Math 117: Honours Calculus I

Fall, 2012 List of Theorems

Theorem 1.1 (Binomial Theorem): *For all* $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Theorem 2.1 (Convergent $\Rightarrow$ Bounded): *A convergent sequence is bounded.*

Theorem 2.2 (Properties of Limits): *Let* $\{a_n\}$ *and* $\{b_n\}$ *be convergent sequences. Let* $L = \lim_{n\to\infty} a_n$ *and* $M = \lim_{n\to\infty} b_n$. *Then*

(a) $\lim_{n\to\infty} (a_n + b_n) = L + M$;

(b) $\lim_{n\to\infty} a_n b_n = LM$;

(c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ *if* $M \neq 0$.

Corollary 2.2.1 (Case $L \neq 0$, $M = 0$): Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. If $\lim_{n\to\infty} a_n \neq 0$ and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n}$ does not exist.

Theorem 2.3 (Monotone Sequences: Convergent $\iff$ Bounded): *Let* $\{a_n\}$ *be a monotone sequence. Then* $\{a_n\}$ *is convergent* $\iff$ $\{a_n\}$ *is bounded.*

Theorem 2.4 (Convergent $\iff$ All Subsequences Convergent): *A sequence* $\{a_n\}_{n=1}^{\infty}$ *is convergent with limit* L $\iff$ **each** *subsequence* $\{a_{n_k}\}_{k=1}^{\infty}$ *of* $\{a_n\}_{n=1}^{\infty}$ *is convergent with limit* L.

Theorem 2.5 (Bolzano–Weierstrass Theorem): *A bounded sequence has a convergent subsequence.*

Theorem 2.6 (Cauchy Criterion): $\{a_n\}$ *is convergent* $\iff$ $\{a_n\}$ *is a Cauchy sequence.*

Theorem 3.1 (Equivalence of Function and Sequence Limits): $\lim_{x\to a} f(x) = L \iff f$ *is defined near* a *and* **every** *sequence of points* $\{x_n\}$ *in the domain of* f, *with* $x_n \neq a$ *but* $\lim_{n\to\infty} x_n = a$, *satisfies* $\lim_{n\to\infty} f(x_n) = L$.

Corollary 3.1.1 (Properties of Function Limits): Suppose $L = \lim_{x\to a} f(x)$ and $M = \lim_{x\to a} g(x)$. Then

(a) $\lim_{x\to a}(f(x)+g(x)) = L+M$;

(b) $\lim_{x\to a} f(x)g(x) = LM$;

(c) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$.

Corollary 3.1.2 (Cauchy Criterion for Functions): $\lim_{x\to a} f(x)$ exists $\iff$ for **every** $\epsilon > 0$, $\exists\delta > 0$ such that whenever

$$0 < |x-a| < \delta \quad \textbf{and} \quad 0 < |y-a| < \delta$$

then $|f(x) - f(y)| < \epsilon$.

Corollary 3.1.3 (Squeeze Principle for Functions): Suppose $f(x) \leq h(x) \leq g(x)$ when $0 < |x-a| < r$ for some positive real number r. Then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L \Rightarrow \lim_{x\to a} h(x) = L.$$

Corollary 3.1.4 (Properties of Continuous Functions): Suppose f and g are continuous at a. Then $f+g$ and fg are continuous at a and f/g is continuous at a if $g(a) \neq 0$.

Corollary 3.1.5 (Continuity of Rational Functions): A rational function is continuous at all points of its domain.

Corollary 3.1.6 (Continuous Functions of Sequences): f is continuous at an interior point a of the domain of f $\iff$ each sequence $\{x_n\}$ in the domain of f with $\lim_{n\to\infty} x_n = a$ satisfies $\lim_{n\to\infty} f(x_n) = f(a)$.

Corollary 3.1.7 (Composition of Continuous Functions): Suppose g is continuous at a and f is continuous at $g(a)$. Then $f \circ g$ is continuous at a.

Theorem 3.2 (Intermediate Value Theorem [IVT]): *Suppose*

(i) *f is continuous on [a,b],*

(ii) $f(a) < 0 < f(b)$.

Then there exists a number $c \in (a,b)$ such that $f(c) = 0$.

Corollary 3.2.1 (Generalized Intermediate Value Theorem): Suppose

(i) f is continuous on [a,b],

(ii) $f(a) < y < f(b)$.

Then there exists a number $c \in (a, b)$ such that $f(c) = y$.

Theorem 3.3 (Boundedness of Continuous Functions on Closed Intervals): *If f is continuous on $[a, b]$ then f is bounded on $[a, b]$.*

Theorem 3.4 (Weierstrass Max/Min Theorem): *If f is continuous on $[a, b]$ then it achieves both a maximum and minimum value on $[a, b]$.*

Corollary 3.4.1 (Image of a Continuous Function on a Closed Interval): If f is continuous on $[a, b]$ then $f([a, b])$ is either a closed interval or a point.

Theorem 4.1 (Differentiable $\Rightarrow$ Continuous): *If f is differentiable at a then f is continuous at a.*

Theorem 4.2 (Properties of Differentiation): *If f and g are both differentiable at a, then*

(a) $(f + g)'(a) = f'(a) + g'(a)$,

(b) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$,

(c) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$ *if* $g(a) \neq 0$.

Theorem 4.3 (Chain Rule): *Suppose $h = f \circ g$. Let a be an interior point of the domain of h and define $b = g(a)$. If $f'(b)$ and $g'(a)$ both exist, then h is differentiable at a and*

$$h'(a) = f'(b)g'(a).$$

Theorem 4.4 (Interior Local Extrema): *Suppose*

(i) *f has an interior local extremum (maximum or minimum) at c,*

(ii) *$f'(c)$ exists.*

Then $f'(c) = 0$.

Corollary 4.4.1 (Rolle's Theorem): Suppose

(i) f is continuous on $[a, b]$,

(ii) f' exists on (a, b),

(iii) $f(a) = f(b)$.

Then there exists a number $c \in (a, b)$ for which $f'(c) = 0$.

Corollary 4.4.2 (Mean Value Theorem [MVT]): Suppose

(i) f is continuous on $[a, b]$,

(ii) f' exists on (a, b).

Then there exists a number $c \in (a, b)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 4.4.3 (Zero Derivative on an Interval): Suppose $f'(x) = 0$ for every x in an interval I (of nonzero length). Then f is constant on I.

Corollary 4.4.4 (Equal Derivatives): Suppose $f'(x) = g'(x)$ for every x in an interval I (of nonzero length). Then $f(x) = g(x) + k$ for all $x \in I$, where k is a constant.

Corollary 4.4.5 (Monotonic Functions): Suppose f is differentiable on an interval I. Then

(i) f is increasing on $I \iff f'(x) \geq 0$ on I;

(ii) f is decreasing on $I \iff f'(x) \leq 0$ on I.

Corollary 4.4.6 (Horse-Race Theorem): Suppose

(i) f and g are continuous on $[a, b]$,

(ii) f' and g' exist on (a, b),

(iii) $f(a) \geq g(a)$,

(iv) $f'(x) \geq g'(x) \quad \forall x \in (a, b)$.

Then $f(x) \geq g(x) \quad \forall x \in [a, b]$.

Corollary 4.4.7 (First Derivative Test): Suppose f is differentiable near a critical point c (except possibly at c, provided f is continuous at c). If there exists a $\delta > 0$ such that

(i) $f'(x) \begin{cases} \leq 0 & \forall x \in (c - \delta, c) \quad (f \text{ decreasing}), \\ \geq 0 & \forall x \in (c, c + \delta) \quad (f \text{ increasing}), \end{cases}$ then f has a local minimum at c;

(ii) $f'(x) \begin{cases} \geq 0 & \forall x \in (c - \delta, c) \quad (f \text{ increasing}), \\ \leq 0 & \forall x \in (c, c + \delta) \quad (f \text{ decreasing}), \end{cases}$ then f has a local maximum at c;

(iii) $f'(x) > 0$ on $(c - \delta, c) \cup (c, c + \delta)$ or $f'(x) < 0$ on $(c - \delta, c) \cup (c, c + \delta)$, then f does not have a local extremum at c.

Corollary 4.4.8 (Second Derivative Test): Suppose f is twice differentiable at a critical point c (this implies $f'(c) = 0$). If

(i) $f''(c) > 0$, then f has a local minimum at c;

(ii) $f''(c) < 0$, then f has a local maximum at c.

Corollary 4.4.9 (Cauchy Mean Value Theorem): Suppose

(i) f and g are continuous on $[a, b]$,

(ii) f' and g' exist on (a, b).

Then there exists a number $c \in (a, b)$ for which

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Corollary 4.4.10 (L'Hôpital's Rule for $\frac{0}{0}$): Suppose f and g are differentiable on (a, b), $g'(x) \neq 0$ for all $x \in (a, b)$, $\lim_{x \to b^-} f(x) = 0$, and $\lim_{x \to b^-} g(x) = 0$. Then

$$\lim_{x \to b^-} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to b^-} \frac{f(x)}{g(x)} = L.$$

This result also holds if

(i) $\lim_{x \to b^-}$ is replaced by $\lim_{x \to a^+}$;

(ii) $\lim_{x \to b^-}$ is replaced by $\lim_{x \to \infty}$ and b is replaced by ∞;

(iii) $\lim_{x \to b^-}$ is replaced by $\lim_{x \to -\infty}$ and a is replaced by $-\infty$.

Corollary 4.4.11 (L'Hôpital's Rule for $\frac{\infty}{\infty}$): Suppose f and g are differentiable on (a, b), $g'(x) \neq 0$ for all $x \in (a, b)$, and $\lim_{x \to b^-} f(x) = \infty$, and $\lim_{x \to b^-} g(x) = \infty$. Then

$$\lim_{x \to b^-} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to b^-} \frac{f(x)}{g(x)} = L.$$

This result also holds if

(i) $\lim_{x \to b^-}$ is replaced by $\lim_{x \to a^+}$;

(ii) $\lim_{x \to b^-}$ is replaced by $\lim_{x \to \infty}$ and b is replaced by ∞;

(iii) $\lim_{x \to b^-}$ is replaced by $\lim_{x \to -\infty}$ and a is replaced by $-\infty$.

Corollary 4.4.12 (Taylor's Theorem): Let $n \in \mathbb{N}$. Suppose

(i) $f^{(n-1)}$ exists and is continuous on $[a, b]$,

(ii) $f^{(n)}$ exists on (a, b).

Then there exists a number $c \in (a, b)$ such that

$$f(b) = \sum_{k=0}^{n-1} \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{(b-a)^n}{n!} f^{(n)}(c).$$

Theorem 4.5 (First Convexity Criterion): *Suppose f is differentiable on an interval I. Then*

(i) *f is convex $\iff$ f' is increasing on I;*

(ii) *f is concave $\iff$ f' is decreasing on I.*

Corollary 4.5.1 (Second Convexity Criterion): Suppose f is twice differentiable on an interval I. Then

(i) f is convex on $I \iff f''(x) \geq 0 \quad \forall x \in I$;

(ii) f is concave on $I \iff f''(x) \leq 0 \quad \forall x \in I$.

Corollary 4.5.2 (Tangent to a Convex Function): If f is convex and differentiable on an interval I, the graph of f lies above the tangent line to the graph of f at every point of I.

Corollary 4.5.3 (Global Second Derivative Test): Suppose f is twice differentiable on I and $f'(c) = 0$ at some $c \in I$. If

(i) $f''(x) \geq 0 \quad \forall x \in I$, then f has a global minimum at c;

(ii) $f''(x) \leq 0 \quad \forall x \in I$, then f has a global maximum at c.

Theorem 4.6 (Continuous Invertible Functions): *Suppose f is continuous on I. Then f is one-to-one on $I \iff f$ is strictly monotonic on I.*

Corollary 4.6.1 (Continuity of Inverse Functions): Suppose f is continuous and one-to-one on an interval I. Then its inverse function f^{-1} is continuous on $f(I) = \{f(x) : x \in I\}$.

Corollary 4.6.2 (Differentiability of Inverse Functions): Suppose f is continuous and one-to-one on an interval I and differentiable at $a \in I$. Let $b = f(a)$ and denote the inverse function of f on I by g. If

(i) $f'(a) = 0$, then g is **not** differentiable at b;

(ii) $f'(a) \neq 0$, then g is differentiable at b and $g'(b) = \dfrac{1}{f'(a)}$.