dfrac{\partial w}{\partial \tau}(0, 0, \varepsilon) + \varepsilon^2 \nu \dfrac{\partial w}{\partial T}(0, 0, \varepsilon) = 0. \end{cases}$$

We look for a solution to this partial differential equation of the form

$$w(T, \tau, \varepsilon) = w_0(T, \tau) + \varepsilon w_1(T, \tau) + \varepsilon^2 w_2(T, \tau) + o(\varepsilon^2).$$

We find

$$\varepsilon^0 : \begin{cases} \dfrac{\partial^2 w_0}{\partial T^2} + w_0 = 0, \\ w_0(0, 0) = 1, \\ \dfrac{\partial w_0}{\partial T}(0, 0) = 0. \end{cases}$$

$$\varepsilon^1 : \begin{cases} \dfrac{\partial^2 w_1}{\partial T^2} + w_1 = -2\dfrac{\partial^2 w_0}{\partial T \partial \tau} - 2\dfrac{\partial w_0}{\partial T}, \\ w_1(0, 0) = 0, \\ \dfrac{\partial w_1}{\partial T}(0, 0) = -\dfrac{\partial w_0}{\partial \tau}(0, 0), \end{cases}$$

$$\varepsilon^2 : \begin{cases} \dfrac{\partial^2 w_2}{\partial T^2} + w_2 = -2\nu \dfrac{\partial^2 w_0}{\partial T^2} - \dfrac{\partial^2 w_0}{\partial \tau^2} - 2\dfrac{\partial w_0}{\partial \tau} - 2\dfrac{\partial^2 w_1}{\partial T \partial \tau} - 2\dfrac{\partial w_1}{\partial T}, \\ w_2(0,0) = 0, \\ \dfrac{\partial w_2}{\partial T}(0,0) = -\dfrac{\partial w_1}{\partial \tau}(0,0) - \nu \dfrac{\partial w_0}{\partial T}(0,0). \end{cases}$$

The solution to the ε^0 problem is

$$w_0(T,\tau) = A(\tau)\cos T + B(\tau)\sin T,$$

where the initial conditions on w_0 imply that $A(0) = 1$ and $B(0) = 0$.

The ε^1 equation is then

$$\frac{\partial^2 w_1}{\partial t^2} + w_1 = 2(A' + A)\sin T - 2(B' + B)\cos T. \tag{7.4}$$

We avoid secular terms by setting

$$\begin{aligned} A' + A &= 0, \\ B' + B &= 0, \end{aligned}$$

which, using the initial conditions $A(0) = 1$ and $B(0) = 0$, yields $A(\tau) = e^{-\tau}$ and $B(\tau) = 0$. We thus find

$$\begin{aligned} w_0(T,\tau) &= e^{-\tau}\cos T \\ w_1(T,\tau) &= C(\tau)\cos T + D(\tau)\sin T. \end{aligned}$$

The initial conditions on w_1 then imply that $C(0) = 0$ and $D(0) = 1$.

The ε^2 equation thus becomes

$$\frac{\partial^2 w_2}{\partial t^2} + w_2 = \left[-2(D' + D) + (1 + 2\nu)e^{-\tau}\right]\cos T + 2(C' + C)\sin T.$$

We remove secular terms by setting

$$\begin{aligned} C' + C &= 0, \\ D' + D &= \frac{1}{2}(1 + 2\nu)e^{-\tau}, \end{aligned}$$

with the initial conditions $C(0) = 0$ and $D(0) = 1$. We thus find $C(\tau) = 0$ and $D(\tau) = [1 + \frac{1}{2}(1 + 2\nu)\tau]e^{-\tau}$, so that

$$w(T,\tau) \sim w_0 + \varepsilon w_1 = e^{-\tau}\left\{\cos T + \varepsilon\left[1 + \frac{1}{2}(1 + 2\nu)\tau\right]\sin T\right\}.$$

We see, however, that this solution still contains a secular term. Fortunately we still have enough freedom to suppress this secularity: we need only choose $\nu = -1/2$, so that $D(\tau) = e^{-t}$ and

$$w(T,\tau) \sim w_0 + \varepsilon w_1 = e^{-\tau}(\cos T + \varepsilon \sin T).$$

Thus

$$u(t,\varepsilon) \sim w\left(\left(1 - \frac{1}{2}\varepsilon^2\right)t, \varepsilon t\right) = e^{-\varepsilon t}\left[\cos\left(1 - \frac{1}{2}\varepsilon^2\right)t + \varepsilon \sin\left(1 - \frac{1}{2}\varepsilon^2\right)t\right].$$

We observe in Fig. 7.1 for a relatively large value of ε that this solution reproduces the exact solution Eq. (7.2) more closely than the approximation obtained previously by the derivative expansion method.

• (Rayleigh Oscillator) [Bender & Orszag 1999, p. 554]

For $a > 0$ consider the nonlinear problem

$$\begin{cases} u'' + u = \varepsilon(u' - \frac{1}{3}u'^3), \\ u(0,\varepsilon) = 0, \qquad u'(0,\varepsilon) = 2a. \end{cases}$$

To remove secularities at leading-order, it is sufficient to introduce a single slow variable, in which case the two-variable expansion and the derivative expansion methods (with $N = 1$) are equivalent.

Letting $\tau = \varepsilon t$, we look for a solution of the form

$$u(t,\varepsilon) = w(t,\tau,\varepsilon) = w_0(t,\tau) + \varepsilon w_1(t,\tau) + o(\varepsilon) \qquad \varepsilon \to 0.$$

We find

$$\varepsilon^0 : \begin{cases} \dfrac{\partial^2 w_0}{\partial t^2} + w_0 = 0, \\ w_0(0,0) = 0, \\ \dfrac{\partial w_0}{\partial t}(0,0) = 2a, \end{cases}$$

$$\varepsilon^1 : \begin{cases} \dfrac{\partial^2 w_1}{\partial t^2} + w_1 = \dfrac{\partial w_0}{\partial t} - 2\dfrac{\partial^2 w_0}{\partial t \partial \tau} - \dfrac{1}{3}\left(\dfrac{\partial w_0}{\partial t}\right)^3, \\ w_1(0,0) = 0, \\ \dfrac{\partial w_1}{\partial t}(0,0) = 0. \end{cases}$$

Let us express the solution to the ε^0 problem as

$$w_0(t,\tau) = A(\tau)\sin(t+\theta(\tau)),$$

where the initial conditions imply that $A(0)\sin\theta(0) = 0$ and $A(0)\cos\theta(0) = 2a$. Without loss of generality we find $A(0) = 2a > 0$ and $\theta(0) = 0$.

The ε^1 problem then becomes

$$\frac{\partial^2 w_1}{\partial t^2} + w_1 = \left(A - 2A' - \frac{1}{4}A^3\right)\cos(t+\theta) + 2A\theta'\sin(t+\theta) - \frac{1}{12}A^3\cos 3(t+\theta),$$

where we have expressed the right-hand side directly in terms of Fourier harmonics using the relation

$$\cos^3 t = \left(\frac{e^{it}+e^{-it}}{2}\right)^3 = 2\,\mathrm{Re}\,\frac{e^{3it}+3e^{2it}e^{-it}}{8} = \frac{1}{4}(\cos 3t + 3\cos t).$$

To avoid secular terms we must set

$$\begin{aligned} A - 2A' - \frac{1}{4}A^3 &= 0,\\ 2A\theta' &= 0. \end{aligned}$$

That is,

$$\begin{aligned} 8A' &= 4A - A^3 = A(2-A)(2+A),\\ \theta' &= 0. \end{aligned}$$

Thus

$$\int d\tau = \int \frac{-8dA}{A(A-2)(A+2)} = \int \left(\frac{2}{A} - \frac{1}{A-2} - \frac{1}{A+2}\right) dA = \log\frac{A^2}{A^2-4} + \log\alpha.$$

Hence

$$\frac{A^2-4}{A^2} = \alpha e^{-\tau},$$

where the constant α is seen to equal $\alpha = (a^2-1)/a^2$ because of the initial condition $A(0) = 2a$. Thus

$$A(\tau) = \frac{2a}{\sqrt{a^2-(a^2-1)e^{-\tau}}} > 0.$$

The equation $\theta' = 0$, along with the initial condition $\theta(0) = 0$, implies that $\theta(\tau) = 0$ for all τ. Hence

$$u(t,\varepsilon) \sim \frac{2a\sin t}{\sqrt{a^2-(a^2-1)e^{-\varepsilon t}}} \qquad (\varepsilon \to 0).$$

Since $\lim_{\tau\to\infty} A(\tau) = 2$, we see that the solution approaches a limit cycle as $t \to \infty$, as illustrated in Fig. 7.2.

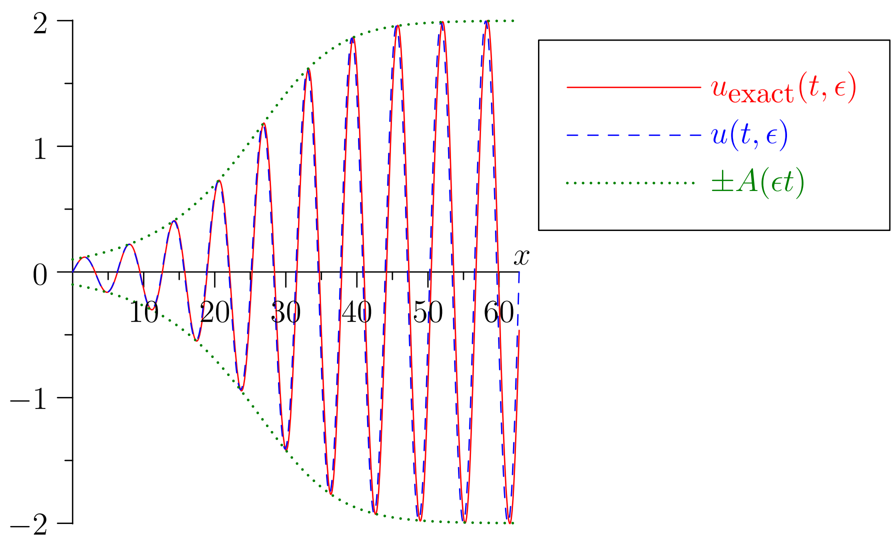


Figure 7.2: Multiple-scale solution (u) of the Rayleigh Oscillator versus exact solution u_{exact} for $a = 0.05$ and $\varepsilon = 0.2$.

Appendix A

Series Reversion

[Morse & Feshbach 1953, p. 411]

Given the power series of a holomorphic function,

$$w = f(z) = \sum_{n=1}^{\infty} a_n z^n,$$

with $f(0) = 0$ and $f'(0) = a_1 \neq 0$, we now derive a general formula for the power series of its (holomorphic) inverse function in a neighbourhood of 0:

$$z = f^{-1}(w) = \sum_{n=1}^{\infty} b_n w^n.$$

Let C be a contour enclosing w. Then, using the substitution $\zeta = f(z)$ we find

$$f^{-1}(w) = \frac{1}{2\pi i} \int_C \frac{f^{-1}(\zeta)}{\zeta - w}\, d\zeta = \frac{1}{2\pi i} \int_C \frac{z f'(z)}{f(z) - w}\, dz.$$

Hence

$$(f^{-1})^{(n)}(w) = \frac{n!}{2\pi i} \int_C \frac{z f'(z)}{(f(z) - w)^{n+1}}\, dz = \frac{(n-1)!}{2\pi i} \int_C \frac{1}{(f(z) - w)^n}\, dz,$$

on integrating by parts. Thus

$$\begin{aligned} b_n &= \frac{1}{n!}(f^{-1})^{(n)}(0) \\ &= \frac{1}{2n\pi i} \int_C \frac{1}{f^n(z)}\, dz = \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \frac{z^n}{f^n(z)}\bigg|_{z=0} \\ &= \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \left(\sum_{k=1}^{\infty} a_k z^{k-1} \right)^{-n} \Bigg|_{z=0}. \end{aligned}$$

For example,

$$b_1 = a_1^{-1}, \qquad b_2 = -a_1^{-3} a_2, \qquad b_3 = a_1^{-5}(2a_2^2 - a_1 a_3).$$

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