Math 118: Honours Calculus II
Winter, 2005 List of Theorems

Lemma 5.1 (Partition Refinement): *If P and Q are partitions of $[a,b]$ such that $Q \supset P$, then*
$$\mathcal{L}(P,f) \le \mathcal{L}(Q,f) \le \mathcal{U}(Q,f) \le \mathcal{U}(P,f).$$

Lemma 5.2 (Upper Sums Bound Lower Sums): *Let f be bounded on $[a,b]$. If P and Q are* **any** *partitions of $[a,b]$, then*
$$\mathcal{L}(P,f) \le \mathcal{U}(Q,f).$$

Lemma 5.3 (Lower Integrals vs. Upper Integrals): *Let f be bounded on $[a,b]$. Then*
$$\underline{\int_a^b} f \le \overline{\int_a^b} f.$$

Theorem 5.1 (Integrability): *$\int_a^b f$ exists and equals $\alpha \iff$ there exists a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ of $[a,b]$ such that*
$$\lim_{n\to\infty} \mathcal{L}(P_n,f) = \alpha = \lim_{n\to\infty} \mathcal{U}(P_n,f).$$

Theorem 5.2 (Cauchy Criterion for Integrability): *Suppose f is bounded on $[a,b]$. Then $\int_a^b f$ exists $\iff$ for each $\epsilon > 0$ there exists a partition P of $[a,b]$ such that*
$$U(P,f) - L(P,f) < \epsilon.$$

Corollary 5.2.1 (Piecewise Integration): Suppose $a < c < b$. Then
$$\int_a^b f \,\exists \iff \int_a^c f \,\exists \text{ and } \int_c^b f \,\exists.$$

Furthermore, when either side holds,
$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Theorem 5.3 (Darboux Integrability Theorem): *$\int_a^b f$ exists and equals $\alpha \iff$ for* **any** *sequence of partitions P_n having subinterval widths that go to zero as $n \to \infty$, all Riemann sums $\mathcal{S}(P_n,f)$ converge to α.*

Theorem 5.4 (Linearity of Integral Operator): *Suppose* $\int_a^b f$ *and* $\int_a^b g$ *exist. Then*

(i) $\int_a^b (f+g) \exists = \int_a^b f + \int_a^b g$

(ii) $\int_a^b (cf) \exists = c\int_a^b f$ *for any constant* $c \in \mathbb{R}$.

Theorem 5.5 (Integral Bounds): *Suppose*

(i) $\int_a^b f \,\exists$,

(ii) $m \le f(x) \le M$ *for* $x \in [a, b]$.

Then

$$m(b-a) \le \int_a^b f \le M(b-a).$$

Corollary 5.5.1 (Preservation of Non-Negativity): If $f(x) \ge 0$ for all $x \in [a, b]$ and $\int_a^b f$ exists then $\int_a^b f \ge 0$.

Corollary 5.5.2 (Continuity of Integrals): Suppose $\int_a^b f$ exists. Then the function $F(x) = \int_a^x f$ is continuous on $[a, b]$.

Theorem 5.6 (Integrability of Continuous Functions): *If* f *is continuous on* $[a, b]$ *then* $\int_a^b f$ *exists.*

Theorem 5.7 (Integrability of Monotonic Functions): *If* f *is monotonic on* $[a, b]$ *then* $\int_a^b f$ *exists.*

Lemma 5.4 (Families of Antiderivatives): *Let* $F_0(x)$ *be an antiderivative of* f *on an interval* I. *Then* F *is an antiderivative of* f *on* $I \iff F(x) = F_0(x) + C$ *for some constant* C.

Theorem 5.8 (Antiderivatives at Points of Continuity): *Suppose*

(i) $\int_a^b f$ *exists;*

(ii) f *is continuous at* $c \in (a, b)$.

Then f *has the antiderivative* $F(x) = \int_a^x f$ *at* $x = c$.

Corollary 5.8.1 (Antiderivative of Continuous Functions): If f is continuous on $[a, b]$ then f has an antiderivative on $[a, b]$.

Theorem 5.9 (Fundamental Theorem of Calculus [FTC]): *Let* f *be integrable and have an antiderivative* F *on* $[a, b]$. *Then*

$$\int_a^b f = F(b) - F(a).$$

Corollary 5.9.1 (FTC for Continuous Functions): Let f be continuous on $[a, b]$ and let F be any antiderivative of f on $[a, b]$. Then

$$\int_a^b f = F(b) - F(a).$$

Theorem 5.10 (Mean Value Theorem for Integrals): *Suppose f is continuous on $[a, b]$. Then*

$$\int_a^b f = f(c)(b - a)$$

for some number $c \in [a, b]$.

Theorem 7.1 (Change of Variables): *Suppose g' is continuous on $[a, b]$ and f is continous on $g([a, b])$. Then*

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Theorem 7.2 (Integration by Parts): *Suppose f' and g' are continuous functions on $[a, b]$. Then*

$$\int_a^b fg' = [fg]_a^b - \int_a^b f'g.$$

Lemma 7.1 (Polynomial Factors): *If z_0 is a root of a polynomial $P(z)$ then $P(z)$ is divisible by $(z - z_0)$.*

Lemma 7.2 (Linear Partial Fractions): *Suppose that $P(x)/Q(x)$ is a proper rational function such that $Q(x) = (x - a)^n Q_0(x)$, where $Q_0(a) \neq 0$ and $n \in \mathbb{N}$. Then there exists a constant A and a polynomial P_0 with $\deg P_0 < \deg Q - 1$ such that*

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-a)^n} + \frac{P_0(x)}{(x-a)^{n-1}Q_0(x)}.$$

Lemma 7.3 (Quadratic Partial Fractions): *Let $x^2 + \gamma x + \lambda$ be an irreducible quadratic polynomial (i.e. $\gamma^2 - 4\lambda < 0$). Suppose that $P(x)/Q(x)$ is a proper rational function such that $Q(x) = (x^2 + \gamma x + \lambda)^m Q_0(x)$, where $Q_0(x)$ is not divisible by $(x^2 + \gamma x + \lambda)$ and $m \in \mathbb{N}$. Then there exists constants Γ and Λ and a polynomial P_0 with $\deg P_0 < \deg Q - 2$ such that*

$$\frac{P(x)}{Q(x)} = \frac{\Gamma x + \Lambda}{(x^2 + \gamma x + \lambda)^m} + \frac{P_0(x)}{(x^2 + \gamma x + \lambda)^{m-1}Q_0(x)}.$$

Theorem 7.3 (Linear Interpolation Error): *Let f be a twice-differentiable function on $[0,h]$ satisfying $|f''(x)| \le M$ for all $x \in [0,h]$. Let*

$$L(x) = f(0) + \frac{f(h)-f(0)}{h}x.$$

Then

$$\int_0^h |L(x) - f(x)|\, dx \le \frac{Mh^3}{12}.$$

Corollary 7.3.1 (Trapezoidal Rule Error): Let P be a uniform partition of $[a,b]$ into n subintervals of width $h=(b-a)/n$, and f be a twice-differentiable function on $[a,b]$ satisfying $|f''(x)| \le M$ for all $x \in [a,b]$. Then the error $E_n^{\mathcal{T}} \doteq \mathcal{T}_n - \int_a^b f$ of the uniform Trapezoidal Rule

$$\mathcal{T}_n = h \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}$$

satisfies

$$\left|E_n^{\mathcal{T}}\right| \le \frac{nMh^3}{12} = \frac{M(b-a)^3}{12n^2}.$$

Theorem 8.1 (Pappus' Theorems): *Let $\mathcal{L}$ be a line in a plane.*

(i) If a curve lying entirely on one side of $\mathcal{L}$ is rotated about $\mathcal{L}$, the area of the surface generated is the product of the length of the curve times the distance travelled by the centroid.

(ii) If a region lying entirely on one side of $\mathcal{L}$ is rotated about $\mathcal{L}$, the volume of the solid generated is the product of the area of the region times the distance travelled by the centroid.

Theorem 9.1 (Increasing Functions: Bounded $\iff$ Asymptotic Limit Exists): *Let f be a monotonic increasing function on $[a,\infty)$. Then f is bounded on $[a,\infty)$ $\iff$ $\lim_{x\to\infty} f$ exists.*

Corollary 9.1.1 (Improper Integrals of Non-Negative Functions): Let f be a non-negative function that is integrable on $[a,T]$ for all $T \ge a$. If there exists a bound B such that $\int_a^T f \le B$ for all $T \ge a$, then $\int_a^\infty f$ converges.

Corollary 9.1.2 (Comparison Test): Suppose $0 \le f(x) \le g(x)$ and $\int_a^T f$ and $\int_a^T g$ exist for all $T \ge a$. Then

(i) $\int_a^\infty g \in \mathcal{C} \Rightarrow \int_a^\infty f \in \mathcal{C}$;

(ii) $\int_a^\infty f \in \mathcal{D} \Rightarrow \int_a^\infty g \in \mathcal{D}$.

Corollary 9.1.3 (Limit Comparison Test): Let f and g be positive integrable functions satisfying

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

(i) For $0 < L < \infty$ we have $\int_a^\infty g \in \mathcal{C} \iff \int_a^\infty f \in \mathcal{C}$.

(ii) When $L = 0$ we can only say $\int_a^\infty g \in \mathcal{C} \Rightarrow \int_a^\infty f \in \mathcal{C}$.

Theorem 9.2 (Cauchy Criterion for Improper Integrals): *Let f be a function.*

(i) Suppose $\int_a^t f$ exists for all $t \in (a, b)$. Then $\int_a^{b-} f \in \mathcal{C} \iff \forall\epsilon > 0,\ \exists\delta > 0$ such that

$$x, y \in (b - \delta, b) \Rightarrow \left|\int_x^y f\right| < \epsilon;$$

(ii) Suppose $\int_a^T f$ exists for all $T > a$. Then $\int_a^\infty f \in \mathcal{C} \iff \forall\epsilon > 0,\ \exists T$ such that

$$T_2 \geq T_1 \geq T \Rightarrow \left|\int_{T_1}^{T_2} f\right| < \epsilon.$$

Theorem 9.3 (Cauchy Criterion for Infinite Series): *The infinite series $\sum_{k=1}^\infty a_k$ converges if and only if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$m > n \geq N \Rightarrow \left|\sum_{k=n}^m a_k\right| < \epsilon.$$

Theorem 9.4 (Divergence Test): *If* $\displaystyle\sum_{k=1}^\infty a_k \in \mathcal{C}$ *then* $\displaystyle\lim_{n\to\infty} a_n = 0$.

Theorem 9.5 (Non-Negative Terms: Convergence $\iff$ Bounded Partial Sums): *If $a_k \geq 0$ and $S_n = \sum_{k=1}^n a_k$ then $\sum_{k=1}^\infty a_k \in \mathcal{C} \iff \{S_n\}_{n=1}^\infty$ is a bounded sequence.*

Corollary 9.5.1 (Comparison Test): If $0 \leq a_k \leq b_k$ for $k \in \mathbb{N}$ then

(i) $\displaystyle\sum_{k=1}^\infty b_k \in \mathcal{C} \Rightarrow \sum_{k=1}^\infty a_k \in \mathcal{C}$;

(ii) $\displaystyle\sum_{k=1}^\infty a_k \in \mathcal{D} \Rightarrow \sum_{k=1}^\infty b_k \in \mathcal{D}$.

Corollary 9.5.2 (Limit Comparison Test): Suppose $a_k \geq 0$ and $b_k > 0$ for $k \in \mathbb{N}$ and $\lim\limits_{k\to\infty} a_k/b_k = L$. Then

(i) if $0 < L < \infty$: $\displaystyle\sum_{k=1}^{\infty} a_k \in \mathcal{C} \iff \sum_{k=1}^{\infty} b_k \in \mathcal{C}$;

(ii) if $L = 0$: $\displaystyle\sum_{k=1}^{\infty} b_k \in \mathcal{C} \Rightarrow \sum_{k=1}^{\infty} a_k \in \mathcal{C}$.

Corollary 9.5.3 (Ratio Comparison Test): If $a_k > 0$ and $b_k > 0$ and

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k}$$

for all $k \geq N$, then

(i) $\displaystyle\sum_{k=1}^{\infty} b_k \in \mathcal{C} \Rightarrow \sum_{k=1}^{\infty} a_k \in \mathcal{C}$;

(ii) $\displaystyle\sum_{k=1}^{\infty} a_k \in \mathcal{D} \Rightarrow \sum_{k=1}^{\infty} b_k \in \mathcal{D}$.

Corollary 9.5.4 (Ratio Test): Suppose $a_k > 0$ and $b_k > 0$.

(i) If $\exists$ a number $x < 1$ such that $\frac{a_{k+1}}{a_k} \leq x$ for all $k \geq N$, then $\sum_{k=1}^{\infty} a_k \in \mathcal{C}$.

(ii) If $\exists$ a number $x \geq 1$ such that $\frac{a_{k+1}}{a_k} \geq x$ for all $k \geq N$, then $\sum_{k=1}^{\infty} a_k \in \mathcal{D}$.

Corollary 9.5.5 (Limit Ratio Test): Suppose $a_k > 0$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = c.$$

Then

(i) $\displaystyle 0 \leq c < 1 \Rightarrow \sum_{k=1}^{\infty} a_k \in \mathcal{C}$,

(ii) $\displaystyle c > 1 \Rightarrow \sum_{k=1}^{\infty} a_k \in \mathcal{D}$,

(iii) $c = 1 \Rightarrow$?

Theorem 9.6 (Integral Test): *Suppose f is continuous, decreasing, and non-negative on $[1, \infty)$. Then*

$$\sum_{k=1}^{\infty} f(k) \in \mathcal{C} \iff \int_1^{\infty} f \in \mathcal{C}.$$

Theorem 9.7 (Absolute Convergence): *An absolutely convergent series is convergent.*

Theorem 9.8 (Radius of Convergence): *For each power series $\sum_{k=0}^{\infty} c_k x^k$ there exists a number R, called the radius of convergence, with $0 \le R \le \infty$, such that*

$$\sum_{k=0}^{\infty} c_k x^k \in \begin{cases} \mathrm{Abs}\,\mathcal{C} & \text{if } |x| < R, \\ \mathcal{D} & \text{if } |x| > R, \\ ? & \text{if } |x| = R. \end{cases}$$

Lemma A.1 (Complex Conjugate Roots): *Let P be a polynomial with real coefficients. If z is a root of P, then so is $\bar{z}$.*

Theorem A.1 (Fundamental Theorem of Algebra): *Any non-constant polynomial $P(z)$ with complex coefficients has a complex root.*

Corollary A.1.1 (Polynomial Factorization): Every complex polynomial $P(z)$ of degree $n \ge 0$ has exactly n complex roots z_1, z_2, …, z_n and can be factorized as $P(z) = A(z - z_1)(z - z_2)\ldots(z - z_n)$, where $A \in \mathbb{C}$.

Corollary A.1.2 (Real Polynomial Factorization): Every polynomial with real coefficients can be factorized as

$$P(x) = A(x - a_1)^{n_1} \ldots (x - a_k)^{n_k} (x^2 + \gamma_1 x + \lambda_1)^{m_1} \ldots (x^2 + \gamma_\ell x + \lambda_\ell)^{m_\ell}.$$