Review of Math 117

1 Real Numbers

Induction: Show first case and that case n implies case $n+1$.

Binomial Theorem: For $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

$\mathbb{R}$ **is complete:** Every *nonempty* subset of $\mathbb{R}$ with an upper bound has a *least* upper bound in $\mathbb{R}$.

2 Limits

Limit: $\lim_{x \to a} f(x) = L$ means for every $\epsilon > 0$ we can find a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

One-Sided Limit: $\lim_{x \to a+} f(x) = L$ means for every $\epsilon > 0$ we can find a $\delta > 0$ such that

$$x \in (a, a+\delta) \Rightarrow |f(x) - L| < \epsilon.$$

Vertical Asymptote: $\lim_{x \to a+} f(x) = \infty$ means for every $M > 0$ we can find a $\delta > 0$ such that

$$x \in (a, a+\delta) \Rightarrow f(x) > M.$$

Horizontal Asymptote: $\lim_{x \to \infty} f(x) = L$ means for every $\epsilon > 0$ we can find a number N such that

$$x > N \Rightarrow |f(x) - L| < \epsilon.$$

Infinite Limit: $\lim_{x \to \infty} f(x) = \infty$ means for every $M > 0$ we can find a number N such that

$$x > N \Rightarrow f(x) > M.$$

Cauchy Criterion: $\lim_{x \to a} f(x)$ exists $\iff$ for every $\epsilon > 0$ we can find a $\delta > 0$ such that $x, y \in (a - \delta, a) \cup (a, a + \delta) \Rightarrow |f(x) - f(y)| < \epsilon$.

Sequences: $a_n = f(n)$ is a function on the domain $\mathbb{N}$.

Cauchy Criterion for Sequences: $\lim_{n \to \infty} a_n$ exists $\iff$ for every $\epsilon > 0$ we can find a number N such that $m, n > N \Rightarrow |a_m - a_n| < \epsilon$.

Convergent $\Rightarrow$ Bounded.

Monotone Sequences: Convergent $\iff$ Bounded.

Convergent $\iff$ All Subsequences Convergent.

Bounded $\Rightarrow \exists$ Convergent Subsequence.

Limit Properties: $\lim_{x \to a}(f(x)+g(x))\exists = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$ **if** these individual limits exist.

Continuity: $\lim_{x \to a} f(x) = f(\lim_{x \to a} x) = f(a)$.

Intermediate Value Theorem: If

(i) f is continuous on [a,b],

(ii) $f(a) < y < f(b)$,

then there exists a number $c \in (a, b)$ such that $f(c) = y$.

Closed intervals: Continuous $\Rightarrow$ bounded; maximum and minimum values achieved.

3 Derivatives

Derivative:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Differentiable $\Rightarrow$ Continuous.

Derivative Properties: At a point a, if f and g are differentiable then

(a) $(f + g)' = f' + g'$,

(b) $(fg)' = f'g + fg'$,

(c) $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$ if $g(a) \neq 0$.

Chain Rule: If $y = f(g(x))$, then $\dfrac{dy}{dx} = f'\big(g(x)\big)\, g'(x)$.

Taylor's Theorem: If

(i) $f^{(n-1)}$ is continuous on $[a,b]$,

(ii) $f^{(n)}$ exists on (a,b),

then $\exists c \in (a,b)$ such that

$$f(b) = \sum_{k=0}^{n-1} \frac{(b-a)^k}{k!} f^{(k)}(a) + \underbrace{\frac{(b-a)^n}{n!} f^{(n)}(c)}_{R_n}.$$

Mean Value Theorem: Case $n=1$. Suppose

(i) f is continuous on $[a,b]$,

(ii) f' exists on (a,b).

Then there exists a number $c \in (a,b)$ for which

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Rolle's Theorem: Case $f(a) = f(b)$.

Monotonic Functions: Suppose f is differentiable on I. Then f is increasing on $I \iff f'(x) \geq 0$ on I.

Extrema: Extrema can occur either at

(i) an end point,

(ii) a point where f' does not exist,

(iii) a point where $f' = 0$.

First Derivative Test: If f is decreasing to the left of c and increasing to the right of c, then f has a minimum at c.

Second Derivative Test: If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

L'Hôpital's Rule for $\frac{0}{0}$ Form: Suppose f and g are differentiable, with $g' \neq 0$ on (a,b), $\lim\limits_{x\to a^+} f(x) = 0$, $\lim\limits_{x\to a^+} g(x) = 0$. Then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} \exists = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

if the limit on the RHS exists. (There is a similar L'Hôpital's Rule for the $\frac{\infty}{\infty}$ form.)

Continuous Functions: Invertible (1–1) $\iff$ Strictly Monotonic.

Continuous Invertible Functions Have Continous Inverses.

Differentiable Invertible Functions Have Differentiable Inverses:

$$f^{-1\prime}(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))},$$

unless $f'(a) = 0$.

Convexity Criterion: A twice differentiable function f on I is convex $\iff$ $f'' \geq 0$ on I.