Math 225 (A1) Solution to Homework Assignment 6

1. Using the eigenvalues and their corresponding eigenvectors of $A$, the matrix $A$ can be diagonlized as follows: $P^{-1}AP = D$, where $D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $P = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}$. Using the change of variables $\underline{x} = P\underline{y}$, the differential equation $\underline{x}' = A\underline{x}$ becomes $\underline{y}' = D\underline{y}$. This system is

$$y_1' = -y_1$$

$$y_2' = y_2$$

which is decoupled. Solving each equation independently, we get $y_1 = c_1e^{-t}$ and $y_2 = c_2e^t$, where $c_1, c_2$ are arbitrary constants. Thus, $\underline{y} = \begin{pmatrix} c_1e^{-t} \\ c_2e^t \end{pmatrix}$. Finally,

$$\underline{x} = P\underline{y} = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1e^{-t} \\ c_2e^t \end{pmatrix} = \begin{pmatrix} -c_1e^{-t} - 3c_2e^t \\ c_1e^{-t} + c_2e^t \end{pmatrix} = c_1e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2e^t\begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

2. Let $A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$. Using eigenvalues and their corresponding eigenvectors, we see that $A$ can be diagonalized as follows: $P^{-1}AP = D$, where $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & -2 & 1 \end{pmatrix}$. Make the change of variable$\underline{x} = P\underline{y}$, we see that $\underline{y}' = D\underline{y}$ which can be solved as: $y_1 = c_1e^t$, $y_2 = c_2e^{2t}$ and $y_3 = c_3e^{3t}$. Thus,

$$\underline{x} = c_1e^t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2e^{2t}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} + c_3e^{3t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Using the initial conditions, we see that

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \underline{x}(0) = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} + c_3\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_2 - c_3 \\ c_1 - 2c_2 + c3 \\ -2c_2 + c_3 \end{pmatrix}.$$

Solving for $c_1, c_2, c_3$, we get $c_1 = 1$, $c_2 = 1$ and $c_3 = 2$. Thus,

$$\underline{x} = e^t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} + 2e^{3t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{2t} - 2e^{3t} \\ e^t - 2e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{pmatrix}.$$

3. Since $A^T = A$ (the transpose of $A$ is $A$), therefore $A$ is symmetric and $A$ can be orthogonally diagonalized. The eigenvalues of $A$ are: $\lambda_1 = 5$, $\lambda_2 = 2$ and $\lambda_3 = -2$ and their corresponding eigenvectors are: $\underline{u_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\underline{u_2} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\underline{u_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Let $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. Normalize $\underline{u_1}, \underline{u_2}, \underline{u_3}$ to get the unit eigenvectors:

$$\underline{v_1} = \frac{1}{||\underline{u_1}||}\underline{u_1} = \frac{1}{\sqrt{3}}\underline{u_1} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Similarly, $\underline{v_2} = \frac{1}{||\underline{u_2}||}\underline{u_2} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$ and $\underline{v_3} = \frac{1}{||\underline{v_3}||}\underline{v_3} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. Thus,

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

4.

(a) First of all, the matrix $A$ is symmetric (that is, $A^T = A$) so that $A$ is orthogonally diagonalizable. We find that the eigenvalues of $A$ are: $\lambda_1 = 2$ and $\lambda_2 = 4$ and their corresponding eigenvectors are: $\underline{u_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\underline{u_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Since $\underline{u_1} \cdot \underline{u_2} = 0$, the vectors $\underline{u_1}$ and $\underline{u_2}$ are orthogonal. (We don't really need to check that since $\underline{u_1}$ and $\underline{u_2}$ are eigenvectors corresponding to distinct eigenvalues). Now, we normalize and get $\underline{v_1} = \frac{1}{||\underline{u_1}||}\underline{u_1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\underline{v_2} = \frac{1}{||\underline{u_2}||}\underline{u_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. Thus, $P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

(b) $D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$.

(c) The spectral decomposition of $A$ is:

$$A = \lambda_1 \underline{v_1}\ \underline{v_1}^T + \lambda_2 \underline{v_2}\ \underline{v_2}^T = 2\underline{v_1}\ \underline{v_1}^T + 4\underline{v_2}\ \underline{v_2}^T.$$

5.

(a) The characteristic equation is: $\det(A - \lambda I) = 0$ which can be simplified as: $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$. Using the fact, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 3 & -2 & 1 \\ 2 & -2 & 2 \\ 3 & -6 & 5 \end{pmatrix}$, $A^2 = \begin{pmatrix} 8 & -8 & 4 \\ 8 & -12 & 8 \\ 12 & -24 & 16 \end{pmatrix}$ and $A^3 = \begin{pmatrix} 20 & -24 & 12 \\ 24 & -40 & 24 \\ 36 & -72 & 44 \end{pmatrix}$, we can check that $A^3 - 6A^2 + 12A - 8I = 0$, the zero matrix.

(b) Multiplying $A^3 - 6A^2 + 12A - 8I = 0$ by $A$, we get $A^4 - 6A^3 + 12A^2 - 8A = 0$ so that

$$A^4 = 6A^3 - 12A^2 + 8A = 6(6A^2 - 12A + 8I) - 12A^2 + 8A = 24A^2 - 64A + 48I$$
$$= \begin{pmatrix} 48 & -64 & 32 \\ 64 & -112 & 64 \\ 96 & -192 & 112 \end{pmatrix}.$$

(c) Multiplying $A^3 - 6A^2 + 12A - 8I = 0$ by $A^{-1}$, we get $A^2 - 6A + 12I - 8A^{-1} = 0$. Thus, $A^{-1} = \frac{1}{8}A^2 - \frac{3}{4}A + \frac{3}{2}I$.