Math 373: Mathematical Programming and Optimization I

Fall, 2023 Assignment 3

November 8, due November 25

1. Let $P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_i^\intercal \boldsymbol{x} \geq b_i, i = 1, \ldots, m\}$ be a convex polyhedron.

   Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are distinct basic feasible solutions that satisfy $\boldsymbol{a}_i^\intercal \boldsymbol{u} = \boldsymbol{a}_i^\intercal \boldsymbol{v} = b_i$ for $i = 1, \ldots, n-1$, so that $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent, where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$ are linearly independent.

   Let $L_1 = \{t\boldsymbol{u} + (1-t)\boldsymbol{v} : 0 \leq t \leq 1\}$ be the segment that joins $\boldsymbol{u}$ and $\boldsymbol{v}$ and $L_2 = \{\boldsymbol{w} \in P : \boldsymbol{a}_i^\intercal \boldsymbol{w} = b_i,\ i = 1, \ldots, n-1\}$. Prove that $L_1 = L_2$.

   We first show that $L_1 \subset L_2$. For any vector $t\boldsymbol{u} + (1-t)\boldsymbol{v} \in L_1$, we have

   $$a_i^\intercal \Big(t\boldsymbol{u} + (1-t)\boldsymbol{v}\Big) = ta_i^\intercal \boldsymbol{u} + (1-t)a_i^\intercal \boldsymbol{v} = tb_i + (1-t)b_i = b_i.$$

   Hence $t\boldsymbol{u} + (1-t)\boldsymbol{v} \in L_2$. That is, $L_1 \subset L_2$.

   Next, we show $L_2 \subset L_1$. The vectors $\boldsymbol{w} \in L_2$ satisfy a system of linear equations $\boldsymbol{a}_i^\intercal \boldsymbol{w} = b_i$ for $i = 1, \ldots, n-1$. Since the constraint vectors $\boldsymbol{a}_i$ are linearly independent, they span a space of dimension $n-1$; the rank-nullity theorem then implies that L_2 is a subset of the one-dimensional space $\{\boldsymbol{w} \in \mathbb{R}^n : \boldsymbol{a}_i^\intercal \boldsymbol{w} = b_i,\ i = 1, \ldots, n-1\}$, which is a line. Since $\boldsymbol{u}, \boldsymbol{v} \in L_2$, we know that this is the line through $\boldsymbol{u}$ and $\boldsymbol{v}$. Let $\boldsymbol{w} \in L_2 \subset P$. Then $\boldsymbol{w} = t\boldsymbol{u} + (1-t)\boldsymbol{v}$, for some $t \in \mathbb{R}$. If $t > 1$, then $\boldsymbol{u}$ lies between $\boldsymbol{w} \in P$ and $\boldsymbol{v} \in P$ and would therefore not be an extreme point (basic feasible solution) of P. If $t < 0$, then $\boldsymbol{v}$ would lie between $\boldsymbol{u} \in P$ and $\boldsymbol{w} \in P$, and would not be an extreme point of P. These contradictions establish that $t \in [0, 1]$, so that $L_2 \subset L_1$. It follows that $L_1 = L_2$. $\boxed{5}$

2. If a linear programming problem in standard form has a non-degenerate basic feasible solution that is optimal, prove that the dual problem has a unique optimal solution. Hint: consider complementary slackness. $\boxed{4}$

   Let $\boldsymbol{x}^*$ be a non-degenerate basic optimal solution to the primal problem. Since the primal problem has an optimal solution, the dual has an optimal solution $\boldsymbol{p}$. Let $j_1, \ldots, j_m$ be a set of basic indices corresponding to $\boldsymbol{x}^*$ and consider the complementary slackness condition

   $$(c_j - \boldsymbol{p}^\intercal \boldsymbol{A}_j)x_j^* = 0, \qquad j = j_1, \ldots, j_m.$$

   Since $\boldsymbol{x}^*$ is nondegenerate, we know that each basic variable x_j^* is positive, so

   $$c_j = \boldsymbol{p}^\intercal \boldsymbol{A}_j, \qquad j = j_1, \ldots, j_m.$$

   This is just the system of equations

   $$\boldsymbol{c}_B^\intercal = \boldsymbol{p}^\intercal \boldsymbol{B},$$

   which has a unique solution $\boldsymbol{p}^\intercal = \boldsymbol{c}_B^\intercal \boldsymbol{B}^{-1}$ since the basis $\boldsymbol{B} = \{\boldsymbol{A}_j : j = j_1, \ldots, j_m\}$ is invertible.