Game Theory

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players: 1 & 2

pure strategies

Strategies: {Paper, Scissors, Rock}

Payoff matrix of player 1 (row-player) A $(=\pi_1)$ $= \begin{array}{c|ccc} & R & P & S \\ \hline R & 0 & -1 & 1 \\ P & 1 & 0 & -1 \\ S & -1 & 1 & 0 \end{array}$

each row corresponds to a strategy of the firs player

Payoff matrix of player 2 (column player) B $(=\pi_2)$ $= \begin{array}{c|ccc} & R & P & S \\ \hline R & 0 & 1 & -1 \\ P & -1 & 0 & 1 \\ S & 1 & -1 & 0 \end{array}$ $= \pi_1^T$

strategy (vector) of player 1: $x \in \left\{ \overset{R}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{P}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{S}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$

" " " " 2: $y \in$

pure strategies

Payoff of player 1: $(u_1 =)$ $x^T A y$ (utility)

Payoff of player 2: $(u_2 =)$ $x^T A^T y$ (utility)

Mixed - strategies

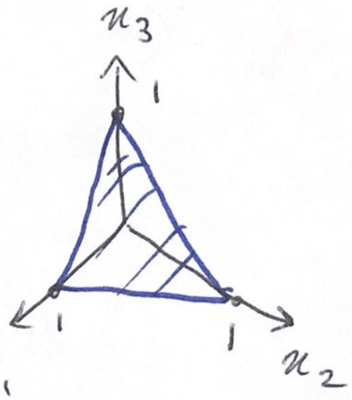
	1	2	3	4	5	6	7	8	9	10
P_1:	R	R	S	P	R	R	R	P	P	S
P_2:	S	P	R	S	P	S	R	S	P	R

Mixed-strategy of player 1: $x_1 = \begin{array}{c} R \\ P \\ S \end{array}\begin{bmatrix} 5/10 \\ 3/10 \\ 2/10 \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} 3/10 \\ 3/10 \\ 4/10 \end{bmatrix}$

general mixed strategy: x

$$x \in \Delta_3 = \left\{ x \in \mathbb{R}^3 \;\middle|\; x_1 + x_2 + x_3 = 1 \;,\; x_1, x_2, x_3 \geq 0 \right\}$$

$$\Delta_n = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_i x_i = 1 \;,\; x_i \geq 0 \;\; \forall i \right\}$$



Average payoff of player 1 : $\bar{u}_1 = x_1^T A \, x_2$

Expected : $E[u_1] =$ "

Example (Prisoner's Dilemma)

$$A = \begin{array}{ccc} & C & D \\ \text{cooperation } C & -1 & -3 \\ \text{detection } D & 0 & -2 \end{array} \qquad B = \begin{array}{ccc} & C & D \\ C & -1 & 0 \\ D & -3 & -2 \end{array}$$

The 'best strategy' for player 1 (& also player 2) is to defect,

i.e, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So the 'best' pair of strategies is $\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

So the payoff of player 1 (& also 2) is :

$$x^T A y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -2$$

Example (Battle of the sexes)

$$A = \begin{array}{ccc} & \text{theatre} & \text{football} \\ \text{theatre} & 1 & 0 \\ \text{football} & 0 & 3 \end{array} \qquad B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

modification $A = B = \begin{bmatrix} a>0 & 0 \\ 0 & b>0 \end{bmatrix}$ (Coordination game)

Example (Snowdrift game)

$$A = \begin{array}{ccc} & C & D \\ C & r - c/2 & r - c \\ D & r & 0 \end{array} \qquad \begin{array}{l} r>0,\ c>0 \\ r-c>0 \end{array}$$

AMA

(anti-coordination game)

Classification of 2×2 games

payoff matrix $A = \begin{bmatrix} R & S \\ T & P \end{bmatrix}$ $R, S, T, P \in \mathbb{R}$

1) Prisoner's Dilemma (PD) $T > R > P > S$

2) Coordination Game (CD) $R > T \underset{=}{>} P > S$

3) Anti Coordination Game ($\underset{\text{Snow}}{\underset{?}{S}}$D) $T > R \underset{=}{>} S > P$

Best response

Given a strategy y (of the second player), we denote the best-respons to y by $BR(y) : \Delta_n \to \Delta_n$, and define it to be that strategy of the first player that results in the highest payoff (of player 1) against y.

Example (PD game)

$$A = \begin{array}{c|cc} & C & D \\ \hline C & -1 & -3 \\ D & 0 & -2 \end{array}$$

What is the best-response to Cooperation?

$x^* = BR(y) = ?$ $y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$u_1 = x^{*T} A y = \begin{bmatrix} x_1^* & x_2^* \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1^* & x_2^* \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -x_1^*$$

$$x^* = \underset{x \in \Delta_2}{\arg\max}\ u_1 = \underset{x \in \Delta_2}{\arg\max}(-x_1) = \underset{\substack{0 \leq x_1 \leq 1 \\ 0 \leq x_2 \leq 1 \\ x_1 + x_2 = 1}}{\arg\max}(-x_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\Rightarrow x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\max_{x \in \Delta_2} u_1 = \max_{0 \le x_1 \le 1} (-x_1) = 0$$

$$x_1^* = 0 \Rightarrow x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \overset{x_1^* + x_2^* = 1}{\Rightarrow} x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Player 1 : R → ~~π~~ Payoff = -1 P

Player ~~1~~ 2 : P

$$[1 \quad 0 \quad 0] \begin{array}{c|ccc} & R & P & S \\ \hline R & 0 & -1 & 1 \\ P & 1 & 0 & -1 \\ S & -1 & 1 & 0 \end{array} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -1$$

R

A

$$\left[\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$[1 \quad 0 \quad 0] \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1$$

$$u_1 = x^T A y \Rightarrow u_1^T = y^T A^T x \neq u_2$$

$$u_2 = y^T A x \Rightarrow \underbrace{u_2^T = u_2}_{} = \cancel{y}\, x^T A^T y \quad ✓$$

$$u_1 = \underbrace{\sum_{i=1}^{10}}_{\text{number of moves}} u_{1,i} = \underbrace{\sum_{i=1}^{10} x_1^{i^T}}_{\text{str. pl. 1 at R.1}} A x_2^i = \dots = x_1^T A x_2$$

Exercise ↑

$$\max_{x\in\Delta_2} u_1 = \max_{0\le x_1\le 1}(-x_1) = 0$$

$$x_1^* = 0 \Rightarrow x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \overset{x_1^*+x_2^*=1}{\Rightarrow} x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

what is the best-response to defection $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

$$\max u_1 \quad \text{s.t } x\in\Delta_2 \qquad u_1 = x^T A y = [x_1 \ \ x_2]\begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -3x_1 - 2x_2 \quad (**)$$

So the LPP is max ~~Max~~ $(-3x_1 - 2x_2)$

$$\text{s.t.} \quad x_1 + x_2 = 1, \quad x_1, x_2 \geq 0$$

standard form:

$$\min \ 3x_1 + 2x_2$$
$$\text{s.t.} \ x_1 + x_2 = 1$$
$$x_1, x_2 \geq 0$$

optimal solution (Best Response $BR(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$)

$$x^* = \arg\max_{\substack{x_1+x_2=1 \\ x_1,x_2\geq 0}} (-3x_1 - 2x_2) \quad (*)$$

(*) & (**)

$$x^* = \arg\max_{\substack{x_1\leq 1 \\ x_1\geq 0}} (-2 - x_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

maximized when $x_1=0$, which results in $x_2=1$

$x_1+x_2=1$

what is the best-response to any strategy $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$?

$$u_1 = x^T A y = [x_1 \;\; x_2] \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [x_1 \;\; x_2] \begin{bmatrix} -y_1 - 3y_2 \\ -2y_2 \end{bmatrix}$$

$$= [x_1 \;\; x_2] \begin{bmatrix} -3 + 2y_1 \\ \wedge \\ -2 + 2y_1 \end{bmatrix} = x_1 (-3 + 2y_1) + x_2 (-2 + 2y_1)$$

✓ we know $y_1 + y_2 = 1$

x_1 & x_2 are like a weighted average. So the max happens when $x_1 = 0$ & $x_2 = 1$. Hence, $x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$(x_1 + x_2 = 1)$

So in any case, the optimal (best-response) is to defect $\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

Nash Equilibrium

We call a pair of strategies (x^*, y^*) a Nash Equilibrium (NE) if any othe strategy of the first (resp. second) player does not earn her a higher payoff.

$$u_1(x, y^*) \leq u_1(x^*, y^*) \;\; \forall x \in \Delta(1)$$

$$\& \;\; u_2(x^*, y) \leq u_2(x^*, y^*) \;\; \forall y \in \Delta(2)$$

Eq.

$$x^T A y^* \leq x^{*T} A y^*$$

$$\& \;\; x^{*T} B y \leq x^{*T} B y^*$$

Basically

$$x^* \in BR(y^*)$$

$$\& \;\; y^* \in BR(x^*)$$

Example (PD)

$$BR\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Leftrightarrow$$

Step 1

The pair $\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ is a NE. ✓

$\Downarrow$

There is no other NE.

Example (CD)

$$A = \begin{array}{c|cc} & C & D \\ \hline C & 2 & 0 \\ D & 0 & 1 \end{array} \qquad B = \begin{array}{c|cc} & C & D \\ \hline C & 1 & 0 \\ D & 0 & 2 \end{array}$$

Intuitively, $\left(\underset{C}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{C}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}\right)$ & $\left(\underset{D}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \underset{D}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}\right)$

What about other(?) NE?

To find a pair at NE, we need both (1) & (2) to be satisfied.

(1): $u_1(x, y^*) \leq u_1(x^*, y^*) \quad \forall x \in \Delta$

↳ Eq. $x^* \in BR(y^*)$

$$x^* = \arg\max_{x \in \Delta} x^T A y^* = \arg\max_{x \in \Delta_2} [x_1 \;\; x_2] \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1^* \\ y_2^* \end{bmatrix}$$

$$= \arg\max_{x \in \Delta_2} [x_1 \;\; x_2] \begin{bmatrix} 2y_1^* \\ y_2^* \end{bmatrix} = \arg\max_{x \in \Delta_2} [x_1 \;\; x_2] \begin{bmatrix} 2y_1^* \\ 1 - y_1^* \end{bmatrix}$$

$$= \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & 2y_1^* > 1 - y_1^* \\ \in \Delta_2 & 2y_1^* = 1 - y_1^* \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & 2y_1^* < 1 - y_1^* \end{cases} = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & y_1^* > 1/3 \\ \in \Delta_2 & y_1^* = 1/3 \;\; (\Rightarrow y_2^* = 2/3) \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & y_1^* < 1/3 \end{cases} \quad (*)$$

(2) $y^* \in BR(x^*)$

$$y^* = \arg\max_{y \in \Delta_2} x^{*T} B \overset{\text{not a star :(}}{y} = \arg\max_{y \in \Delta_2} [x_1^* \quad x_2^*] \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \arg\max_{y \in \Delta_2} [x_1^* \quad 2x_2^*] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [x_1^* \quad 2-2x_1^*] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\overset{y_1^*}{=} \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & x_1^* > 2-2x_1^* \iff x_1^* > 2/3 \\ \in \Delta_2 & x_1^* = 2-2x_1^* \iff x_1^* = 2/3 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & x_1^* < 2-2x_1^* \iff x_1^* < 2/3 \end{cases} \qquad (**)$$

Comparing (*) & (**):

(*) $x^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a BR. Then $y_1^* > 1/3$ which corresponds to either Δ_2 or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in (**). ~~None are~~ acceptable. 1st is not

$$y^* = \begin{cases} \Delta_2: \rightarrow x_1^* = 2/3 \quad \times \quad (x_1^* = 1) \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}: \rightarrow x_1^* > 2/3 \quad \checkmark \quad (x_1^* = 1 > 2/3) \end{cases}$$

So $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$ is a NE.

If $x^* = \begin{bmatrix}0\\1\end{bmatrix}$ is a BR. Then $y^* < 1/3$

$$\Rightarrow y^* = \begin{cases} \Delta_2 \to x_1^* = 2/3 \quad \times \\ \begin{bmatrix}0\\1\end{bmatrix} \to \checkmark\ x_1^* < 2/3 \quad (x_1^* = 0 < 2/3). \end{cases}$$

So $\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$ is a NE.

If $x^* \in \Delta_2$, then it is still a BR. Then $y_1^* = 1/3$

From (**), the only option for $y_1^* = 1/3$ is $y^* \in \Delta_2$, which implies $x_1^* = 2/3$.

So $\left(\begin{bmatrix}2/3\\1/3\end{bmatrix}, \begin{bmatrix}1/3\\2/3\end{bmatrix}\right)$ is a NE.

These 3 are the only NE.

Note:

$$\overset{A=}{\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}}\begin{pmatrix}1/3\\2/3\end{pmatrix} = \begin{pmatrix}2/3\\2/3\end{pmatrix}$$

$$\begin{pmatrix}2/3 & 1/3\end{pmatrix}\overset{B=}{\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}} = \begin{pmatrix}2/3 & 2/3\end{pmatrix}$$

the set of all NE are

$$\left\{ \left(\underbrace{\underset{\text{pure}}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{\text{pure}}{\begin{bmatrix}1\\0\end{bmatrix}}}\right), \left(\underbrace{\underset{\text{pure}}{\begin{bmatrix}0\\1\end{bmatrix}}, \underset{\text{pure}}{\begin{bmatrix}0\\1\end{bmatrix}}}\right), \left(\underset{\text{mixed}}{\begin{bmatrix}2/3\\1/3\end{bmatrix}}, \underset{\text{mixed}}{\begin{bmatrix}1/3\\2/3\end{bmatrix}}\right) \right\}$$

a NE in pure strategies

$\max(-x^2) = 0$ $x^* = 0$

Eq. $x^* = \underbrace{\arg}_{\text{optimal solution}} \max(-x^2) = 0$

$\max_{x \in \mathbb{R}} -(1-x)^2 * = 0$ $x^* = \arg\max_{x \in \mathbb{R}} -(1-x)^2 = 1$

$\max_{x \in \mathbb{R}} (1-x^2) = 1$ $x^* = \arg\max_{x \in \mathbb{R}} (1-x^2) = 0$

$\max_{-1 \leq x \leq 1} x^2 = 1$ $x^* = \arg\max_{-1 \leq x \leq 1} x^2 = \{-1, 1\}$

$\max_{-1 < x < 1} x^2 = ?$ $x^* = \arg\max_{-1 < x < 1} x^2 = ?$

$$u_1 = \sum_{i=1}^{10} u_{1,i} = \sum_{i=1}^{10} x_1^{i\,T} A x_2^i = \dots = x_1^T A x_2$$

number of models

str. pl. 1 of R.1

Exercise: