Government Debt Control: Optimal Currency Portfolio and Payments

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Motivated by empirical facts, we develop a theoretical model for optimal currency government debt portfolio and debt payments, which allows both government debt aversion and jumps in the exchange rates. We obtain first a realistic stochastic differential equation for public debt, and then solve explicitly the optimal currency debt problem. We show that higher debt aversion and jumps in the exchange rates lead to a lower proportion of optimal debt in foreign currencies. Furthermore, we show that for a government with extreme debt aversion it is optimal not to issue debt in foreign currencies. To the best of our knowledge, this is the first theoretical model that provides a rigorous explanation of why developing countries have reduced consistently their proportion of foreign debt in their debt portfolios.

Key words: Debt control, optimal currency debt portfolio, optimal debt payments, stochastic control.

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Public debt is a key macroeconomic variable. In particular, the currency composition of public debt is an important variable that can exacerbate a debt crisis, such as in Mexico in 1994. Since a government debt portfolio is, in most cases, the largest financial portfolio in a country, high foreign currency debt (especially, short term) exposes the country to the fluctuations of the exchange rates, and this becomes dramatic when unexpected and huge depreciations (or devaluations) of the domestic currency occur.

Panizza (2008) points out that developing countries have been reducing consistently the proportion of foreign debt in their portfolios in favor of local currency debt. This empirical fact is happening in the context in which most developing countries have access to the international capital markets. That is, although countries can borrow in external currencies, there is a deliberate tendency to borrow in domestic currency. A natural question arises. What is the explanation for this type of government behavior? There is a consensus that the development of the domestic capital markets has played a key role. According to Borensztein et al. (2008), debt crises are the factors that have urged countries to pursue such strength in the domestic markets. In other words, the underlying factor that explains this tendency is the goal of reducing the exchange rate vulnerabilities, and hence the chances of a debt crisis. This can be interpreted as the countries have become more debt averse due to their past experiences.

What does the theoretical literature on currency government debt management say about the fact above? Surprisingly, although the order of magnitude of the debt of a country can be of trillions of dollars and, as we mentioned above, the government debt portfolio is in general the largest in the country, the theoretical literature has paid almost no attention to currency government debt portfolios.

As far as we know, Licandro and Masoller (2000) and Giavazzi and Missale (2004) are the only references which deal with currency debt portfolios\(^1\). These approaches have the following limitations: (1) the debt dynamics is not realistic because they consider only one period models, (2) the jumps in the exchange rates are not considered explicitly, and (3) the role of debt aversion is not included. Thus, important elements of the currency debt analysis have not been included.

To the best of our knowledge, for the first time in the literature, we present a model for debt management that includes jumps in the exchange rates and debt aversion. We obtain explicitly the optimal currency debt portfolio and optimal debt payments. We use this model to show that the behavior of developing countries of reducing their proportion of foreign debt in their debt portfolios is consistent with a high debt aversion. Moreover, we show that an extremely high debt aversion can lead to issue only government debt in local currency. That is, it would be optimal for such a country to have no debt in foreign currency.

\(^1\) There exist numerical approaches as well. See Melecky (2007) for a survey.
We model the currency debt problem as a stochastic control model in continuous time with infinite horizon. We believe that is the suitable framework to study debt portfolios. In fact, Bolder (2003) shows that the government debt problem can be conceptualized in this manner. However, his problem is different from ours. He assumes that the government issues only local currency debt and his goal is to find the optimal proportion of the different terms of debt. In contrast, our focus is on finding the optimal currency debt composition. That is why we do not consider debt in different terms; instead, we assume that the government issues bonds in different currencies.

To get a realistic debt dynamics, we extend the model of government debt for a single currency, presented in macroeconomic textbooks (see Blanchard and Fischer 1989, for example), to a multicurrency setting. We model the exchange rates dynamics as stochastic process that present jumps, and thus consider a more general framework than Zapatero (1995) and Cadenillas and Zapatero (1999). We succeed in finding a stochastic differential equation for government debt, in which its current value depends on the present and past values of variables such as the interest rates, exchange rates, debt payments and the proportions of debt in different currencies.

The running cost, or what we call here the debt disutility of the government, depends on both the debt payments and the debt itself. The existence of the former cost comes from the fact that, in order to get additional positive fiscal results to repay debt, the government has to cut spendings or increase taxes. The rationale for considering the latter cost stems from the fact that the government acknowledges that having high debt can lead to debt problems, which can end up in a debt crisis, for instance. Since we are interested in analyzing the effects of debt aversion on the optimal currency debt portfolio, we have to consider a disutility function in which the debt aversion is a parameter such that: the greater the debt aversion, the higher the disutility generated by the debt itself and debt payments. In our model, the disutility function presents constant relative risk aversion (CARA) in debt payments and the debt itself. We have two reasons for such choice. First, we have a unique parameter that fully characterizes government debt aversion. Second, our model with constant relative risk aversion is a generalization of the quadratic function, which is the disutility function most widely used in different areas of economics (see, for example, Kydland and Prescott 1977, Taylor 1979, Cadenillas and Zapatero 1999, and Cadenillas et al. 2013).

The goal of the government debt manager is to choose both the currency debt portfolio and the payments that minimize the expected total cost (disutility).

We succeed in solving the problem explicitly. Thus, we can perform some comparative statics to analyze the effects of some parameters (such as the debt aversion and the size and frequency of the jumps of the exchange rates) on the optimal currency debt portfolio and the

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2 Besides, the consideration of different terms makes the problem intractable from an analytical perspective. That is why Bolder (2003) has to pursue a simulation approach.
optimal debt payments.

The paper is organized as follows. We present the model for government debt control in Section 2. In Section 3 we present a theorem which states a sufficient condition for a debt control to be optimal. In Section 4 we find a candidate for solution and then verify that this candidate is indeed the solution. In Section 5 we perform some comparative economic analysis to show, among other results, the role played by government debt aversion. We write the conclusions in Section 6.

2 The Government Debt Model

In this section we derive the stochastic differential equation for the debt of a country and, based on it, we then state the government currency debt control problem. Our goal here is to obtain a realistic debt dynamics, that considers debt in a finite number of foreign currencies, and includes jumps in the exchange rates.

2.1 The Debt Ratio Dynamics

The government debt is defined by

\[ X(t) := \text{gross public debt expressed in local currency at time } t. \]

The gross public debt is the cumulative total of all government borrowings less repayments. That is, it includes the central and local government debt, and the domestic and external debt.

In this subsection, our goal is to extend the following version for one single currency debt presented in macroeconomic textbooks (see, for instance, Blanchard and Fischer 1989):

\[
X(t) = X(0) + \int_0^t r_0 X(s) ds - \int_0^t p(s) ds,
\]

(2.1)

where \( r_0 \) is the (continuous) interest rate, and \( p \) stands for the process of debt rate payments.

We consider a government that issues bonds in local and \( m \) foreign currencies. Let \( \Lambda_0(t) \) denote the number of bonds held in local currency at time \( t \), and \( \Lambda_j(t) \) the number of bonds held in foreign currency \( j \) for \( j \in \{1, \cdots, m\} \). The prices of the bonds are denoted by \( R_j(t) \) for \( j \in \{0, 1, \cdots, m\} \). Thus, \( \Lambda_j(t) R_j(t) \) is the amount of debt held in currency \( j \). For instance, if \( j = 1 \) represents Euros, and this currency is a foreign currency for the country, then \( \Lambda_1(t) R_1(t) \) is the amount of debt in Euros at time \( t \).

We require \( m \) exchange rates to express the total debt in local currency. For \( j \in \{1, \cdots, m\} \), let \( Q_j(t) \) be the exchange rate of the currency \( j \) with respect to the local currency. To be more
precise,

$$Q_j(t) := \text{local currency units per unit of foreign currency } j \text{ at time } t.$$

Then, the total debt in terms of local currency $X$ can be written as

$$X(t) = \Lambda_0(t)R_0(t) + \sum_{j=1}^{m} \Lambda_j(t)R_j(t)Q_j(t); \quad (2.2)$$

or equivalently,

$$X(t) = Z_0(t) + \sum_{j=1}^{m} Z_j(t)Q_j(t),$$

where $Z_i(t) := \Lambda_i(t)R_i(t), \forall i \in \{0, 1, \cdots, m\}$.

To find the debt dynamics, given the evolution of the number of bonds in each currency, we require the dynamics of the price of the bonds and the dynamics of the exchange rates. Since we are interested in studying a currency debt portfolio, in our model the source of randomness will come from the exchange rates. We will assume that the exchange rates follow a process driven by Brownian motions and Poisson processes. For technical reasons, we need to specify a suitable probability space in which these processes are defined.

Consider a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0, \infty)\}$, which is the $P$-augmentation of the filtration generated by both an $m$-dimensional Brownian motion $W = \{W_1, \cdots, W_m\}$, and an independent $m$-dimensional Poisson process $N = \{N_1, \cdots, N_m\}$ with corresponding intensities $\{\lambda_1, \cdots, \lambda_m\}$.

Let $Q = \{(Q_1(t), \cdots, Q_m(t)), t \geq 0\}$ be an $\mathbb{F}$-adapted process. Following Jeanblanc-Picqué and Pontier (1990), Cadenillas (2002) and Guo and Xu (2004), we generalize the model in Zapatero (1995), and Cadenillas and Zapatero (1999), and consider the following multidimensional setting for the exchange rates$^3$.

For $j \in \{1, 2, \cdots, m\}$ :

$$dQ_j(t) = \mu_j Q_j(t)dt + \sigma_j Q_j(t)dW(t) + \tilde{\varphi}_j Q_j(t^-)d\tilde{N}(t), \quad (2.3)$$

with initial exchange rates $Q_j(0) = q_j > 0$. Here $\sigma_j$ and $\tilde{\varphi}_j$ are the $j$-th row of the $m \times m$-matrices $\sigma = [\sigma_{ij}]$ and $\varphi = [\varphi_{ij}]$, respectively. The parameter $\mu_j \in (-\infty, \infty)$. Throughout the paper $A^T$ denotes the transpose of matrix $A$. We will assume that $\sigma \sigma^T$ and the family of matrices $\{\varphi_j \varphi_j^T : j \in \{1, \cdots, m\}\}$ are all positive definite, where $\varphi_j$ stands for the $j$-th column of matrix $\varphi$. Furthermore, we denote by $\tilde{N}$ the compensated Poisson process. That is, for $^3$ Cadenillas and Zapatero (1999) assume that, in the absence of government interventions in the exchange markets, the exchange rate follows a geometric Brownian motion. That is, they assume equation (2.3) without the component that corresponds to the compensated Poisson process $\tilde{N}$.
where $ \lambda_j > 0$ and $N_j$ is a Poisson process. The left continuous version of any process $Y(t)$ is denoted by $Y(t^-)$. We note that the exchange rate $Q_j$ has right-continuous with left-limits paths, a property that is determined by the stochastic integral with respect to the jump process. For more information about stochastic differential equations like (2.3), see for example, Ikeda and Watanabe (1981), Protter (2004), Cont and Tankov (2004), and Applebaum (2009).

**Remark 1.** Equation (2.3) implies that the process $Q_j$ jumps at time $t > 0$ if and only if some of the Poisson processes in $N = \{N_1, \cdots, N_m\}$ do so. That is why $Q_j$ is right-continuous with left-limits. Moreover, since the number of jumps of a Poisson process is finite on each finite interval $[0, t]$, then the sample paths of both the process $Q_j(\cdot)$ and its left-continuous version $Q_j(\cdot^-)$ are bounded on any finite interval $[0, t]$.

The model for the exchange rates, equation (2.3), is appropriate to describe sudden depreciations (or devaluations) of the local currency. The time of the jumps in the exchange rates are random, driven by Poisson processes. For instance, if a Poisson event occurs in the exchange rate $j, j \in \{1, 2, \cdots, m\}$, then it generates an effect on all the exchange rates via $\phi_{ij}$. The occurrence of this event is random, but we know that the intensity of the event is given by $\lambda_j$. This parameter measures the frequency of the jumps in the exchange rate. The information about the size of the jumps is contained in the matrix $\phi$. In particular, if the element in the diagonal $\phi_{jj}$ is positive, this will be consistent with empirical currency crises (such as Mexico in 1994, Asia in 1997, and Russia in 1998) in which the exchange rate went up dramatically. Thus, if a jump in the exchange rate $j$ occurs, then an increase in the rate of depreciation (or devaluation) of size $\phi_{jj} > 0$ takes place. For example, $\phi_{jj} = 0.3$ means that the exchange rate went up unexpectedly 30%. Thus, our model accounts for the fact that a government that has foreign currencies in its debt portfolio faces a risk of depreciation (or devaluation) of its local currency. Since we also want to analyze the effects of the size of jumps on the optimal currency debt portfolio, the parameters in matrix $\phi$ give us a precise, direct and intuitive measure of the magnitude of the jumps. This is the reason why we choose the modeling given in equation (2.3) over other jump processes models.

We assume that the prices of the bonds $R_j$ satisfy

$$dR_j(t) = R_j(t) r_j dt, \quad \forall j \in \{0, 1, \cdots, m\};$$

where $R_j(0) = 1$, and $r_j \in (0, \infty)$ is the (continuous) interest rate on debt issued in currency $j$. We also assume that the process $R = \{(R_0(t), R_1(t), \cdots, R_m(t)), t \geq 0\}$ is $\mathbb{F}$-adapted.

In Appendix A, we derive the debt dynamics for a discrete time model with one foreign
currency. The continuous-time version with \( m \) foreign currencies is

\[
X(t) = X(0) + \int_0^t \Lambda_0(s^-)dR_0(s) + \sum_{j=1}^m \int_0^t \Lambda_j(s^-)d(R_j(s)Q_j(s)) - \int_0^t p(s). \tag{2.5}
\]

We assume that the process of rate payment \( p \) and the \( m \)-dimensional process \( \Lambda = (\Lambda_1, \cdots, \Lambda_m) \) are \( \mathbb{F} \)-adapted and right-continuous with left-limits. In addition, we assume \( p \) non-negative, and the technical condition \( E_x[\int_0^t p(s)ds] < \infty \) for every \( t > 0 \) to be satisfied.

Using both the dynamics of the exchange rates (2.3) and the dynamics of the price of the bonds (2.4) in equation (2.5), we obtain the stochastic differential equation for the debt process \( X \):

\[
dX(t) = r_0 X(t)dt + \sum_{j=1}^m (r_j + \mu_j - r_0) \Lambda_j(t) R_j(t)Q_j(t)dt + \sum_{j=1}^m \Lambda_j(t) R_j(t)Q_j(t)\sigma_j dW(t)
\]

\[
+ \sum_{j=1}^m \Lambda_j(t^-) R_j(t^-) \tilde{\varphi}_j d\tilde{N}(t) - p(t)dt. \tag{2.6}
\]

Here we point out that since the process \( R_j \) is continuous, its left-continuous version coincides with the process itself, and hence they are interchangeable.

The debt portfolio vector process \( \pi = (\pi_1, \cdots, \pi_m) \) is defined for \( j \in \{1, \cdots, m\} \) by

\[
\pi_j(t) := \frac{\Lambda_j(t) R_j(t)Q_j(t)}{X(t)}, \quad \forall X(t) > 0; \tag{2.7}
\]

from which we deduce that

\[
\pi_j(t^-) = \frac{\Lambda_j(t^-) R_j(t^-)Q_j(t^-)}{X(t^-)}, \quad \forall X(t^-) > 0. \tag{2.8}
\]

For completeness, we define \( \pi(t) := 0 \) if \( X(t) = 0 \), and \( \pi(t^-) := 0 \) if \( X(t^-) = 0 \). Here, \( \pi \in \mathbb{R}^m \) is \( \mathbb{F} \)-adapted and \( \pi_j \) represents the proportion of debt issued in foreign currency \( j \in \{1, \cdots, m\} \). Obviously, the proportion of debt in local currency \( \pi_0 \in \mathbb{R} \) is given by \( \pi_0(t) = 1 - \sum_{j=1}^m \pi_j(t) \).

Expressing equation (2.6) in a compact manner, we have the following government debt dynamics:

\[
dX(t) = X(t) r_0 dt + X(t) \pi^T(t) b dt + X(t) \pi^T(t) \sigma dW(t) + X(t^-) \pi^T(t^-) \varphi d\tilde{N}(t) - p(t) dt, \tag{2.9}
\]

where \( X(0) = x > 0, b = r + \mu - r_0 \mathbf{1} \), with \( \mu = (\mu_1, \cdots, \mu_m) \) and \( r = (r_1, \cdots, r_m) \), with \( \mathbf{1} \) a vector of ones in \( \mathbb{R}^m \); We require the following technical assumption: \( E_x[\int_0^t \pi^T \sigma \pi ds] < \infty \) for every \( t > 0 \).

If we set \( \pi_j(t) = 0 \) for all \( j \in \{1, \cdots, m\} \) and \( t \geq 0 \) in equation (2.9), we recover the
dynamics of debt in one single currency given by equation (2.1). Thus, equation (2.9) is indeed an extension to the multi-currency debt dynamics.

We state our result in the next proposition.

**Proposition 1.** The stochastic differential equation for the government debt dynamics is given by

$$
\frac{dX(t)}{dt} = X(t) r_0 dt + X(t) \pi^T(t)b dt + X(t) \pi^T(t)\sigma dW(t) + X(t^-) \pi^T(t^-) \varphi d\hat{N}(t) - p(t) dt,
$$

(2.10)

where $X(0) = x > 0$; $b = r + \mu - r_0 I$; $\pi = (\pi_1, \cdots, \pi_m)$ is the vector of proportions of debt in foreign currencies, and $p$ is the debt payment rate process, i.e., the debt payment rate expressed in local currency.

We recall that $\varphi_j$ denote the $j$-th column of matrix $\varphi$. If we impose the technical condition that for every $s \geq 0$

$$
1 + \pi^T(s^-) \varphi_j > 0, \quad \forall j \in \{1, 2, \cdots, m\},
$$

(2.11)

the above linear stochastic differential equation for the debt $X$ possesses a unique explicit solution. Indeed, an application of Ito's formula to $X(t)/\xi(t)$ gives

$$
X(t) = \xi(t) \left( x - \int_0^t p(s)\xi(s)^{-1} ds \right), \quad \forall t \geq 0.
$$

(2.12)

Here,

$$
\xi(t) := \exp \left\{ \int_0^t \beta(s) ds + \int_0^t \pi^T(s)\sigma dW(s) + \sum_{j=1}^m \int_0^t \log \left( 1 + \pi^T(s^-) \varphi_j \right) dN_j(s) \right\},
$$

where

$$
\beta(s) := r_0 + \pi^T(s)b - \sum_{j=1}^m \lambda_j \pi^T(s) \varphi_j - \frac{1}{2} \pi^T(s)\sigma\sigma^T \pi(s),
$$

is the unique solution of the homogenous equation

$$
\frac{d\xi(t)}{dt} = \xi(t) r_0 dt + \xi(t) \pi^T(t)b dt + \xi(t) \pi^T(t)\sigma dW(t) + \xi(t^-) \pi^T(t^-) \varphi d\hat{N}(t),
$$

with initial condition $\xi(0) = 1$. Here $\log a$ stands for the natural logarithm of $a > 0$. We note that the process $\xi$ can be interpreted as the debt dynamics with initial value equal to one, and without any debt payments.

We now discuss the nature of the jumps of the debt process. Suppose the Poisson event $N_i$ occurs at the random time $J$. Then, equation (2.12) implies that

$$
\frac{X(J)}{X(J^-)} = \frac{\xi(J)}{\xi(J^-)}.
$$

Since
\[ \frac{\xi(J)}{\xi(J^-)} = \exp \left\{ \int_0^J \beta(s) ds + \int_0^J \pi^T(s) \sigma dW(s) + \int_0^J \log \left( 1 + \pi^T(s^-) \varphi_i \right) dN_i(s) \right\} \]

\[ = \left( 1 + \pi^T(J^-) \varphi_i \right), \]

we have

\[ X(J) = X(J^-) \left( 1 + \pi^T(J^-) \varphi_i \right). \quad (2.13) \]

By (2.11), the debt values before and after the jump \( X(J^-) \) and \( X(J) \), respectively, have the same sign. In particular, if the value before the jump is positive the debt process will never jump to a negative or zero value. Notice that the jump in \( X \) can be upward or downward.

From Proposition 1, the debt at time \( t \), \( X(t) \), depends on the following variables: interest rates of the bonds \( (r_0, r_1, \cdots, r_m) \), exchange rate depreciation \( (\mu_1, \cdots, \mu_m) \), portfolio currency composition process \( \pi = (\pi_1, \cdots, \pi_m) \), and debt payment rate process \( p \). Moreover, it also depends on the realizations of the random components of the exchange rates, i.e, the Brownian motion and the Poisson process, and their corresponding parameters \( \sigma \), \( \varphi \), and \( (\lambda_1, \cdots, \lambda_m) \). Thus, we have a realistic debt dynamics.

From the perspective of a developing country, the foreign interest rates and the randomness of the exchange rates are given. Furthermore, for a debt policy maker, the local interest rate is essentially exogenous\(^4\). On the other hand, the debt manager can exert control on the debt portfolio process \( \pi \) and, to some degree, on the debt payment rate process \( p \). They are precisely the control variables in our government debt problem that we will describe in the next subsection.

### 2.2 The Debt Problem

In reality there exists a debt problem as long as public debt is positive. Thus, we consider the following stopping time

\[ \Theta(\omega) := \inf \{ s \geq 0 : X(s^- , \omega) \leq 0 \} . \quad (2.14) \]

Since after \( \Theta \) there is no debt problem, we consider the following: if \( \Theta(\omega) \) is finite, we impose \( X_{\Theta}(\omega) := X_{\Theta^-} (\omega) \) and \( X_t(\omega) := 0 \) for all \( t > \Theta(\omega) \). In view of (2.11) and (2.13), we must have \( X_{\Theta^-} = 0 \). Thus, the dynamics of \( X \) is given by equation (2.10) for every \( t \in [0, \Theta) \),

\(^4\) From the country perspective, of course the government (but not necessarily the debt manager) can influence, to some extent, the local interest rate.
Now we turn our attention to the running costs, or the disutility function of the government. This cost depends on the debt payments and the level of the existing debt. The existence of the former cost comes from the fact that, in order to get positive fiscal results, countries have to cut spendings and increase taxes. On the other hand, the cost linked to the existing debt exists because high debt can lead to debt problems in the future, a default or a debt crisis, for example.

We observe that the disutility function most widely used in different areas of economics is quadratic (see, for example, Kydland and Prescott 1977, Taylor 1979, Cadenillas and Zapatero 1999, and Cadenillas et al. 2013). This function represents an agent with risk aversion. More precisely, it has constant relative risk aversion (CRRA) equal to 1 (see Remark 2 below). Since we are interested in analyzing the effects of debt aversion on the optimal currency debt portfolio, we consider a function in which the debt aversion is a parameter. Specifically, for \( \gamma \in (0, \infty) \), we define the cost (or loss) function by

\[
h(x, p) := \alpha x^{\gamma+1} + p^{\gamma+1},
\]

where \( x \) represents the public debt and \( p \) the debt payment rate. Here, \( \alpha \in (0, \infty) \) is a parameter that represents the importance that the government gives to the existing debt \( x \) relative to the debt rate payment \( p \). The function \( h \) has the property of CRRA equal to \( \gamma \) in \( x \) and \( p \). Thus, the parameter \( \gamma \) represents the aversion of the debt manager with respect to the existing debt and the debt payment. That is, for a given debt level and debt payment, the bigger the parameter \( \gamma \) the higher the disutility of the government. For instance, countries that have never had a default or have never suffered a severe debt crisis (such as Canada and USA) have a lower \( \gamma \) than countries that have experienced serious debt problems (such as Argentina and Greece). Thus, our specification of disutility function not only provides us with a unique parameter that characterizes fully the debt aversion of the government, but also generalizes the most common disutility function used in economics, namely, the quadratic function.

**Remark 2.** For a utility function \( u(y) \) the relative risk aversion is defined by \(-yu''(y)/u'(y)\) (see, for instance, MasCollel et al. 1995 or Pratt 1964). Similarly, for a disutility function \( d(y) \) we define the relative risk aversion as \(yd''(y)/d'(y)\).

Since \( X(t) = 0 \) for \( t > \Theta \), there is no debt problem after \( \Theta \). Hence, we also impose \( p(t) = 0 \) for \( t > \Theta \). Thus, the total cost after the debt becomes zero is null. Hence

\[
E_x \left[ \int_{\Theta}^{\infty} e^{-\delta t} \left( \alpha X_t^{\gamma+1} + p_t^{\gamma+1} \right) dt \right] = 0.
\]

Here \( \delta > 0 \) represents the discount rate.

As we discussed above, we will assume that the debt policy maker can exert control on both
the currency debt portfolio and the debt payment rate. We provide below the formal definition of control process.

Definition 1. Let \( u : [0, \infty) \times \Omega \to \mathbb{R}^m \times \mathbb{R} \) be a process defined by \( u(t, \omega) := (\pi(t, \omega), p(t, \omega)) \), where \( \pi \) is a portfolio debt process and \( p \) is a payment rate process, which are right-continuous with left-limits and adapted to \( \mathbb{F} \). For a given \( x > 0 \), the process \((\pi, p)\) will be called an admissible control process if it satisfies:

\[
(i) \quad \mathbb{E}[\int_0^\Theta e^{-\delta t} h(X_t, p_t) \, dt] < \infty, \\
(ii) \quad \left(1 + \pi^T(s^-) \varphi_j\right) > 0, \quad \forall j \in \{1, \cdots, m\}, \quad \forall s \geq 0, \\
(iii) \quad \mathbb{E}\left[\int_0^t \left(1 + \pi^T(s^-) \varphi_j\right)^{2(\gamma+1)} ds\right] < \infty, \quad \forall j \in \{1, \cdots, m\}, \quad \forall t \geq 0.
\]

The set of all admissible controls will be denoted by \( A(\xi) = A \).

Remark 3. We observe that condition (2.16) implies \( \mathbb{E}[\int_0^\Theta e^{-\delta t} X_t^{\gamma+1} \, dt] < \infty \). Then, we have

\[
\lim_{t \to \infty} \mathbb{E}\left[e^{-\delta t} X_t^{\gamma+1}\right] = 0. \tag{2.19}
\]

To complete this section, we state the debt problem.

Problem 1. Consider the debt dynamics given in Proposition 1. We want to select the admissible control \( \hat{u} = (\hat{\pi}, \hat{p}) \) that minimizes the performance functional given by

\[
J(x; u) = J(x; \pi, p) := \mathbb{E}_x \left[\int_0^\Theta e^{-\delta t} \left(\alpha X_t^{\gamma+1} + p_t^{\gamma+1}\right) \, dt\right]. \tag{2.20}
\]

The control \( \hat{u} = (\hat{\pi}, \hat{p}) \) will be called the optimal debt control.

From a mathematical point of view, Problem 1 is a stochastic control problem with jumps. This theory has been studied and/or applied, for instance, in Jeanblanc-Picqué and Pontier (1990), Cadenillas (2002), Guo and Xu (2004), and Oksendal and Sulem (2008).

To illustrate our framework, let us consider two examples. In the first one the debt manager chooses not to issue debt in foreign currencies and to pay only the interest rate of their current debt. The second example is a more general version of the first one.

Example 1. Suppose the debt manager chooses the following debt policy: \( \hat{\pi}(t) = 0 \) and \( \hat{p}(t) = r_0 X(t) \) for every \( t \geq 0 \). According to equation (2.10), the debt at every point in time equals the initial debt. That is, \( \forall t \geq 0 \)

\[
X(t) = x.
\]

This implies that \( X(t) > 0 \) for all \( t \geq 0 \), and hence \( \Theta = \infty \). Thus, the total discounted
government cost (disutility) is

\[ J(x; \tilde{\pi}, \tilde{p}) = \left( \frac{\alpha + r_0^{\gamma+1}}{\delta} \right) x^{\gamma+1}. \tag{2.21} \]

We point out that the debt policy \((\tilde{\pi}, \tilde{p})\) is admissible if and only if \(\delta > 0\). Since the latter condition is satisfied, this policy is admissible.

**Example 2.** Let \(\pi_c\) be an arbitrary constant vector in \(\mathbb{R}^m\) whose components are positive, and such that condition (2.17) is satisfied. Let \(p\) be an arbitrary positive real number. Suppose the debt manager considers the following debt policy: for every \(t \geq 0\), \(\tilde{\pi}(t) = \pi_c \in \mathbb{R}^m\) and \(\tilde{p}(t) = \rho X(t)\). Then, considering this debt policy in equation (2.10), the dynamics of the debt becomes

\[ dX(t) = (r_0 - \rho)X(t)dt + X(t)\pi_c^T b dt + X(t)\pi_c^T \sigma dW(t) + X(t^-)\pi_c^T \phi \tilde{N}(t), \]

with \(X_0 = x\). We note that the above stochastic differential equation (SDE) has the form of equation (2.10), except for the last term. Consequently, using equation (2.12) with \(p(s) = 0\) for \(s \geq 0\), the solution to the above SDE is given by

\[ X(t) = x \exp \left\{ \int_0^t \beta ds + \int_0^t \pi_c^T \sigma dW(s) + \sum_{j=1}^m \int_0^t \log \left( 1 + \pi_c^T \phi_j \right) dN_j(s) \right\}, \tag{2.22} \]

with

\[ \beta := r_0 - \rho - \sum_{j=1}^m \lambda_j \pi_c^T \phi_j - \frac{1}{2} \pi_c^T \sigma \sigma^T \pi_c; \]

from which we observe that \(X(t) > 0\) for all \(t \geq 0\). Hence \(\Theta = \infty\). Then, the discounted government disutility is

\[ J(x; \pi_c, p) = \mathbb{E}_x \left[ \int_0^\infty e^{-\delta t} h(X_t) dt \right] = \mathbb{E}_x \left[ \int_0^\infty e^{-\delta t} \left( \alpha X_t^{\gamma+1} + \rho^{\gamma+1} X_t^{\gamma+1} \right) dt \right] = (\alpha + \rho^{\gamma+1}) \int_0^\infty e^{-\delta t} \mathbb{E}_x \left[ X_t^{\gamma+1} \right] dt = (\alpha + \rho^{\gamma+1}) x^{\gamma+1} \int_0^\infty e^{-(\gamma+1)\rho + \zeta} t dt \]

\[ = \left\{ \begin{array}{ll} \frac{(\alpha + \rho^{\gamma+1}) x^{\gamma+1}}{(\gamma+1)\rho + \zeta} & \text{if } (\gamma+1)\rho + \zeta > 0, \\ \infty & \text{if } (\gamma+1)\rho + \zeta \leq 0. \end{array} \right. \]

To get the fourth equality above, we have used the result computed in Lemma 1 (Appendix B), with

\[ \zeta := \delta - 1/2 (\gamma+1)\pi_c^T \sigma \sigma^T \pi_c - \sum_{j=1}^m \lambda_j \left\{ (1 + \pi_c^T \phi_j)^{(\gamma+1)} - 1 - (\gamma+1)\pi_c^T \phi_j \right\} - (\gamma+1)(\pi_c^T b + r_0). \]
Thus, the debt policy given in Example 2 is admissible if and only if \((\gamma + 1)\rho + \zeta > 0\). It would be interesting to compare those policies with the optimal debt policy that we will obtain in Section 4. Certainly, we expect this type of arbitrary debt policies not to be, in general, optimal. We will confirm this fact in Section 5.

3 The Value Function and a Verification Theorem

The main purpose of this section is to state a sufficient condition that an optimal solution of the debt problem must satisfy. The value function is a key instrument to achieve that goal.

We define the value function \(V : (0, \infty) \rightarrow \mathbb{R}\) by

\[
V(x) := \inf_{(\pi, p) \in A} J(x; \pi, p). \quad (3.1)
\]

This represents the smallest expected cost that can be achieved when the initial debt is \(x > 0\) and we consider all the admissible debt controls.

**Proposition 2.** The value function \(V\) is non-negative and homogeneous of degree \(\gamma + 1\). Therefore, it is increasing, convex and \(V(0^+) = 0\).

**Proof.** See Appendix C.

We require some notation to define the Hamilton-Jacobi-Bellman equation. Let \(g : (0, \infty) \rightarrow \mathbb{R}\) be a function in \(C^2((0, \infty))\). For \(\pi \in \mathbb{R}^m\) and \(p \in \mathbb{R}\), let us define the operator \(\mathcal{L}(\pi, p)\) by

\[
\mathcal{L}(\pi, p)g(x) := \frac{1}{2}\pi^T \sigma \sigma^T \pi x^2 g''(x) + (\pi^T bx + r_0 x - p) g'(x) - \delta g(x) + \sum_{j=1}^m \lambda_j \left( g(x + \pi^T \varphi_j x) - g(x) - g'(x) \pi^T \varphi_j x \right), \quad (3.2)
\]

where we recall that \(\varphi_j\) is the \(j\)-th column of matrix \(\varphi\).

For a function \(v : (0, \infty) \rightarrow \mathbb{R}\) in \(C^2((0, \infty))\), consider the Hamilton-Jacobi-Bellman (HJB) equation

\[
\min_{(\pi, p) \in A} \left\{ \mathcal{L}(\pi, p)v(x) + h(x, p) \right\} = 0. \quad (3.3)
\]

Next we state a sufficient condition for a debt policy to be optimal.

**Theorem 1.** Let \(v \in C^2((0, \infty))\) be an increasing and convex function on \((0, \infty)\) with \(v(0^+) = 0\). Suppose that \(v\) satisfies the HJB equation (3.3) for every \(x \in (0, \infty)\), and the polynomial growth condition

\[
v(x) \leq C(1 + x^{\gamma+1}), \quad (3.4)
\]
for some constant $C$. Then, for every $x \in (0, \infty)$, we have the following two results.

(a) For every $(\pi, p) \in \mathcal{A}(x)$:

$$v(x) \leq J(x; \pi, p).$$

(b) Suppose that the stochastic control $\hat{u} = (\hat{\pi}, \hat{p})$, defined by

$$\hat{u} = (\hat{\pi}, \hat{p}) := \arg\min_{(\pi, p) \in \mathcal{A}} \left\{ \mathcal{L}(\pi, p) v(x) + h(x, p) \right\}, \quad (3.5)$$

is admissible for $X = \hat{X}$ and $\theta = \hat{\Theta}$. Then

$$v(x) = J(x; \hat{\pi}, \hat{p}).$$

In other words, $(\hat{\pi}, \hat{p})$ is the optimal debt control and $V = v$ is the value function for Problem 1. Here, $\hat{X}$ is the debt process generated by the control $(\hat{\pi}, \hat{p})$, and $\hat{\Theta}$ is the corresponding stopping time defined in (2.14).

Proof. See Appendix D. \hfill \Box

4 The Explicit Solution

At the beginning of this section we are going to make conjectures to obtain a candidate for optimal debt control and a candidate for value function. At the end of this section, we are going to apply Theorem 1 to prove rigorously that the candidate for optimal control is indeed the optimal control, and the candidate for value function is indeed the value function.

We want to find a control $(\pi, p)$ and the corresponding function $v$ that satisfy the conditions of Theorem 1. According to equation (3.5) in that theorem,

$$\hat{\pi} = \arg\min_{\pi} \left[ \frac{1}{2} x^2 \pi^T \sigma \sigma^T \pi v''(x) + x \pi^T b v'(x) + \sum_{j=1}^{m} \lambda_j \left( v(x + \pi^T \varphi_j x) - v'(x) \pi^T \varphi_j x \right) \right],$$

$$\hat{p} = \arg\min_{p} \left[ -pv'(x) + h(x, p) \right].$$

Let us conjecture that $v$ is strictly convex. Then $\hat{\pi} \in \mathbb{R}^m$ satisfies the equation

$$\pi^T \sigma \sigma^T x^2 v''(x) + b^T x v'(x) + \sum_{j=1}^{m} \lambda_j \left( v'(x + \pi^T \varphi_j x) \varphi_j^T x - v'(x) \varphi_j^T x \right) = 0; \quad (4.1)$$

and for $\hat{p} \in \mathbb{R}$ we have:

$$-v'(x) + \frac{\partial h(x, p)}{\partial p} = 0.$$
Thus, if we know the value function \( v \), we can characterize the candidate for optimal debt control \((\hat{\pi}, \hat{p})\). Based on both the proof of Proposition 2 and the form of the disutility function \( h \), we conjecture that the value function for \( \gamma \in (0, \infty) \) is given by

\[
v(x) = Kx^{\gamma+1}, \tag{4.2}\]

for every \( x > 0 \) and some constant \( K > 0 \).

By means of this conjecture, using (4.1), we characterize \( \hat{\pi} \in \mathbb{R}^m \) as the vector that solves the following equation in \( \pi \):

\[
\gamma\sigma\sigma^T \pi + b + \sum_{j=1}^{m} \lambda_j \varphi_j \left\{(1 + \pi^T \varphi_j)^\gamma - 1\right\} = 0; \tag{4.3}\]

and for \( \hat{p} \in \mathbb{R} \) we have:

\[
\hat{p}(t) = K^{\frac{1}{\gamma}} \hat{X}(t), \tag{4.4}\]

where \( \hat{X} \) is the debt dynamics generated by the debt policy \((\hat{\pi}, \hat{p})\). Since all the entries in \( \sigma \) and \( \varphi \) are constants, we observe that the process \( \hat{\pi} \) is indeed a constant vector in \( \mathbb{R}^m \).

To complete the specification of the candidates, it remains to characterize the constant \( K \). By Theorem 1, we know that for the function \( v \) and the process \( \hat{u} = (\hat{\pi}, \hat{p}) \) to be suitable candidates for solutions, they should satisfy the HJB equation (3.3). Considering it along with equation (3.5), we must have

\[
\mathcal{L}(\hat{\pi}, \hat{p})v(x) + h(x, \hat{p}) = 0. \tag{4.5}\]

After simplifying the previous equation, we obtain the equivalent form:

\[
\gamma K^{\frac{\gamma+1}{\gamma}} + \hat{\zeta}K - \alpha = 0, \tag{4.6}\]

where

\[
\hat{\zeta} := \delta - \sum_{j=1}^{m} \lambda_j \left\{(1 + \pi^T \varphi_j)^{\gamma+1} - 1 - (\gamma + 1)\pi^T \varphi_j \right\} - 1/2 (\gamma + 1)\gamma\pi^T \sigma \sigma^T \pi
\]

\[- (\gamma + 1)(\pi^T b + r_0). \tag{4.7}\]

Here, we emphasize that the choice of \( K \) in (4.6) guarantees that \( v(x) = Kx^{\gamma+1} \) satisfies the HJB equation (3.3) for \((\hat{\pi}, \hat{p})\). Let us prove that indeed \( K > 0 \). Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by

\[
f(z) := \gamma z^{\frac{\gamma+1}{\gamma}} + \hat{\zeta}z - \alpha.
\]

We note that \( f \) is convex on \((0, \infty)\). Since \( f(0^+) = -\alpha < 0 \) and \( f(+\infty) = +\infty \), then there
exists a unique real number $K > 0$ such that $f(K) = 0$, as required.

To complete this section, we are going to prove rigorously that for $\gamma \in (0, \infty)$ the above candidate for optimal debt control (4.3)-(4.4) is indeed the optimal control, and the above candidate for value function (4.2) is indeed the value function. For such proof we need $(\gamma + 1)K^{1/\gamma} + \hat{\chi} > 0$, i.e.,

$$
\delta > - (\gamma + 1)K^{1/\gamma} + \sum_{j=1}^{m} \lambda_j \left\{ (1 + \hat{\pi}^T \varphi_j)^{\gamma+1} - 1 - (\gamma + 1)\hat{\pi}^T \varphi_j \right\} + 1/2 (\gamma + 1)\hat{\pi}^T \sigma \sigma^T \hat{\pi} - (\gamma + 1) \left( \hat{\pi}^T b + r_0 \right).
$$

We point out that the above condition is consistent with the real world, where governments are by far more concerned about the present than about the future. Indeed, the larger the $\delta$ the more concerned the government is about the present costs than the future costs. We also remark that imposing a condition on the discount rate is a common practice in infinite horizon models in macroeconomics and finance (see, for example, Romer 2002, Merton 1969, and Sotomayor and Cadenillas 2009).

**Theorem 2.** Let us assume condition (4.8). Suppose $\hat{\pi} \in \mathbb{R}^m$ satisfies:

(i) $\gamma \sigma \sigma^T \hat{\pi} + b + \sum_{j=1}^{m} \lambda_j \varphi_j \left\{ (1 + \hat{\pi}^T \varphi_j)^{\gamma} - 1 \right\} = 0$, \hspace{1cm} (4.9)

(ii) $1 + \hat{\pi}^T \varphi_j > 0$, for every $j \in \{1, \cdots, m\}$. \hspace{1cm} (4.10)

Let $\hat{p}$ be the process defined by

$$
\hat{p}(t) := K^{\frac{1}{\gamma}} \hat{X}(t),
$$

where $K$ is the unique positive real solution to equation (4.6), and $\hat{X}$ denotes the debt process generated by the debt control $\hat{u} = (\hat{\pi}, \hat{p})$, which is given by

$$
\hat{X}_t = x \exp \left\{ \beta t + \hat{\pi}^T \sigma W(t) + \sum_{j=1}^{m} \log \left( 1 + \hat{\pi}^T \varphi_j \right) N_j(t) \right\},
$$

with

$$
\beta := r_0 - K^{1/\gamma} + \hat{\pi}^T b - \sum_{j=1}^{m} \lambda_j \hat{\pi}^T \varphi_j - \frac{1}{2} \hat{\pi}^T \sigma \sigma^T \hat{\pi}.
$$

Then $\hat{u} = (\hat{\pi}, \hat{p})$ is optimal, and $V(x) = v(x) = Kx^{\gamma+1}$ is the value function for Problem 1.

**Proof.** To prove this theorem, it suffices to show that all conditions of Theorem 1 are satisfied. Regarding the candidate for value function, we get immediately that $v \in C^2((0, \infty))$ and $v(0+) = 0$. Moreover, since $K > 0$, $v$ is increasing and strictly convex on its domain. On the
other hand, by construction, we observe that the candidate $v$ also satisfies the HJB equation (3.3),

$$L(\hat{\pi}, \hat{p})v(x) + h(x, \hat{p}) = 0.$$ 

Taking $C = K$, condition (3.4) also holds. It remains to verify that conditions (2.16) and (2.18) are satisfied for $(\hat{\pi}, \hat{p})$. Since $\hat{\pi}$ is a constant vector, condition (2.18) is immediate. To show condition (2.16), we will use the result in Appendix B, equation (B.1), with $\rho = K^{1/\gamma}$. We note that, since $(\hat{\pi}, \hat{p})$ is one case of the class of debt policies given in Example 2, we know that $\hat{\Theta} = \infty$. Thus,

$$E_x \left[ \int_0^{\hat{\Theta}} e^{-\delta t} h(\hat{X}_t, \hat{p}_t) dt \right] = E_x \left[ \int_0^\infty e^{-\delta t} \left( \alpha X_t^{\gamma+1} + K^{(\gamma+1)/\gamma} \hat{X}_t^{\gamma+1} \right) dt \right]
= \left( \alpha + K^{(\gamma+1)/\gamma} \right) \int_0^\infty E[e^{-\delta t} X_t^{\gamma+1}] dt
= \left( \alpha + K^{(\gamma+1)/\gamma} \right) x^{\gamma+1} \int_0^\infty \exp \left\{ - \left( (\gamma + 1) K^{\frac{1}{\gamma}} \hat{\zeta} \right) t \right\}
< \infty,$$

where the last inequality follows from condition (4.8), that is, $(\gamma + 1) K^{1/\gamma} + \hat{\zeta} > 0$.

By virtue of Theorem 1, $(\hat{\pi}, \hat{p})$ is optimal and $v(x) = K x^{\gamma+1}$ is the value function for Problem 1.

We observe that the explicit solution for the debt process $\hat{X}$ given in Theorem 2 is an extension of a geometric Brownian motion to include jumps. This model is general enough to reflect the debt evolution in reality. For instance, if $\beta > 0$, then the theoretical trend generated by the model is consistent with the recent trend of most countries in which debt increases over time. Moreover, it is also a generalization of the basic stochastic model for the debt evolution presented in Greiner and Fincke (2009).

As an application of Theorem 2, we present below two particular cases in which the government debt portfolio can be obtained explicitly.

**Remark 4.** Let $\gamma = 1$. Then the value function is $V(x) = K x^2$, and the optimal debt control is given by

$$\hat{\pi}(s) = [\sigma \sigma^T + \sum_{j=1}^m \lambda_j \varphi_j \varphi_j^T]^{-1} (r_0 1 - r - \mu),$$

$$\hat{p}(s) = K \hat{\Theta}(s).$$

Here $\hat{X}$ is the debt process generated by the debt policy $(\hat{\pi}, \hat{p})$. The parameter $K$ is given by

$$K := \frac{\sqrt{c^2 + 4\alpha - c}}{2} > 0,$$
where
\[ c := \delta - 2r_0 + b^T \Gamma^{-1} b, \]
with \( \Gamma = [\sigma \sigma^T + \sum_{j=1}^{m} \lambda_j \varphi_j \varphi_j^T] \).

**Remark 5.** Suppose that there are no jumps in the exchange rates. Then the value function is \( V(x) = K x^{\gamma+1} \), and the optimal debt control is given by

\[
\hat{\pi}(s) = \frac{(\sigma \sigma^T)^{-1}(r_0 1 - r - \mu)}{\gamma},
\]
\[
\hat{p}(s) = K^{1/\gamma} \hat{X}(s).
\]

Here \( \hat{X} \) is the debt process generated by the debt policy \((\hat{\pi}, \hat{p})\). The parameter \( K \) is given by

\[
K := \frac{\sqrt{c^2 + 4\gamma \alpha - c}}{2\gamma} > 0,
\]
where
\[
c := \delta - (\gamma + 1)r_0 + \frac{\gamma + 1}{2\gamma} b^T (\sigma \sigma^T)^{-1} b.
\]

5  
**Economic Analysis**

In this section we analyze the effects of some parameters on both the optimal debt control and the value function. To simplify the analysis and facilitate the interpretation of the implications of the model, in this section we will consider two currencies: the local and one foreign currency. In other words, throughout this section \( m = 1 \).

We recall that the source of the jumps in our model comes from a Poisson process, whose jumps have size one. The parameter \( \varphi \) allows us to introduce arbitrary sizes of the jump. In other words, anytime the Poisson process jumps, the effect on the exchange rate depreciation (or devaluation) is \( \varphi \). Thus, to be consistent with empirical currency crises, we are going to assume \( \varphi > 0 \).

5.1  
**Optimal solution with two currencies**

Under the previous considerations, we have the following straightforward corollary of Theorem 2.

**Corollary 1.** Let \( m = 1 \) and \( \gamma \in (0, \infty) \). Suppose \( \hat{\pi}_1 \in \mathbb{R} \) satisfies:

\[
(i) \quad \gamma \sigma^2 \hat{\pi}_1 + b + \lambda \varphi \left( 1 + \hat{\pi}_1 \varphi \right) \gamma - 1 = 0, \tag{5.1}
\]
\[
(ii) \quad 1 + \hat{\pi}_1 \varphi > 0. \tag{5.2}
\]
Let \( \hat{p} \) be the process defined by
\[
\hat{p}(t) = K^{\frac{1}{\gamma}} \hat{X}(t),
\]
where \( \hat{X} \) denotes the debt process generated by the debt control \( \hat{u} = (\hat{\pi}_1, \hat{p}) \), and \( K \) is the unique positive real solution to equation
\[
\gamma K^{\frac{\gamma+1}{\gamma}} + \hat{\zeta}_1 K - \alpha = 0,
\]
where
\[
\hat{\zeta}_1 := \delta - \lambda \left\{ (1 + \hat{\pi}_1 \varphi)^{\gamma+1} - 1 - (\gamma + 1) \hat{\pi}_1 \varphi \right\} - 1/2(\gamma + 1)\gamma \sigma^2 \hat{\pi}_1^2 - (\gamma + 1)(\hat{\pi}_1 b + r_0).
\]

Let us assume \((\gamma + 1)K^{1/\gamma} + \hat{\zeta}_1 > 0\). Then \((\hat{\pi}_1, \hat{p})\) is the optimal debt control, and \( V(x) = v(x) = Kx^{\gamma+1} \) is the value function for Problem 1.

**Proof.** Set \( m = 1 \) in Theorem 2.

We know that developing countries hold positive proportions of foreign currency in their debt portfolio. With this fact in mind, we provide below a proposition which states a sufficient and necessary condition for a positive solution \( \hat{\pi}_1 \in \mathbb{R} \) to exist.

**Proposition 3.** Suppose all the assumptions in Corollary 1 are satisfied. Then we have \( b < 0 \) if and only if there exists a unique \( \hat{\pi}_1 > 0 \) such that (5.1) and (5.2) are satisfied. In either case \( \hat{\pi}_1 \in (0, \bar{\eta}) \), where \( \bar{\eta} := \frac{\lambda \varphi - b}{\gamma \sigma^2} \).

**Proof.** Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by
\[
f(z) := \gamma \sigma^2 z + \lambda \varphi (1 + z \varphi)^\gamma - (\lambda \varphi - b).
\]
Note first that \( f(0) = b \) and \( f(+\infty) = +\infty \). Suppose \( b < 0 \). Since \( f \) is strictly increasing and continuous, there exists a unique \( \hat{\pi}_1 > 0 \) such that \( f(\hat{\pi}_1) = 0 \). Now suppose that there exists a unique \( \hat{\pi}_1 > 0 \) such that (5.1) and (5.2) are satisfied. Then, from
\[
-b = f(\hat{\pi}_1) - b = \gamma \sigma^2 \hat{\pi}_1 + \lambda \varphi (1 + \hat{\pi}_1 \varphi)^\gamma - \lambda \varphi > 0,
\]
we conclude that \( b < 0 \). To show the \( \bar{\eta} \) is an upper bound, observe that condition (5.1) implies
\[
\lambda \sigma^2 (\bar{\eta} - \hat{\pi}_1) = (\lambda \varphi - b) - \gamma \sigma^2 \hat{\pi}_1 = \lambda \varphi (1 + \hat{\pi}_1 \varphi)^\gamma > 0.
\]
\[\square\]
Proposition 3 says that to capture the reality of developing countries we need to assume $b < 0$. Otherwise, we do not get the empirical fact that the proportion of foreign debt in the government debt portfolio is positive. Accordingly, from now on we assume $b = r + \mu - r_0 < 0$. That is, the interest rate of domestic currency debt $r_0$ is greater than the interest rate of foreign currency debt $r$ plus the rate of depreciation (or devaluation) of the exchange rate $\mu$.

5.2 Optimal policy versus arbitrary debt policies

We compare numerically the optimal debt policy we have obtained with the ones given in Examples 1 and 2 of Section 2. We will use the parameter values in Table 1 for the numerical computations.

<table>
<thead>
<tr>
<th>Table 1: Parameter values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi$</td>
</tr>
<tr>
<td>0.2</td>
</tr>
</tbody>
</table>

Specifically, we will consider three debt policies chosen arbitrarily, and denoted by $(\pi^{(i)}, p^{(i)})$, with $i \in \{1, 2, 3\}$. For every $t \geq 0$ we have $\pi^{(1)}(t) = 0$, $\pi^{(2)}(t) = 50\%$, $\pi^{(3)}(t) = 100\%$, and $p^{(i)}(t) = r_0 X(t)$, for $i \in \{1, 2, 3\}$. The corresponding disutilities are given by $J^{(i)}(x) := J(x; \pi^{(i)}, p^{(i)})$. We will compare them with the optimal debt policy $(\hat{\pi}, \hat{p})$ given in Corollary 1.

We recall that the value function $V$ represents the minimum disutility, and is a function of the initial debt. Hence, the cost $J(x; u^{(i)})$ must be greater than or equal to $V(x)$.

<table>
<thead>
<tr>
<th>Table 2: Disutility values of some debt policies and value function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
</tr>
<tr>
<td>0.8</td>
</tr>
<tr>
<td>1.0</td>
</tr>
<tr>
<td>1.2</td>
</tr>
</tbody>
</table>

*See Table 1 for the values of the other parameters used in these computations.

Table 2 shows that indeed $J^{(i)}(x) > V(x)$ for $i \in \{1, 2, 3\}$. As shown in Table 3, the optimal currency debt portfolio and the optimal payments are sensitive to the degree of debt aversion $\gamma$. In particular, the higher the degree of debt aversion, the lower the proportion of foreign currency debt in the government portfolio. Indeed, this result will be proved rigorously in the next subsection.

After illustrating that the government gets the best result by following the optimal debt policy given in Corollary 1, we proceed with the economic analysis.
Table 3: The optimal solution

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\hat{\pi}(t)$</th>
<th>$\hat{p}(t)$</th>
<th>$V(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.8$</td>
<td>$0.59459$</td>
<td>$0.06655,X(t)$</td>
<td>$0.11443,x^{1.8}$</td>
</tr>
<tr>
<td>$1.0$</td>
<td>$0.47058$</td>
<td>$0.10889,X(t)$</td>
<td>$0.10889,x^{2.0}$</td>
</tr>
<tr>
<td>$1.2$</td>
<td>$0.38936$</td>
<td>$0.14952,X(t)$</td>
<td>$0.10225,x^{2.2}$</td>
</tr>
</tbody>
</table>

$^*$See Table 1 for the values of the other parameters used in these computations.

5.3 Economic results

We will state and prove two economic results. We will use the Intermediate Value Theorem to prove them.

ECONOMIC RESULT 1 This economic result shows the effects of jumps in the exchange rates on the optimal currency debt portfolio. Let $b < 0$. Let $\hat{\pi}(\lambda, \varphi)$ represent the optimal proportion of foreign currency debt when the intensity of the Poisson process $N$ is $\lambda$, and the size of the jumps is $\varphi$. Then

(i) For every $\varphi > 0$, $\lambda_1 > \lambda_0$ implies $\hat{\pi}(\lambda_1, \varphi) < \hat{\pi}(\lambda_0, \varphi)$,

(ii) For every $\lambda > 0$, $\varphi_1 > \varphi_0$ implies $\hat{\pi}(\lambda, \varphi_1) < \hat{\pi}(\lambda, \varphi_0)$.

The Economic Result 1 implies that countries with debt in a foreign currency, whose exchange rate shows recurrent and big depreciations (or devaluations), will reduce its proportion of this type of debt in their portfolio in favor of debt in local currency.

Proof. Economic Result 1. (i) Let $f: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$f(z, \lambda) := \gamma \sigma^2 z + \lambda \varphi (1 + z \varphi) - (\lambda \varphi - b).$$

We note that $f$ is strictly increasing in each of its arguments. By Corollary 1, and definition of $\hat{\pi}(\lambda_0, \varphi)$, we have $f(\hat{\pi}(\lambda_0, \varphi), \lambda_0) = 0$. Since $f$ is strictly increasing in $\lambda$,

$$f(\hat{\pi}(\lambda_0, \varphi), \lambda) > f(\hat{\pi}(\lambda_0, \varphi), \lambda_0) = 0.$$

On the other hand, we observe that

$$f(0, \lambda_1) = b < 0.$$

Applying the Intermediate Value Theorem to $f(\cdot, \lambda_1)$, there exists a unique $z_0 \in (0, \hat{\pi}(\lambda_0, \varphi))$ such that $f(z_0, \lambda_1) = 0$. By the uniqueness of the solution given in Proposition 3, we must have $z_0 = \hat{\pi}(\lambda_1, \varphi)$. Hence the claim holds.
(ii) This claim can be established in a similar manner.

For a special case, we provide below the explicit effect of $\lambda$ and $\varphi$ on the optimal proportion of foreign debt.

**Remark 6.** Let us consider $\gamma = 1$ and $m = 1$. Then, using Remark 4 of Section 4, we have

\[
\frac{\partial \hat{\pi}}{\partial \lambda} = \frac{b\varphi^2}{(\sigma^2 + \lambda\varphi^2)^2} < 0,
\]

\[
\frac{\partial \hat{\pi}}{\partial \varphi} = \frac{2\lambda\varphi b}{(\sigma^2 + \lambda\varphi^2)^2} < 0.
\]

**Economic Result 2** This economic assertion is concerned with the effects of debt aversion on the optimal currency debt portfolio. Let $b < 0$. For $\gamma \in (0, \infty)$, let $\hat{\pi}(\gamma)$ be the optimal proportion of foreign currency debt when the degree of debt aversion is $\gamma$. Then $\hat{\pi}(\cdot)$ is strictly decreasing. Moreover, $\lim_{\gamma \to \infty} \hat{\pi}(\gamma) = 0$.

The Economic Result 2 states the role played by the degree of debt aversion on the optimal currency debt portfolio. We observe that an increase in debt aversion leads to a decrease in the proportion of foreign currency in the portfolio, and hence an increase of the proportion of debt in local currency. In other words, an increase in the degree of debt aversion discourages countries from borrowing debt in foreign currency. In addition, it is interesting to note that an extreme debt aversion leads to decide not to borrow in foreign currency.

**Proof.** (Economic Result 2). Let $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be defined by

\[
f(z, \gamma) := \gamma\sigma^2 z + \lambda\varphi(1 + z\varphi)^{\gamma} - (\lambda\varphi - b).
\]

Consider $\gamma_1 > \gamma_0$. We note first that, for each fixed $z > 0$ we have $f(z, \gamma_1) > f(z, \gamma_0)$ and $\lim_{\gamma \to \infty} f(z, \gamma) = \infty$. We also note that for each fixed $\gamma > 0$, the function $f(\cdot, \gamma)$ is strictly increasing. Moreover, by definition of $\hat{\pi}(\gamma_0)$, $f(\hat{\pi}(\gamma_0), \gamma_0) = 0$. Then

\[
f(\hat{\pi}(\gamma_0), \gamma_1) > f(\hat{\pi}(\gamma_0), \gamma_0) = 0.
\]

On the other hand, $f(0, \gamma_1) = b < 0$. Now applying the Intermediate Value Theorem to $f(\cdot, \gamma_1)$, there exists a unique real number $z \in (0, \hat{\pi}(\gamma_0))$ such that $f(z, \gamma_1) = 0$. By the uniqueness of the solution given in Proposition 3, we must have $z = \hat{\pi}(\gamma_1)$. Hence,

\[0 < \hat{\pi}(\gamma_1) < \hat{\pi}(\gamma_0).
\]

To prove the second claim, we note that from Proposition 3, we have $0 < \hat{\pi}_1(\gamma) < \bar{\eta}$. The proof concludes after observing that $\lim_{\gamma \to \infty} \bar{\eta} = \lim_{\gamma \to \infty} \frac{\lambda\varphi - b}{\lambda\varphi} = 0$. 


We show below that, in a special case, the effect of the degree of debt aversion can be computed explicitly.

**Remark 7.** Suppose $m = 1$ and no jumps. Using Remark 5 of Section 4, we get

$$\frac{d\hat{\pi}}{d\gamma} = \frac{b}{\gamma^2 \sigma^2} < 0,$$

$$\lim_{\gamma \to \infty} \hat{\pi}(\gamma) = \lim_{\gamma \to \infty} -\frac{b}{\gamma \sigma^2} = 0.$$ 

6 Conclusion

We have made two contributions in this paper. On the theoretical side, we have developed for the first time a model for government debt control (that includes jumps in the exchange rates and debt aversion), to find explicitly the optimal currency debt portfolio and optimal debt payments. On the applied side, this model provides a rigorous explanation of the consistent reduction in the proportion of foreign currency in government debt portfolios in favor of local currency. Specifically, we have found that high debt aversion and jumps in the exchange rates explain that behavior.

A Debt dynamics in discrete time

To motivate the derivation of the multi-currency debt dynamics in continuous time, in this appendix we present a discrete-time example. Before doing so, we discuss the connection between the price of a bond and the interest charges on the corresponding debt. Suppose that at time $t$ the government issues one unit of a bond in local currency. That is, the debt issued is $R_0(t)$. We want to calculate the interest charges of this debt at time $t + \Delta t$. From equation (2.4) we know that $R_0(t + \Delta t) - R_0(t) = R_0(t)(e^{r_0\Delta t} - 1) \approx R_0(t)r_0\Delta t$. Thus, the difference of the prices of the bond in local currency represents the interest charges on debt in local currency. Now suppose that the government issues one unit of a bond in foreign currency $j$ instead. We want to calculate the interest charges of this debt at time $t + \Delta t$ expressed in local currency. In this case the answer is given by the difference $R_j(t + \Delta t)Q_j(t + \Delta t) - R_j(t)Q_j(t)$.

Suppose that $m = 1$ and the change of bond prices and debt payments occur only at times $t \in \{0, 1, 2\}$. By definition of total debt given in equation (2.2), at time $t = 0$,

$$X(0) = \Lambda_0(0)R_0(0) + \Lambda_1(0)R_1(0)Q_1(0).$$

Considering the prices of the bonds, the exchange rates, and possible payments, the debt at
time $t = 1$ is:

$$X(1) = \Lambda_0(0)R_0(1) + \Lambda_1(0)R_1(1)Q_1(1) - p(1).$$

We note that the debt at time $t = 1$ is determined by the debt issued at time 0, i.e. $\Lambda_0(0)$ and $\Lambda_1(0)$, the new prices of the bonds $R_0(1)$ and $R_1(1)$, and the exchange rate $Q_1(1)$. Moreover, the payment $p(1)$ reduce the total debt. Based on the previous two equations, we can recast the total debt at time $t = 1$ as

$$X(1) = X(0) + \Lambda_0(0)[R_0(1) - R_0(0)] + \Lambda_1(0)[R_1(1)Q_1(1) - R_0(0)Q_0(0)] - p(1). \quad (A.1)$$

This form states that the total debt at time 1 is the sum of the initial total debt $X(0)$, the interest of debt in local currency $\Lambda_0(0)[R_0(1) - R_0(0)]$, and the interest of debt in the foreign currency expressed in local currency $\Lambda_1(0)[R_1(1)Q_1(1) - R_0(0)Q_0(0)]$, minus the debt payment $p(1)$.

At time $t = 1$ the number of bonds in both currencies need not be the same as at time 0. In fact, using equation (2.2) we have

$$X(1) = \Lambda_0(1)R_0(1) + \Lambda_1(1)R_1(1)Q_1(1),$$

where we recall that $\Lambda_0(1)$ and $\Lambda_1(1)$ are the number of bonds in local currency and foreign currency held at time 1, respectively. Considering the prices of the bonds, the exchange rates, and possible payments, the total debt at time $t = 2$ is

$$X(2) = \Lambda_0(1)R_0(2) + \Lambda_1(1)R_1(2)Q_1(2) - p(2).$$

Combining the previous two equations, we arrive at

$$X(2) = X(1) + \Lambda_0(1)[R_0(2) - R_0(1)] + \Lambda_1(1)[R_1(2)Q_1(2) - R_1(1)Q_1(1)] - p(2). \quad (A.2)$$

Using equations (A.1) and (A.2), we see that the total debt at time $t = 2$ can be written as

$$X(2) = X(0) + \sum_{i=1}^{2} \Lambda_0(i-1)[R_0(i) - R_0(i-1)]$$

$$+ \sum_{i=1}^{2} \Lambda_1(i-1)[R_1(i)Q_1(i) - R_1(i-1)Q_1(i-1)] - \sum_{i=1}^{2} p(i). \quad (A.3)$$
Lemma 1. Let $\rho$ be a positive real number, and let $\pi$ be a constant vector in $\mathbb{R}^m$ such that

$$(1 + \pi^T \varphi_j) > 0, \quad \forall j \in \{1, \cdots, m\}.$$ 

Consider the following debt policy: $\pi(t) = \pi$ and $p(t) = \rho X(t)$ for every $t \geq 0$. Then

$$E_x \left[ e^{-\delta t} X_t^{\gamma+1} \right] = x^{\gamma+1} \exp\{-((\gamma + 1)\rho + \zeta)t\}dt,$$

(B.1)

where

$$\zeta := \delta - 1/2 \left( \gamma + 1 \right) \gamma \pi^T \sigma \sigma^T \pi - \sum_{j=1}^{m} \lambda_j \left((1 + \pi^T \varphi_j)^{\gamma+1} - 1 - (\gamma + 1)\pi^T \varphi_j \right) - (\gamma + 1)\left(\pi^T b + r_0\right).$$

Proof. According to equation (2.22), the solution of the SDE that corresponds to $(\pi, p)$ is given by

$$X(t) = x \exp \left\{ \int_0^t \beta ds + \int_0^t \pi^T \sigma dW(s) + \sum_{j=1}^{m} \int_0^t \log (1 + \pi^T \varphi_j) dN_j(s) \right\},$$

where

$$\beta := r_0 - \rho + \pi^T b - \sum_{j=1}^{m} \lambda_j \pi^T \varphi_j - \frac{1}{2} \pi^T \sigma \sigma^T \pi.$$

Since all the entries in $\sigma$, $\varphi$, and $\pi$ are constants,

$$X(t) = x \exp \left\{ \beta t + \pi^T \sigma W(t) + \sum_{j=1}^{m} \log (1 + \pi^T \varphi_j) N_j(t) \right\}.$$

Thus, using the fact that $N = \{N_j : j = 1, \cdots, m\}$ and $W = \{W_j : j = 1, \cdots, m\}$ are independent, we have

$$E_x \left[ e^{-\delta t} X_t^{\gamma+1} \right] =$$

$$x^{\gamma+1} \exp \left\{ (\gamma + 1)\beta - \delta \right\} E_x \left[ \exp \left\{ (\gamma + 1)\pi^T \sigma W(t) \right\} \right] E_x \left[ \exp \left\{ \sum_{j=1}^{m} \log(1 + \pi^T \varphi_j)^{\gamma+1} N_j(t) \right\} \right].$$

Computing the expected values on the right hand side of the above equation, we obtain

$$E_x \left[ \exp\{(\gamma + 1)\pi^T \sigma W(t)\} \right] = \exp\{1/2 (\gamma + 1)^2 \pi^T \sigma \sigma^T \pi t\};$$

and for $j \in \{1, \cdots, m\}$:

$$E_x \left[ \exp \left\{ \log(1 + \pi^T \varphi_j)^{\gamma+1} N_j(t) \right\} \right] = \exp \left\{ \lambda_j t \left( (1 + \pi^T \varphi_j)^{\gamma+1} - 1 \right) \right\}.$$
Consequently,
\[ E_x \left[ e^{-\delta t} X_t^\gamma \right] = x^{\gamma + 1} \exp \left\{ - \left( (\gamma + 1) \rho + \xi \right) t \right\}. \]

C Proof of Proposition 2

Proof. \( V \) is non-negative by definition. To show the homogeneity property of \( V \), we note that equation (2.12) implies that \( \forall \nu > 0: \)
\[ J(\nu x; \pi, \nu p) = \nu^{\gamma + 1} J(x; \pi, p). \]

Consequently,
\[ V(\nu x) := \inf_{(\pi, \nu p) \in \mathcal{A}} J(\nu x; \pi, \nu p) = \nu^{\gamma + 1} \inf_{(\pi, p) \in \mathcal{A}} J(x; \pi, p) = \nu^{\gamma + 1} V(x). \]

Taking \( \nu = 1/x \) in the above equation, we obtain \( V(x) = x^{\gamma + 1} V(1) \). Since \( x^{\gamma + 1} \) is strictly increasing and convex, the proposition follows.

D Proof of Theorem 1

Proof. Let \((\pi, p)\) be an admissible control process, whose corresponding debt dynamics is given by equation (2.10). Let \( \tau \) be a stopping time such that \( \tau \leq \Theta \), and let us consider the non-negative process \( \{X_t^{\wedge \tau} : t \geq 0\} \). Since \( v \) is twice continuously differentiable, an application of Ito’s formula (Ikeda and Watanabe 1981) yields
\[
e^{-\delta (t \wedge \tau)} v(X_{t \wedge \tau}) = v(X_0) + \int_0^{t \wedge \tau} e^{-\delta s} \left( \mathcal{L}(\pi_s, p_s) v(X_s) \right) ds + M(t) + \sum_{j=1}^m M_j(t), \quad (D.1)
\]
where \( \mathcal{L} \) is the operator defined in (3.2), and
\[
M(t) := \int_0^{t \wedge \tau} e^{-\delta s} v'(X_s) \pi_s^T \varphi X_s dW(s),
\]
\[
M_j(t) := \int_0^{t \wedge \tau} e^{-\delta s} \left( v(X_s^- + \pi_s^T \varphi_j X_s^-) - v(X_s^-) \right) d\tilde{N}_j(s),
\]
for every \( j \in \{1, 2, \cdots, m\} \).

Let \( a, b \) satisfy \( 0 < a < X_0 = x < b < \infty \). We define \( \tau_a := \inf \{ s \geq 0 : X_s^- \leq a \} \),

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Letting $\tau_b := \inf\{s \geq 0 : X_s^- \geq b\}$, and $\tau_{ab} := \tau_a \wedge \tau_b$. We observe that $\tau_{ab} \leq \Theta$. Hence, for $s < \tau_{ab}$ we have that both $X_s$ and $X_{s^-}$ belong to the interval $[a, b]$. In view of (2.11) and (2.13), if $\tau_a < \infty$, then we have $0 < X_{\tau_a} \leq a$.

From now on, we set $\tau = \tau_{ab}$. We claim that the above stochastic integrals $M$ and $M_j$ are martingales. Indeed, recalling that $X_j \in \{a, b\}$, it follows that $\pi^T \sigma \sigma^T \pi_j X_s^2 ds \leq \xi E\left[\int_0^\tau \pi^T \sigma \sigma^T \pi_j X_s^2 ds\right] < \infty$.

Hence the integral with respect to the Brownian motion above is a martingale. Similarly, we prove the other part of the claim, that is, $M_j$ is a martingale. It suffices to show that for each $j \in \{1, \cdots, m\}$

$$E_x\left[\int_0^{\tau_{ab}} e^{-2\delta s} \lambda_j \left(v(X_s^- + \pi^T \varphi_j X_s^-) - v(X_{s^-})\right)^2 ds\right] < \infty.$$ 

Indeed, recalling that $X_s^- \in [a, b]$ for all $s \in [0, \tau)$, and $v$ is continuous, we have that $E_x\left[\int_0^{\tau_{ab}} e^{-2\delta s} \lambda_j \left(v(X_{s^-})\right)^2 ds\right]$ is bounded. Using equations (2.18) and (3.4), we conclude that $E_x\left[\int_0^{\tau_{ab}} e^{-2\delta s} \lambda_j \left(v(X_s^- + \pi^T \varphi_j X_s^-)\right)^2 ds\right]$ is also bounded. Hence, the previous inequality holds. This proves that the integrals with respect to the compensated Poisson process are martingales. This completes the proof of the claim. Consequently, for every $t \geq 0$, we have $E_x[M(t)] = 0$, and $E_x[M_j(t)] = 0$ for each $j \in \{1, \cdots, m\}$.

Taking expectations in equation (D.1),

$$E_x\left[e^{-\delta (t \wedge \tau_{ab})} v(X_{t \wedge \tau_{ab}})\right] = v(x) + E_x\left[\int_0^{t \wedge \tau_{ab}} e^{-\delta s} \left(L(\pi_s, p_s) v(X_s)\right) ds\right].$$

Since the HJB equation (3.3) implies that for $s \in [0, \tau)$

$$L(\pi_s, p_s) v(X_s) \geq -h(X_s, p_s), \quad (D.2)$$

it follows that

$$v(x) \leq E_x\left[e^{-\delta (t \wedge \tau_{ab})} v(X_{t \wedge \tau_{ab}})\right] + E_x\left[\int_0^{t \wedge \tau_{ab}} e^{-\delta s} h(X_s, p_s) ds\right]. \quad (D.3)$$

Letting $b \uparrow \infty$, we have $\tau_b \uparrow \infty$, and hence $\tau_{ab} \uparrow \tau_a$. By the Monotone Convergence Theorem,

$$\lim_{b \to \infty} E_x\left[\int_0^{t \wedge \tau_{ab}} e^{-\delta s} h(X_s, p_s) ds\right] = E_x\left[\int_0^{t \wedge \tau_a} e^{-\delta s} h(X_s, p_s) ds\right].$$
On the other hand, since $v$ is continuous, letting $b \uparrow \infty$, we obtain

$$v(X_{t \wedge \tau_{ab}}) \rightarrow v(X_{t \wedge \tau_{a}}).$$

Moreover, since $v$ satisfies the polynomial growth condition (3.4), using the strategy in Fleming and Soner (2006) (page 157), one can show that the following family of random variables

$$\{v(X_{t \wedge \tau_{ab}}) : b \in (x, \infty)\}$$

is uniformly integrable. Since $e^{-\delta (t \wedge \tau_{ab})}$ is bounded,

$$\lim_{b \to \infty} E_x \left[ e^{-\delta (t \wedge \tau_{ab})} v(X_{t \wedge \tau_{ab}}) \right] = E_x \left[ \lim_{b \to \infty} e^{-\delta (t \wedge \tau_{ab})} v(X_{t \wedge \tau_{ab}}) \right] = E_x \left[ e^{-\delta (t \wedge \tau_{a})} v(X_{t \wedge \tau_{a}}) \right].$$

Consequently, taking the limit as $b \uparrow \infty$ in inequality (D.3), we get

$$v(x) \leq E_x \left[ e^{-\delta (t \wedge \tau_{a})} v(X_{t \wedge \tau_{a}}) \right] + E_x \left[ \int_0^{t \wedge \tau_{a}} e^{-\delta s} h(X_s, p_s) ds \right]. \quad (D.4)$$

Now letting $a \downarrow 0$, we have $\tau_a \uparrow \Theta$. Recalling that $v(0+) = 0$, taking the limit as $a \downarrow 0$, and proceeding as in the case $b \uparrow \infty$, we obtain

$$\lim_{a \to 0} E_x \left[ \int_0^{t \wedge \tau_{a}} e^{-\delta s} h(X_s, p_s) ds \right] = E_x \left[ \int_0^{t \wedge \Theta} e^{-\delta s} h(X_s, p_s) ds \right],$$

and

$$\lim_{a \to 0} E_x \left[ e^{-\delta (t \wedge \tau_{a})} v(X_{t \wedge \tau_{a}}) \right] = E_x \left[ \lim_{a \to 0} e^{-\delta (t \wedge \tau_{a})} v(X_{t \wedge \tau_{a}}) \right] = E_x \left[ e^{-\delta t} v(X_t) I_{\{t < \Theta\}} \right].$$

Taking the limit as $a \downarrow 0$ in inequality (D.4), we get

$$v(x) \leq E_x \left[ e^{-\delta t} v(X_t) I_{\{t < \Theta\}} \right] + E_x \left[ \int_0^{t \wedge \Theta} e^{-\delta s} h(X_s, p_s) ds \right].$$

We note that (2.19) and (3.4) imply

$$\lim_{t \to \infty} E_x \left[ e^{-\delta t} v(X_t) I_{\{t < \Theta\}} \right] = 0.$$

Now letting $t \to \infty$, by the Monotone Convergence Theorem, we conclude that

$$v(x) \leq E_x \left[ \int_0^{\Theta} e^{-\delta s} h(X_s, p_s) ds \right]. \quad (D.5)$$

This proves part (a) of this Theorem.

To show part (b), we observe that, since $(\hat{\pi}, \hat{p})$ satisfies (3.5), the HJB equation (3.3) implies that the inequality (D.2) becomes an equality when $(\pi_s, p_s) = (\hat{\pi}_s, \hat{p}_s)$ and $X(s) = \hat{X}(s)$. Then, inequality (D.5) becomes an equality as well. This completes the proof of this Theorem.
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