

Math 373: Mathematical Programming and Optimization I

Fall, 2024 Assignment 3

October 29, due November 18

1. Let \mathbf{A} be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$. Prove that exactly one of these alternatives holds:

- (i) $\mathbf{Ax} = \mathbf{b}$ has a solution \mathbf{x} ;
- (ii) $\mathbf{A}^\top \mathbf{p} = \mathbf{0}$ has a solution \mathbf{p} with $\mathbf{p}^\top \mathbf{b} \neq 0$.

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Suppose $\mathbf{Ax} = \mathbf{b}$ has a solution \mathbf{x} . If $\mathbf{A}^\top \mathbf{p} = \mathbf{0}$, then

$$\mathbf{p}^\top \mathbf{b} = \mathbf{p}^\top \mathbf{Ax} = (\mathbf{A}^\top \mathbf{p})^\top \mathbf{x} = \mathbf{0}^\top \mathbf{x} = 0.$$

Thus (i) \Rightarrow (ii).

Alternatively, if $\mathbf{Ax} = \mathbf{b}$ does not have a solution \mathbf{x} , then $\mathbf{b} \notin \text{span}\{\mathbf{A}_1, \dots, \mathbf{A}_n\}$. On decomposing $\mathbf{b} = \mathbf{s} + \mathbf{p}$, where $\mathbf{s} \in \text{span}\{\mathbf{A}_1, \dots, \mathbf{A}_n\}$ and $\mathbf{p}^\top \mathbf{A} = \mathbf{0}^\top$, it follows that $\mathbf{p} \neq \mathbf{0}$. Then $\mathbf{p}^\top \mathbf{b} = \mathbf{p}^\top \mathbf{s} + \mathbf{p}^\top \mathbf{p} = 0 + |\mathbf{p}|^2 > 0$. That is, (i) \Rightarrow (ii).

2. Let $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i^\top \mathbf{x} \geq b_i, i = 1, \dots, m\}$ be a convex polyhedron.

Suppose that \mathbf{u} and \mathbf{v} are distinct basic feasible solutions that satisfy $\mathbf{a}_i^\top \mathbf{u} = \mathbf{a}_i^\top \mathbf{v} = b_i$ for $i = 1, \dots, n-1$, so that \mathbf{u} and \mathbf{v} are adjacent, where $\mathbf{a}_1, \dots, \mathbf{a}_{n-1}$ are linearly independent.

Let $L_1 = \{t\mathbf{u} + (1-t)\mathbf{v} : 0 \leq t \leq 1\}$ be the segment that joins \mathbf{u} and \mathbf{v} and $L_2 = \{\mathbf{w} \in P : \mathbf{a}_i^\top \mathbf{w} = b_i, i = 1, \dots, n-1\}$. Prove that $L_1 = L_2$.

We first show that $L_1 \subset L_2$. For any vector $t\mathbf{u} + (1-t)\mathbf{v} \in L_1$, we have

$$\mathbf{a}_i^\top (t\mathbf{u} + (1-t)\mathbf{v}) = t\mathbf{a}_i^\top \mathbf{u} + (1-t)\mathbf{a}_i^\top \mathbf{v} = tb_i + (1-t)b_i = b_i.$$

Hence $t\mathbf{u} + (1-t)\mathbf{v} \in L_2$. That is, $L_1 \subset L_2$.

Next, we show $L_2 \subset L_1$. The vectors $\mathbf{w} \in L_2$ satisfy a system of linear equations $\mathbf{a}_i^\top \mathbf{w} = b_i$ for $i = 1, \dots, n-1$. Since the constraint vectors \mathbf{a}_i are linearly independent, they span a space of dimension $n-1$; the rank-nullity theorem then implies that L_2 is a subset of the one-dimensional space $\{\mathbf{w} \in \mathbb{R}^n : \mathbf{a}_i^\top \mathbf{w} = b_i, i = 1, \dots, n-1\}$, which is a line. Since $\mathbf{u}, \mathbf{v} \in L_2$, we know that this is the line through \mathbf{u} and \mathbf{v} . Let $\mathbf{w} \in L_2 \subset P$. Then $\mathbf{w} = t\mathbf{u} + (1-t)\mathbf{v}$, for some $t \in \mathbb{R}$. If $t > 1$, then \mathbf{u} lies between $\mathbf{w} \in P$ and $\mathbf{v} \in P$ and would therefore not be an extreme point (basic feasible solution) of P . If $t < 0$, then \mathbf{v} would lie between $\mathbf{u} \in P$ and $\mathbf{w} \in P$, and would not be an extreme point of P . These contradictions establish that $t \in [0, 1]$, so that $L_2 \subset L_1$. It follows that $L_1 = L_2$.

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