Math 373: Mathematical Programming and Optimization I

Fall, 2024 Assignment 1

September 24, due October 7

1. Suppose that (x_1, x_2, x_3) is a feasible solution to the linear programming problem

$$\begin{aligned} \text{minimize} \quad & 4x_1 + 2x_2 + x_3 \\ \text{subject to} \quad & x_1 - x_2 \geq 3, \\ & 2x_1 + x_2 + x_3 \geq 4, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Let y_1 and y_2 be non-negative numbers.

(a) Show that

$$x_1(y_1 + 2y_2) + x_2(-y_1 + y_2) + x_3 y_2 \geq 3y_1 + 4y_2.$$

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On summing the first constraint multiplied by y_1 with the second constraint multiplied by y_2, we obtain

$$y_1(x_1 - x_2) + y_2(2x_1 + x_2 + x_3) \geq 3y_1 + 4y_2.$$

The desired result then follows on rearranging to isolate x_1, x_2, and x_3.

(b) Find constraints on y_1 and y_2 so that

$$4x_1 + 2x_2 + x_3 \geq x_1(y_1 + 2y_2) + x_2(-y_1 + y_2) + x_3 y_2$$

at every feasible solution (x_1, x_2, x_3). 1

We require that $y_1 + 2y_2 \leq 4$, $-y_1 + y_2 \leq 2$, and $y_2 \leq 1$.

(c) Use parts (a) and (b) to find a lower bound to the optimal cost in terms of only the variables y_1 and y_2. 1

By transitivity, we find

$$4x_1 + 2x_2 + x_3 \geq x_1(y_1 + 2y_2) + x_2(-y_1 + y_2) + x_3 y_2 \geq 3y_1 + 4y_2.$$

(d) Formulate the linear programming problem in the variables (y_1, y_2) that determines the **largest possible value** for the lower bound to the optimal cost found in part (c). 1

$$\begin{aligned} \text{maximize} \quad & 3y_1 + 4y_2 \\ \text{subject to} \quad & y_1 + 2y_2 \leq 4, \\ & -y_1 + y_2 \leq 2, \\ & y_2 \leq 1, \\ & y_1, y_2 \geq 0. \end{aligned}$$

This is known as the *dual problem* to the original linear programming problem.

2. Let $g : \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex increasing (but not necessarily differentiable) function on $\mathbb{R}$. Show that $f \circ g$ is convex on I.

   Let $\boldsymbol{p}, \boldsymbol{q} \in I$. Since g is convex on $\mathbb{R}$ we know that

$$g(t\boldsymbol{p} + (1-t)\boldsymbol{q}) \leq tg(\boldsymbol{p}) + (1-t)g(\boldsymbol{q}).$$

   Since f is increasing and convex on $\mathbb{R}$,

$$f(g(t\boldsymbol{p} + (1-t)\boldsymbol{q})) \leq f(tg(\boldsymbol{p}) + (1-t)g(\boldsymbol{q})) \leq tf(g(\boldsymbol{p})) + (1-t)f(g(\boldsymbol{q}))$$

   for all $t \in [0,1]$. Hence $f \circ g$ is convex on $\mathbb{R}$. 3

3. Let P be a convex polyhedron. Complete the proof of the following theorem.

   Theorem 1: *A point $\boldsymbol{x} \in P$ is an extreme point of P if and only if the set $P \setminus \{\boldsymbol{x}\}$ (the set obtained by removing $\boldsymbol{x}$ from P) is convex.*

   Proof: Let $\boldsymbol{x} \in P$. Suppose $P \setminus \{\boldsymbol{x}\}$ is convex. Then it contains every convex combination of points $\boldsymbol{y}, \boldsymbol{z} \in P \setminus \{\boldsymbol{x}\}$. Since $\boldsymbol{x}$ does not belong to $P \setminus \{\boldsymbol{x}\}$ it cannot be expressed as a convex combination of points $\boldsymbol{y}$ and $\boldsymbol{z}$ in $P \setminus \{\boldsymbol{x}\}$. That is, $\boldsymbol{x}$ is an extreme point of P. 4

   Suppose $\boldsymbol{x} \in P$ is an extreme point of P. If $P \setminus \{\boldsymbol{x}\}$ were not convex, there would exist points $\boldsymbol{y}, \boldsymbol{z} \in P \setminus \{\boldsymbol{x}\} \subset P$ and $t \in (0,1)$ such that $t\boldsymbol{y} + (1-t)\boldsymbol{z} \notin P \setminus \{\boldsymbol{x}\}$. But since P is convex, we know that $t\boldsymbol{y} + (1-t)\boldsymbol{z} \in P$. Thus $t\boldsymbol{y} + (1-t)\boldsymbol{z} = \boldsymbol{x}$, contradicting the definition of an extreme point. Thus $P \setminus \{\boldsymbol{x}\}$ must be convex. 5