Book Review

Rudolf Taschner, The Continuum. A Constructive Approach to Basic Concepts of Real Analysis. Vieweg-Verlag Wiesbaden 2005, XI, 136 pp., Hardcover, EUR 36.90, ISBN 3-8348-0040-6.

The real numbers, visualised as the unbroken, perfectly homogeneous real line, form the foundation upon which all of analysis rests. The logically consistent and rigorous construction of real numbers, starting out with nothing more than the naturals and their evident properties, ranks among the greatest intellectual achievements of mankind. It has not been achieved easily though. From the ancient Greeks' disturbing discovery of irrational numbers via the vagueness of infinitesimals to the fierce foundational debate of the early twentieth century, the understanding of what exactly real numbers are has gone through many a revolution and crisis. While hardly anyone would disagree that, after two and a half millenia, satisfactory clarification has finally been reached, there is truth also in the quote, attributed to G.-C. Rota, that "every generation re-examines the reals in the light of its values and mathematical objectives". It is under this quote that *The Continuum* sets out to present an approach to the real numbers in the decidedly constructive spirit of L.E.J. Brouwer and H. Weyl. To the latter's famed (and identically titled) German text of 1918 the author pays explicit deference.

In which respect is the formal theory of the real numbers, as advocated by G. Cantor and R. Dedekind, deficient? In its first chapter, *The Continuum* explains through striking examples why a statement about real numbers may be easy to prove *formally* yet at the same time may forever remain beyond the reach of even the most powerful *actual* verification. It becomes apparent that for all reasonable purposes most real numbers can only be dealt with by means of approximations, regardless of whether they are as prominent as π and e or as mysterious as γ. Together with historical remarks highlightening Weyl's role in the formation of a constructive theory of the continuum, these examples provide motivation for the subsequent rigorous part of the text. To develop the basic properties of real numbers, first a workable definition has to be given: a real number is simply a sequence (a_n) of n-digit decimal numbers (i.e., $10^n a_n$ is an integer) such that $|a_m - a_n| < 10^{-m} + 10^{-n}$ holds for all m and n. From this pragmatic definition all the well-known algebraic, order and topological properties of the continuum follow. So smoothly do they follow that the reader familiar with classical (i.e., formal) real analysis may fail to notice any discrepancies at all, except perhaps for some slightly unusual terminology. Discrepancies exist, however, and they become more pronounced as the text proceeds. Most of these discrepancies are addressed carefully, e.g. through a discussion of the concepts of apartness and locatedness which, though indispensable in the constructive context, are void in classical analysis. In a few instances, however, the presentation is not as clear. For example, the proof of the important bar theorem dissolves into a rather slick play with words, and the discussion of the amazing Weyl–Brouwer continuity theorem (which, simply put, asserts that a function between complete metric spaces is automatically continuous) would benefit from the simple observation that the classical analyst's obvious counterexample, the sign function, simply is not a function at all according to the constructive definition. Though unedifying, this is the only blemish on an overall pleasant text which leads elegantly and on little more than one hundred pages from the definition of a real number to compactness questions concerning families of continuous functions on arbitrary metric spaces (with the surprising Dini–Brouwer theorem as yet another gem). Since the text is quite fast-paced, and also since it does not offer any concrete exercises or problems, the reader should have absorbed some sound real analysis beforehand.

Much has been said in the past about the overall world outlook of mathematicians being either idealist, formalist, or contructivist, with the latter representing an almost negligible minority attitude. About a century ago, H. Poincaré declared that formalism was a sickness from which mathematics would have to recover. Even at the time, not everybody agreed, and today the formal-axiomatic view of the reals is evidently all-dominant. This, however, may change as computational aspects pervade and influence ever wider parts of mathematics. From a computational perspective, a constructive theory of the continuum is clearly preferable to a theory which for instance has no difficulties yielding the unique existence of a number with some property or other but which at the same time cannot possibly provide a means to find this number. The eminent constructive mathematician D. Bridges has argued that constructivism addresses mathematical reality, whereas idealism and formalism both deal with a form of mathematical virtual reality. Every reader of *The Continuum* is thus in for a reality-check as Taschner's elegant, solid text offers an eye-opening and refreshing view on supposedly well-known facts. It has been the author's objective to provide a natural, constructive and short path to the real numbers. Notwithstanding the minor reservations made earlier, the text largely lives up to its goals; it can be recommended for everyone seriously interested in the foundations of analysis.

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