Chapter 1

Measure Theory

Indicator function:

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Power set:

$$\mathcal{P}(S) = \{s : s \subset S\}.$$

Uncountable summation:

$$\sum_{\alpha \in A} x_\alpha \doteq \sup_{\substack{F \subset A \\ F \text{ finite}}} \sum_{\alpha \in F} x_\alpha.$$

Elementary set: finite union of boxes.

Jordan inner measure:

$$m_{*J}(S) \doteq \sup_{\substack{E \subset S \\ E \text{ elementary}}} m(E).$$

Jordan outer measure:

$$m^{*J}(S) \doteq \inf_{\substack{E \supset S \\ E \text{ elementary}}} m(E).$$

Jordan measurable:

$$m_{*J}(S) = m^{*J}(S).$$

Lebesgue outer measure:

$$m^*(S) \doteq \inf_{\substack{\bigcup_{k=1}^\infty B_k \supset S \\ B_k \text{ boxes}}} \sum_{k=1}^\infty |B_k|.$$

Properties of Lebesgue outer measure:

(i) $m^*(\emptyset) = 0$; *nullity*

(ii) $S \subset T \subset \mathbb{R}^d \Rightarrow m^*(S) \le m^*(T)$; *monotonicity*

(iii) $m^*\left(\bigcup_{k=1}^{\infty} S_k\right) \le \sum_{k=1}^{\infty} m^*(S_k)$, where $S_k \subset \mathbb{R}^d$. *countable subadditivity*

S, T **separated:** $\mathrm{dist}(S,T) > 0$.

Compact disjoint sets in $\mathbb{R}^d$: separated.

Finite additivity for separated sets S and T:

$$m^*(S \cup T) = m^*(S) + m^*(T).$$

Closed dyadic cube in $\mathbb{R}^d$:

$$\left[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}\right] \times \ldots \times \left[\frac{i_d}{2^n}, \frac{i_d+1}{2^n}\right]$$

for integers $n, i_1, i_2, \ldots, i_d$.

Open subset of $\mathbb{R}^d$: countable union of open balls.

Open subset of $\mathbb{R}^d$: countable union of almost disjoint closed cubes.

Outer regularity:

$$m^*(S) = \inf_{\substack{U \supset S \\ U \text{open}}} m^*(U).$$

Lebesgue measurable set S: for every $\epsilon > 0$, there exists an open set $U \supset S$ such that $m^*(U \setminus S) < \epsilon$.

Null set: Lebesgue measure zero.

Boolean algebra: closed under complements and finite unions and intersections.

σ-algebra: closed under complements and countable unions and intersections.

Characterization of measurability. TFAE:

(i) S is Lebesgue measurable;

(ii) given $\epsilon > 0$, there exists an open set U_ϵ **containing** S with $m^*(U_\epsilon \setminus S) < \epsilon$; (outer open approximation)

(iii) given $\epsilon > 0$, there exists an open set U_ϵ with $m^*(U_\epsilon \triangle S) < \epsilon$; (almost open)

(iv) given $\epsilon > 0$, there exists a closed set F_ϵ **contained in** S with $m^*(S \setminus F_\epsilon) < \epsilon$; (inner closed approximation)

(v) given $\epsilon > 0$, there exists a closed set F_ϵ with $m^*(F_\epsilon \triangle S) < \epsilon$; (almost closed)

(vi) given $\epsilon > 0$, there exists a Lebesgue measurable set S_ϵ with $m^*(S_\epsilon \triangle S) < \epsilon$. (almost measurable)

Properties of Lebesgue measure:

(i) $m(\emptyset) = 0$; *nullity*;

(ii) If $S_1, S_2, \ldots \subset \mathbb{R}^d$ is a countable sequence of disjoint Lebesgue measurable sets, then $m(\bigcup_{k=1}^\infty S_k) = \sum_{k=1}^\infty m(S_k)$. *countable additivity*

Upward monotone convergence of increasing sequence of measurable sets:

$$m\left(\bigcup_{k=1}^\infty S_k\right) = \lim_{n\to\infty} m(S_n).$$

Downward monotone convergence of decreasing sequence of finite-measure sets:

$$m\left(\bigcap_{k=1}^\infty S_k\right) = \lim_{n\to\infty} m(S_n).$$

Dominated convergence theorem: Suppose $S_n \subset \mathbb{R}^d$, $n = 1, 2, \ldots$ are Lebesgue measurable sets that converge pointwise to a set S and are all contained in a single Lebesgue measurable set F of finite measure. Then $m(S_n)$ converges to $m(S)$.

Inner regularity:

$$m(S) = \sup_{\substack{K \subset S \\ K \text{ compact}}} m(K).$$

Characterization of finite measurability. Given $\epsilon > 0$, TFAE:

(i) S is Lebesgue measurable with finite measure;

(ii) there exists an open set U_ϵ of finite measure **containing** S with $m^*(U_\epsilon \setminus S) < \epsilon$; (outer open approximation)

(iii) there exists a bounded open set U_ϵ with $m^*(U_\epsilon \triangle S) < \epsilon$; (almost open bounded)

(iv) there exists a compact set K_ϵ **contained in** S with $m^*(S \setminus K_\epsilon) < \epsilon$; (inner compact approximation);

(v) there exists a compact set K_ϵ with $m^*(K_\epsilon \triangle S) < \epsilon$; (almost compact)

(vi) there exists a bounded Lebesgue measurable set S_ϵ with $m^*(S_\epsilon \triangle S) < \epsilon$; (almost bounded measurable)

(vii) there exists a Lebesgue measurable set S_ϵ with finite measure such that $m^*(S_\epsilon \triangle S) < \epsilon$; (almost finite measure)

(viii) there exists an elementary set E_ϵ such that $m^*(E_\epsilon \triangle S) < \epsilon$; (almost elementary)

(ix) there exists a finite union F_ϵ of closed dyadic cubes such that $m^*(F_\epsilon \triangle S) < \epsilon$. (almost dyadic)

Borel σ-algebra on $\mathcal{B}[\mathbb{R}^d]$ generators:

- open;
- closed;
- compact;
- open balls;
- boxes;
- elementary

Caratheodory Criterion: $S \subset \mathbb{R}^d$ is Lebesgue measurable $\iff$ for every $A \subset \mathbb{R}^d$,

$$m^*(A \cap S) + m^*(A \cap S^c) = m^*(A).$$

Useful techniques:

- $$\frac{\epsilon}{2^k} \text{ trick;}$$

- Bound measures from above and below;

- If $x < y + \epsilon$ for all $\epsilon > 0$ and x and y are *independent* of ϵ, then $x \leq y$;

- Use countable unions or intersections to build monotonic sequences of sets;

- Consider finite and infinite measure cases separately;

- Project to countable sequence of balls, annuli, or boxes;

- Supremum over a larger set can never be smaller;

- $$S \subset U \Rightarrow (U \setminus S) \cup S = U;$$

- Draw a Venn diagram!

Chapter 2

The Lebesgue Integral

Almost everywhere: holds outside of a null set.

Support of a function f on $\mathbb{R}^d$: $\{x \in \mathbb{R}^d : f(x) \neq 0\}$.

Simple function: finite linear combination of indicator functions over (w.l.o.g. distinct) measurable sets.

Simple integral:

$$\operatorname{Simp}\int_{\mathbb{R}^d} \sum_{k=1}^{n} c_k 1_{S_k} \doteq \sum_{k=1}^{n} c_k m(S_k).$$

Properties of simple unsigned functions f and g:

(i)
$$\operatorname{Simp}\int_{\mathbb{R}^d} (f+g) = \operatorname{Simp}\int_{\mathbb{R}^d} f + \operatorname{Simp}\int_{\mathbb{R}^d} g$$
and
$$\operatorname{Simp}\int_{\mathbb{R}^d} cf = c\operatorname{Simp}\int_{\mathbb{R}^d} f$$
for every $c \in [0, \infty]$; (unsigned linearity)

(ii) $\operatorname{Simp}\int_{\mathbb{R}^d} f < \infty$ iff f is finite almost everywhere and its support has finite measure; (finiteness)

(iii) $\operatorname{Simp}\int_{\mathbb{R}^d} f = 0$ iff $f = 0$ almost everywhere; (vanishing)

(iv) if f and g agree almost everywhere, $\operatorname{Simp}\int_{\mathbb{R}^d} f = \operatorname{Simp}\int_{\mathbb{R}^d} g$; (equivalence)

(v) if $f(x) \leq g(x)$ for almost every $x \in \mathbb{R}^d$,
$\operatorname{Simp}\int_{\mathbb{R}^d} f \leq \operatorname{Simp}\int_{\mathbb{R}^d} g$; (monotonicity)

(vi) for any Lebesgue measurable set S, $\operatorname{Simp}\int_{\mathbb{R}^d} 1_S = m(S)$. (compatibility)

Absolutely integrable: $|f|_{L^1(\mathbb{R}^d)} \doteq \int_{\mathbb{R}^d} |f| < \infty$.

$f : \mathbb{R}^d \to [0, \infty]$**:**

$$\text{Simp} \int_{\mathbb{R}^d} f \doteq \text{Simp} \int_{\mathbb{R}^d} \max(f, 0) - \text{Simp} \int_{\mathbb{R}^d} \max(-f, 0)$$

$f : \mathbb{R}^d \to \mathbb{C}$**:**

$$\text{Simp} \int_{\mathbb{R}^d} f \doteq \text{Simp} \int_{\mathbb{R}^d} \operatorname{Re} f + i \, \text{Simp} \int_{\mathbb{R}^d} \operatorname{Im} f$$

Measurable function: pointwise limit of a sequence of simple functions.

Relatively open set U in $X \subset \mathbb{R}^d$: $\exists$ open V in $\mathbb{R}^d \ni U = V \cap X$.

Characterization of measurable unsigned functions. TFAE:

(i) f is unsigned Lebesgue measurable;

(ii) f is the pointwise limit of a sequence of unsigned simple functions;

(iii) f is the pointwise almost everywhere limit of unsigned simple functions;

(iv) $f = \sup_n f_n$ for an increasing sequence f_n of bounded unsigned simple functions that have finite-measure support;

(v) for every $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d : f(x) > \lambda\}$ is Lebesgue measurable;

(vi) for every $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d : f(x) \geq \lambda\}$ is Lebesgue measurable;

(vii) for every $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d : f(x) < \lambda\}$ is Lebesgue measurable;

(viii) for every $\lambda \in [0, \infty]$, the set $\{x \in \mathbb{R}^d : f(x) \leq \lambda\}$ is Lebesgue measurable;

(ix) for every interval $I \subset [0, \infty)$, the set $f^{-1}(I) \doteq \{x \in \mathbb{R}^d : f(x) \in I\}$ is Lebesgue measurable;

(x) for every relatively open set $U \subset [0, \infty)$, the set $f^{-1}(U) \doteq \{x \in \mathbb{R}^d : f(x) \in U\}$ is Lebesgue measurable;

(xi) for every relatively closed set $F \subset [0, \infty)$, the set $f^{-1}(F) \doteq \{x \in \mathbb{R}^d : f(x) \in F\}$ is Lebesgue measurable.

Unsigned measurable functions:

- every continuous unsigned function $f : \mathbb{R}^d \to [0, \infty]$;

- every unsigned simple function;

- the supremum, infimum, limit superior, and limit inferior of a sequence of unsigned measurable functions;

- an unsigned function that is almost everywhere equal to an unsigned measurable function;

- the composition $\phi \circ f$ of a continuous function $\phi : [0, \infty] \to [0, \infty]$ and an unsigned measurable function f;

- the sum and product of unsigned measurable functions.

Bounded measurable function with finite support: *uniform* limit of a bounded sequence of simple functions.

Simple functions: measurable and take on finitely many values.

Lower unsigned Lebesgue integral:

$$\underline{\int_{\mathbb{R}^d}} f \doteq \sup_{\substack{h \text{ simple} \\ 0 \le h \le f}} \operatorname{Simp} \int_{\mathbb{R}^d} h.$$

Upper unsigned Lebesgue integral:

$$\overline{\int_{\mathbb{R}^d}} f \doteq \inf_{\substack{h \text{ simple} \\ h \ge f}} \operatorname{Simp} \int_{\mathbb{R}^d} h.$$

Properties of the lower and upper Lebesgue integrals:
Let $f, g : \mathbb{R}^d \to [0, \infty]$ be unsigned (not necessarily measurable) functions. Then

(i) if f is simple, $\underline{\int_{\mathbb{R}^d}} f = \overline{\int_{\mathbb{R}^d}} f = \operatorname{Simp} \int_{\mathbb{R}^d} f$; *compatibility*

(ii) if $f \le g$ pointwise almost everywhere, $\underline{\int_{\mathbb{R}^d}} f \le \underline{\int_{\mathbb{R}^d}} g$ and $\overline{\int_{\mathbb{R}^d}} f \le \overline{\int_{\mathbb{R}^d}} g$; *monotonicity*

(iii) $\underline{\int_{\mathbb{R}^d}} cf = c \underline{\int_{\mathbb{R}^d}} f$ for every $c \in [0, \infty)$; *scaling*

(iv) if f and g agree almost everywhere, $\underline{\int_{\mathbb{R}^d}} f = \underline{\int_{\mathbb{R}^d}} g$ and $\overline{\int_{\mathbb{R}^d}} f = \overline{\int_{\mathbb{R}^d}} g$; *equivalence*

(v) $\underline{\int_{\mathbb{R}^d}} (f + g) \ge \underline{\int_{\mathbb{R}^d}} f + \underline{\int_{\mathbb{R}^d}} g$; *lower superadditivity*

(vi) $\overline{\int_{\mathbb{R}^d}} (f + g) \le \overline{\int_{\mathbb{R}^d}} f + \overline{\int_{\mathbb{R}^d}} g$; *upper subadditivity*

(vii) for any measurable set $S \subset \mathbb{R}^d$,

$$\underline{\int_{\mathbb{R}^d}} f = \underline{\int_{\mathbb{R}^d}} f 1_S + \underline{\int_{\mathbb{R}^d}} f 1_{S^c};$$

complementarity

(viii)

$$\lim_{n\to\infty} \underline{\int_{\mathbb{R}^d}} \min(f(x), n)\, dx = \underline{\int_{\mathbb{R}^d}} f;$$

vertical truncation

(ix)

$$\lim_{n\to\infty} \underline{\int_{\mathbb{R}^d}} f(x)\, 1_{B_n[0]}\, dx = \underline{\int_{\mathbb{R}^d}} f;$$

(use the monotone convergence theorem) *horizontal truncation*

(x) if $f+g$ is a bounded simple function with finite measure support,

$$\text{Simp} \int_{\mathbb{R}^d} (f+g) = \underline{\int_{\mathbb{R}^d}} f + \overline{\int_{\mathbb{R}^d}} g.$$

reflection

Unsigned Lebesgue integral:

$$\int_{\mathbb{R}^d} f \doteq \underline{\int_{\mathbb{R}^d}} f.$$

Markov's inequality: For $f : \mathbb{R}^d \to [0, \infty]$ measurable and every $\lambda \in (0, \infty)$,

$$m(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} f.$$

- If $\int_{\mathbb{R}^d} f < \infty$, then f is finite almost everywhere.

- $\int_{\mathbb{R}^d} f = 0$ iff f is zero almost everywhere.

Triangle inequality: $|f+g|_{L^1(\mathbb{R}^d)} \leq |f|_{L^1(\mathbb{R}^d)} + |g|_{L^1(\mathbb{R}^d)}$

Integral triangle inequality: For $f \in L^1(\mathbb{R}^d)$,

$$\left| \int_{\mathbb{R}^d} f \right| \leq \int_{\mathbb{R}^d} |f|.$$

Approximation of L^1 functions: Let $f \in L^1(\mathbb{R}^d)$ and $\epsilon > 0$. There exists

(i) an absolutely integrable simple function g such that $|f-g|_{L^1(\mathbb{R}^d)} < \epsilon$.

(ii) a step function g such that $|f-g|_{L^1(\mathbb{R}^d)} < \epsilon$.

(iii) a continuous, compactly supported function $g \in L^1(\mathbb{R}^d)$ such that

$$|f-g|_{L^1(\mathbb{R}^d)} < \epsilon.$$

Locally uniform convergence: uniform convergence on bounded subsets.

Egorov's theorem: Let $f_n : \mathbb{R}^d \to \mathbb{C}$ be a sequence of measurable functions that converge pointwise almost everywhere to $f : \mathbb{R}^d \to \mathbb{C}$. Given $\epsilon > 0$, there exists a Lebesgue measurable set S of measure at most ϵ such that f_n converges locally uniformly to f outside of S.

Finite-measurable set: nearly a finite union of boxes.

Absolutely integrable function: nearly continuous.

Pointwise convergent functions: nearly locally uniformly convergent.

Chapter 3

Abstract Measure Spaces

- Let X be a set. A *σ-algebra* on X is a collection $\mathcal{B}$ of X such that
  (i) $\emptyset \in \mathcal{B}$; (empty set)
  (ii) If $S \in \mathcal{B}$, then the complement $S^c \doteq X \setminus S$ is also an element of $\mathcal{B}$; (closure under complement)
  (iii) If $S_1, S_2, \ldots \in \mathcal{B}$, then $\bigcup_{n=1}^{\infty} S_n \in \mathcal{B}$. (closure under countable union)

- Let $\mathcal{B}$ be a σ-algebra on a set X. A *measure* μ on $\mathcal{B}$ is a map $\mu : \mathcal{B} \to [0, \infty]$ such that
  (i) $\mu(\emptyset) = 0$; *nullity*
  (ii) if S_1, S_2, $\ldots$ are disjoint elements of $\mathcal{B}$, then $\mu(\bigcup_{k=1}^{\infty} S_k) = \sum_{k=1}^{\infty} \mu(S_k)$. *countable additivity*

- Let $(X, \mathcal{B}, \mu)$ be a measure space.
  (i) If $S_1, S_2, \ldots$ are $\mathcal{B}$-measurable, then

$$\mu\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{\infty} \mu(S_k).$$

  countable subadditivity

  (ii) If $S_1 \subset S_2 \subset \ldots$ is an increasing sequence of $\mathcal{B}$-measurable sets, then

$$\mu\left(\bigcup_{k=1}^{\infty} S_k\right) = \lim_{n\to\infty} \mu(S_n) = \sup_n \mu(S_n).$$

  upward monotone convergence

(iii) If $S_1 \supset S_2 \supset \ldots$ is a decreasing sequence of $\mathcal{B}$-measurable sets and at least one of the $\mu(S_k)$ is finite, then

$$\mu\left(\bigcap_{k=1}^{\infty} S_k\right) = \lim_{n\to\infty} \mu(S_n) = \inf_n \mu(S_n).$$

downward monotone convergence

- Let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose S_n, $n = 1, 2, \ldots$ are $\mathcal{B}$-measurable sets that converge to a set S. Then

(i) S is $\mathcal{B}$-measurable.

(ii) If the S_n are all contained in another $\mathcal{B}$-measurable set of finite measure, then $m(S_n)$ converges to $m(S)$.

Definition: Let $(X, \mathcal{B})$ be a measurable space and let $f : X \to [0, \infty]$ (or $f : X \to \mathbb{C}$) be an unsigned or complex-valued function. We say that f is *measurable* if $f^{-1}(U)$ is $\mathcal{B}$-measurable for every open subset U of $[0, \infty]$ (or $\mathbb{C}$).

Remark: (*Characterization of measurable functions*)
Let $(X, \mathcal{B})$ be a measurable space. Show that

(i) a function $f : X \to [0, \infty]$ is measurable iff the level sets $\{x \in X : f(x) > \lambda\}$ are measurable for every $\lambda \in [0, \infty)$;

(ii) an indicator function 1_S of a set $S \subset X$ is measurable iff S is measurable;

(iii) a function $f : X \to [0, \infty]$ (or $f : X \to \mathbb{C}$) is measurable iff $f^{-1}(S)$ is measurable for every Borel-measurable subset S of $[0, \infty]$ (or $\mathbb{C}$);

(iv) a function $f : X \to \mathbb{C}$ is measurable iff its real and imaginary parts are measurable;

(v) a function $f : X \to \mathbb{R}$ is measurable iff its positive and negative parts are measurable;

(vi) the pointwise limit f of a sequence of measurable functions $f_n : X \to [0, \infty]$ (or $\mathbb{C}$) is also measurable;

(vii) if $f : X \to [0, \infty]$ (or $\mathbb{C}$) is measurable and $\phi : [0, \infty] \to [0, \infty]$ (or $\mathbb{C} \to \mathbb{C}$) is continuous, then $\phi \circ f$ is measurable;

(viii) the sum or product of two measurable functions in $[0, \infty]$ (or $\mathbb{C}$) is measurable.

Theorem 3.7 (Monotone convergence theorem): *Let* $(X, \mathcal{B}, \mu)$ *be a measure space and* $f_1 \leq f_2 \leq \ldots$ *be an increasing sequence of unsigned measurable functions on* X. *Then*

$$\int_X \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Corollary 3.7.3 (Fatou's lemma): Let $(X, \mathcal{B}, \mu)$ be a measure space and $f_1, f_2, \ldots$ be a sequence of unsigned measurable functions on X. Then

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Theorem 3.8 (Dominated convergence theorem): *Let* $(X, \mathcal{B}, \mu)$ *be a measure space and* $f_1, f_2, \ldots$ *be a sequence of complex-valued measurable functions on* X *that converge pointwise* μ*-almost everywhere on* X. *Suppose that there exists an unsigned absolutely integrable function* $G : X \to [0, \infty]$ *such that for each* $n \in \mathbb{N}$, $|f_n| \leq G$ μ*-almost everywhere. Then*

$$\int_X \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Chapter 4

Modes of Convergence

- pointwise almost everywhere
- uniformly almost everywhere (in L^∞ norm)
- almost uniformly
- in L^1 norm
- in measure

Chapter 5

Differentiation Theorems

Theorem 5.1 (Lebesgue differentiation theorem on $\mathbb{R}$): *Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable function. Then*

$$\lim_{h \to 0^+} \frac{1}{h} \int_{[x,x+h]} f(t)\,dt = f(x)$$

and

$$\lim_{h \to 0^+} \frac{1}{h} \int_{[x-h,x]} f(t)\,dt = f(x)$$

for almost every $x \in \mathbb{R}$.

Theorem 5.3 (Monotone differentiation theorem): *Every monotone function $f : \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere.*

Definition: The *total variation* of a function $F : \mathbb{R} \to \mathbb{R}$ on an (finite or infinite) interval I is

$$|F|_{\mathrm{TV(I)}} \doteq \sup_{\substack{x_0 < \ldots < x_n \\ x_0, \ldots, x_n \in I}} \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|.$$

If $|F|_{\mathrm{TV(I)}}$ is finite, we say that F has *bounded variation on I*. If F has bounded variation on $\mathbb{R}$, we say that F has *bounded variation*.

Theorem 5.4: *A function $F : \mathbb{R} \to \mathbb{R}$ has bounded variation iff it is the difference of two bounded monotone functions.*

Theorem 5.5 (1D Lipschitz differentiation theorem): *Every Lipschitz continuous function is locally of bounded variation, and hence differentiable almost everywhere. Furthermore, its derivative, when it exists, is bounded by its Lipschitz constant.*

Theorem 5.6 (Upper bound for fundamental theorem): *Let $F : [a,b] \to \mathbb{R}$ be increasing, so that the unsigned function $F' : [a,b] \to [0,\infty]$ exists almost everywhere and is measurable. Then*

$$\int_{[a,b]} F' \le F(b) - F(a).$$

Definition: A function $F : \mathbb{R} \to \mathbb{R}$ is said to be *absolutely continuous* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\sum_{k=1}^{n} |F(b_k) - F(a_k)| < \epsilon$ for every finite collection of disjoint intervals $(a_1, b_1) \ldots (a_n, b_n)$ of total length $\sum_{k=1}^{n}(b_k - a_k) < \delta$.

Theorem 5.7 (Fundamental theorem for absolutely continuous functions): *Let $F : [a, b] \to \mathbb{R}$ be absolutely continuous. Then*

$$\int_{[a,b]} F' = F(b) - F(a).$$

Chapter 6

Outer Measures, Premeasures, and Product Measures

Definition: Given a set X, an *outer measure* is a map $\mu^* : \mathcal{P}(X) \mapsto [0, \infty]$ such that

(i) $\mu^*(\emptyset) = 0$; *nullity*

(ii) $S \subset T \subset X \Rightarrow \mu^*(S) \leq \mu^*(T)$; *monotonicity*

(iii) $\mu^*\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{\infty} \mu^*(S_k)$, where $S_k \subset X$.

countable subadditivity

Definition: Let μ^* be an outer measure on a set X. A set $S \subset X$ is said to be *Carathéodory measurable* if the *Carathéodory criterion*

$$\mu^*(A) = \mu^*(A \cap S) + \mu^*(A \cap S^c)$$

holds for every set $A \subset X$.

Markov's inequality: For every $\lambda \in (0, \infty)$,

$$\mu(\{x \in X : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_X f \, d\mu.$$

Theorem 6.1 (Carathéodory lemma): *Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on a set X, let $\mathcal{B}$ be the collection of all subsets of X that are Carathéodory measurable with respect to μ^* and let $\mu : \mathcal{B} \to [0, \infty]$ be the restriction of μ^* to $\mathcal{B}$. Then $\mathcal{B}$ is a σ-algebra and μ is a measure.*

Definition: A *premeasure* on a Boolean algebra $\mathcal{B}_0$ is a finitely additive measure $\mu_0 : \mathcal{B}_0 \to [0, \infty]$ such that $\mu_0(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu_0(E_k)$ whenever $E_1,\ E_2,\ \ldots$ are disjoint subsets of $\mathcal{B}_0$ such that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{B}_0$.

Theorem 6.2 (Hahn–Kolmogorov): *Every premeasure $\mu_0 : \mathcal{B}_0 \to [0, \infty]$ on a Boolean algebra $\mathcal{B}_0$ in X can be extended to a countably additive measure $\mu : \mathcal{B} \to [0, \infty]$.*

Corollary 6.5.3 (Fubini's theorem): Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ be complete σ-finite measure spaces and let $f : X \times Y \to \mathbb{C}$ be absolutely integrable with respect to $\overline{\mathcal{B}_X \times \mathcal{B}_Y}$. Then

$$\int_{X \times Y} f(x, y)\, d\overline{\mu_X \times \mu_Y}(x, y) = \int_X \int_Y f(x, y)\, d\mu_Y(y)\, d\mu_X(x) = \int_Y \int_X f(x, y)\, d\mu_X(x)\, d\mu_Y(y).$$