ANZIAM J. 48(2006), 211–223

A CHARACTERISATION OF NEWTON MAPS

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(Received 10 December, 2005)

Abstract

Conditions are given for a C^k map T to be a Newton map, that is, the map associated with a differentiable real-valued function via Newton's method. For finitely differentiable maps and functions, these conditions are only necessary, but in the smooth case, that is, for $k = \infty$, they are also sufficient. The characterisation rests upon the structure of the fixed point set of T and the value of the derivative T' there, and it is best possible as is demonstrated through examples.

2000 *Mathematics subject classification*: primary 37N30, 39A11, 49M15; secondary 37E05, 65H05.
Keywords and phrases: Newton's method, Newton map, fixed point set, attracting fixed point.

1. Introduction

Newton's method (NM) for computing successive approximations of zeros of functions is one of the most widely used methods in all of applied mathematics; variants and generalisations also play a prominent role in numerous other disciplines [2, 3, 6, 8, 9]. Conceptually, NM becomes especially transparent within a dynamical systems context. The purpose of this brief note is to characterise, in the simplest possible setting, the local properties of the dynamical systems thus encountered.

Throughout, let $f : I \to \mathbb{R}$ be a differentiable function, defined on some open interval $I \subset \mathbb{R}$, and denote by N_f its associated NM transformation, that is,

$$N_f(x) = x - \frac{f(x)}{f'(x)}, \quad \text{for all } x \in I : f'(x) \neq 0; \tag{1.1}$$

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for N_f to be defined for every $x \in I$, set $N_f(x) := x$ whenever $f'(x) = 0$.

NM for finding roots (zeros) of f, that is, real numbers x^* with $f(x^*) = 0$, amounts to picking an initial point $x_0 \in I$ and iterating N_f, thus generating the sequence

$$x_n = N_f(x_{n-1}) = N_f^n(x_0), \quad \text{for all } n \in \mathbb{N},$$

where, here and throughout, for any map $T : I \to \mathbb{R}$ and any $n \in \mathbb{N}$, $T^n(x) = T\big(T^{n-1}(x)\big)$, provided that $T^{n-1}(x) \in I$, and $T^0(x) = x$. Note that $N_f(x) = x$ precisely if $f(x) f'(x) = 0$; that is, the only fixed points of N_f occur where either f or f' vanish. Thus for $f(x_n) f'(x_n) = 0$, and only then, does NM terminate at x_n. If $f(x_n) = 0$, a root has been found, and otherwise (1.1) breaks down due to a horizontal tangent to the graph of f at x_n (see Figure 1).

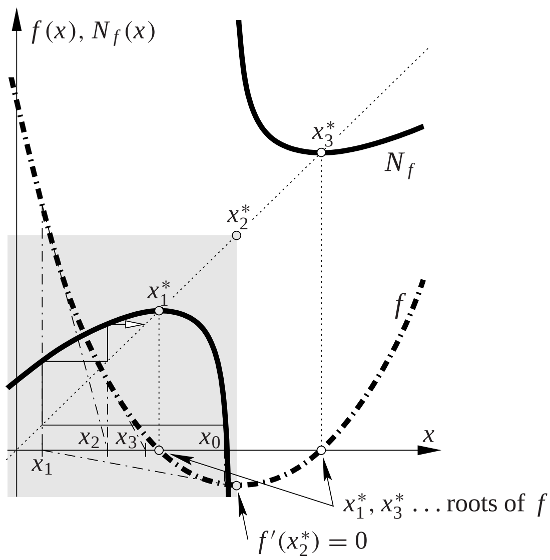

FIGURE 1. Visualising NM: The first few iterates x_1, x_2, x_3 are found graphically, both by means of tangents to the graph of f (broken line) and via the graph of N_f (solid line). Note how the point x_2^* with $f'(x_2^*) = 0$ causes N_f to have a discontinuity.

Clearly, if (x_n) converges to x^*, say, and if N_f is continuous at x^*, then $N_f(x^*) = x^*$, that is, x^* is a fixed point of N_f, and $f(x^*) = 0$. (The trivial alternative $f \equiv$ *const.* is tacitly excluded here, see Lemma 2.4 below.) It is this correspondence between the roots of f and the fixed points of N_f that suggests that NM be studied as a dynamical system. Under a mild assumption, each (isolated) fixed point x^* is *attracting*, that is, $\lim_{n\to\infty} N_f^n(x_0) = x^*$ for all x_0 sufficiently close to x^*. (For x_0 further away from any root, the sequence (x_n) may exhibit a considerably more complicated long-term behaviour [2, 3, 9].) This aspect of NM is put into perspective by the main result of the present paper, Theorem 3.2 below, which completely characterises the local dynamical properties of N_f.

2. Newton maps

The definition of a Newton map given below entails a relationship between the analytic properties of a function f and the analytic properties of its associated NM transformation N_f. It is a simple fact, rarely alluded to in studies of NM, that in general these properties are quite independent.

EXAMPLE 2.1. The function $f(x) = |x|^{3/2}$ is C^1 but not C^2, yet it has a C^∞ NM transformation, namely $N_f(x) = x/3$.

EXAMPLE 2.2. It is easily seen that the function

$$f(x) = \begin{cases} \exp\big(-x^{-2} + |x| + \cos(x^{-2})\big) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is C^∞, and both f and f' vanish only at $x^* = 0$. Nevertheless

$$-1 = \liminf_{x\to 0} N_f(x) < \limsup_{x\to 0} N_f(x) = 1,$$

hence N_f is not even *continuous* at x^*.

Since N_f may fail to be continuous even if f is C^∞, in order to ensure the applicability of NM, some explicit assumption on the smoothness of N_f has to be imposed. To formulate such conditions concisely, let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ and stipulate that $\infty^{-1} := 0$ and $\infty \pm j = \infty$ for all $j \in \mathbb{N}$.

In view of (1.1), for N_f to be C^l for some $l \in \mathbb{N}_\infty$, one might demand that f be at least C^{l+1}, but Examples 2.1 and 2.2 show that this assumption is neither necessary nor sufficient. Simply imposing further conditions on N_f also seems problematic as long as it is not clear whether any such condition is satisfied for a reasonably large class of functions. Thus it is inevitable to address the following general inverse problem: Given a C^l map T, does there exist a function f such that $T = N_f$?

DEFINITION 2.3. Let $I \subset \mathbb{R}$ be an open interval, and $l \in \mathbb{N}_\infty$. A map $T \in C^l(I)$ is called a *Newton map* (associated with f), if $T = N_f$ for some differentiable function $f : I \to \mathbb{R}$.

Clearly, not every $T \in C^l(I)$ is a Newton map, even if $l = \infty$, as the trivial example $T(x) = -x$ shows, for which every f with $N_f = T$ lacks differentiability at

$x^* = 0$. As will become clear shortly, most maps are not Newton, but a satisfactory characterisation is not available for finitely differentiable maps. However, in the smooth case, that is, for $l = \infty$, there is a simple characterisation of Newton maps, as provided by Theorem 3.2 below.

For any map T, denote by $\mathrm{Fix}[T]$ the set of fixed points of T, that is, $\mathrm{Fix}[T] := \{x \in I : T(x) = x\}$, and say that $\mathrm{Fix}[T]$ is *attracting* if $\lim_{n\to\infty} T^n(x_0) \in \mathrm{Fix}[T]$ for all x_0 sufficiently close to $\mathrm{Fix}[T]$.

LEMMA 2.4. *Let $f : I \to \mathbb{R}$ be differentiable, and assume that N_f is continuous. Then $\mathrm{Fix}[N_f]$ is either empty or a (possibly one-point) interval; in the latter case,*

$$\limsup_{x\to x^*} \frac{N_f(x) - x^*}{x - x^*} = \delta \quad \text{for some } \delta \in [0, 1] \tag{2.1}$$

holds for every $x^* \in \mathrm{Fix}[N_f]$.

PROOF. It will first be shown that both sets $Z_0 := \{x \in I : f(x) = 0\}$ and $Z_1 := \{x \in I : f'(x) = 0\}$ of zeros of f and f', respectively, are (possibly empty or one-point) subintervals of I. Moreover, if $Z_1 \neq I$, that is, if f is not constant, then $Z_1 \subset Z_0$; in fact, the two sets coincide unless Z_0 contains exactly one point, in which case Z_1 may be empty. Since $\mathrm{Fix}[N_f] = Z_0 \cup Z_1$ the first part of the lemma follows immediately from this.

If $Z_1 = I$, then $\mathrm{Fix}[N_f] = I$, so let $Z_1 \neq \emptyset$ be different from I. Pick $a \in Z_1$, suppose, by way of contradiction, $f(a) \neq 0$ and, without loss of generality, that $b := \sup\{x \geq a : f(y) = f(a) \text{ for all } y \in [a, x]\}$ belongs to I. Clearly, $f(b) = f(a)$ and $f'(b) = 0$, hence $N_f(b) = b$. By the Mean Value Theorem there exists a sequence $b_n \searrow b$ such that $0 < |f'(b_n)| \leq 1$ for all n. But then

$$\liminf_{n\to\infty} |N_f(b_n) - b| \geq \lim_{n\to\infty} |f(b_n)| = |f(b)| = |f(a)| > 0,$$

clearly contradicting the continuity of N_f. Therefore $f(a) = 0$, hence $Z_1 \subset Z_0$. If $a_1 < a_2$ both belong to Z_0 then, by the previous argument and the Mean Value Theorem, Z_0 contains a point strictly between a_1 and a_2. Since Z_0 is closed, it contains, with any two points, the whole segment joining these points. Thus Z_0 is an interval. If Z_0 is not a singleton then $Z_0 \subset Z_1$ and therefore $Z_0 = Z_1$. The latter equality also holds if Z_0 is one-point because $Z_1 \neq \emptyset$. Finally, if Z_1 is empty then clearly Z_0 cannot contain more than one point.

Assertion (2.1) is trivially true if x^* is an interior point of $\mathrm{Fix}[N_f]$. Without loss of generality therefore assume that x^* is, say, a *right* boundary point of $\mathrm{Fix}[N_f] = Z_0$. Choose $\delta > 0$ so small that $J :=]x^*, x^* + \delta] \subset I$ and, for $0 < t \leq \delta$, let

$$h(t) := \frac{N_f(x^* + t) - x^*}{t}; \tag{2.2}$$

the function h is continuous on $]0,\delta]$, and $h(t) \neq 1$ for all $t > 0$. Since $x \neq N_f(x)$ for $x \in J$,

$$\frac{f'(x)}{f(x)} = \frac{1}{x - N_f(x)}, \quad \text{for all } x \in J,$$

which after integrating both sides from x to $x^* + \delta$, and using the auxiliary function h defined in (2.2), can be written as

$$f(x) = f(x^* + \delta)\exp\left(-\int_{x-x^*}^{\delta} \frac{1}{1-h(t)}\frac{dt}{t}\right), \quad \text{for all } x \in J. \tag{2.3}$$

Assume $f(x^* + \delta) > 0$ without loss of generality. If $h(t) > 1$ for all $t > 0$, then (2.3) implies that $f(x^*) \neq 0$, contradicting $x^* \in Z_0$. Thus $h(t) < 1$ for all $t > 0$, and in particular

$$\limsup_{t \searrow 0} h(t) = \limsup_{x \searrow x^*} \frac{N_f(x) - x^*}{x - x^*} \leq 1.$$

Fix $j \in \mathbb{N}$. Dividing (2.3) by $(x - x^*)^j = \delta^j \exp\left(-j\int_{x-x^*}^{\delta} dt/t\right)$ yields

$$\frac{f(x)}{(x-x^*)^j} = \frac{f(x^*+\delta)}{\delta^j}\exp\left(\int_{x-x^*}^{\delta} \frac{j-1-jh(t)}{1-h(t)}\frac{dt}{t}\right), \quad \text{for all } x \in J. \tag{2.4}$$

To bound $\limsup_{t\searrow 0} h(t)$ from below, pick $\varepsilon > 0$ and assume that $h(t) < -\varepsilon$ for all sufficiently small $t > 0$. In this case, (2.4) with $j = 1$ shows that

$$(x-x^*)^{-1} f(x) \geq f(x^*+\delta)\delta^{-(1+\varepsilon)^{-1}}(x-x^*)^{-\varepsilon(1+\varepsilon)^{-1}} \to \infty, \quad \text{as } x \searrow x^*,$$

which contradicts the differentiability of f at x^*. Since $\varepsilon > 0$ was arbitrary, $\limsup_{t\searrow 0} h(t) \geq 0$. $\square$

REMARK. (i) Lemma 2.4 should be contrasted with the simple fact that for *every* closed set $A \subset \mathbb{R}$ there exists a C^∞ map T with $T(I) \subset I$ and $\mathrm{Fix}[T] = A \cap I$.

(ii) Under the conditions of Lemma 2.4 there is no analogue to (2.1) for the corresponding lim inf which, as simple examples show, can be any number between, and including, the trivial bounds $-\infty$ and δ.

As pointed out earlier, the applicability of NM rests on the correspondence between the roots of f and the fixed points of N_f — *and* the attractiveness of the latter. Mere continuity of N_f does not guarantee that $\mathrm{Fix}[N_f]$ is attracting.

EXAMPLE 2.5. Consider the C^1 function

$$f(x) = \begin{cases} |x|^{3/2} \exp\left(-\int_0^{|x|^{-1}} t^{-1} \sin t \, dt\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

for which the associated NM transformation

$$N_f(x) = \begin{cases} x\left(1 + 2\sin\left(|x|^{-1}\right)\right)/\left(3 + 2\sin\left(|x|^{-1}\right)\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is continuous yet obviously not C^1. The only fixed point of N_f, and correspondingly the only root of f and f', is $x^* = 0$. Since, for every $j \in \mathbb{N}$, the points $\pm 2/(\pi(4j-1))$ are 2-periodic, $\mathrm{Fix}[N_f] = \{0\}$ is *not* attracting.

Thus while $\mathrm{Fix}[N_f]$ is topologically simple whenever N_f is continuous, to make NM practical for approximating zeros, more smoothness is required. Only the case of N_f being at least C^1 will therefore be considered from now on. (For the same reason, the legitimate case $l = 0$ has been excluded from Definition 2.3.) Also, the properties of N_f', albeit not completely determined by the smoothness of f, do depend on the latter. To describe this dependence, for every $k \in \mathbb{N}_\infty$, define the set

$$\Delta_k := \left\{0, 1/2, 2/3, \ldots, 1 - k^{-1}\right\} \cup \left]1 - k^{-1}, 1\right], \tag{2.5}$$

and note that $[0, 1] = \Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_\infty = \{1 - j^{-1} : j \in \mathbb{N}_\infty\}$.

LEMMA 2.6. *Let $f : I \to \mathbb{R}$ be differentiable, and assume that $N_f \in C^1(I)$. Then* $\mathrm{Fix}[N_f]$ *is either empty or an attracting (possibly one-point) interval. Moreover, if* $\mathrm{Fix}[N_f] \neq \emptyset$ *and $f \in C^k(I)$ with $k \in \mathbb{N}_\infty$ then*

$$N_f'(\mathrm{Fix}[N_f]) = \{\delta\} \quad \textit{for some } \delta \in \Delta_k. \tag{2.6}$$

PROOF. The assertions are trivially true if f is constant or $\mathrm{Fix}[N_f] = \emptyset$. Therefore assume that f is not constant and $\mathrm{Fix}[N_f]$ is not empty, hence a subinterval of I, by Lemma 2.4. If x^* is an interior point of $\mathrm{Fix}[N_f]$ then $N_f' \equiv 1$ in a neighbourhood of x^*, and the assertion is again true. Thus assume without loss of generality that x^* is a *right* boundary point of $\mathrm{Fix}[N_f]$. By Lemma 2.4, $N_f'(x^*) \in \Delta_1$, so x^* obviously is attracting from the right, unless perhaps for $N_f'(x^*) = 1$. In the latter case, with the notation introduced in the proof of Lemma 2.4, the function h defined in (2.2), supplemented by $h(0) := N_f'(x^*) = 1$, is continuous on $[0, \delta]$ and can be written as $h(t) = 1 - H(t)$, where H is also continuous on $[0, \delta]$, and $H(t) \neq 0$ unless $t = 0$. With this, (2.3) takes the form

$$f(x) = f(x^* + \delta) \exp\left(-\int_{x-x^*}^{\delta} \frac{dt}{tH(t)}\right), \quad \text{for all } x \in J.$$

Since $f(x^*)=0$ and $f(x^*+\delta)\neq 0$, the integral $\int_0^\delta dt/(tH(t))$ must diverge to $+\infty$. As H is continuous and, except possibly at $t=0$, does not change sign, $H(t)>0$ and so $h(t)<1$ whenever $0<t\leq\delta$. From $N_f(x^*+t)-x^*=th(t)<t$ and $h(0)=0$ it follows that $x^*<N_f(x_0)<x_0$ and therefore $N_f^n(x_0)\searrow x^*$ provided that $x_0\in J$. In other words, x^* is attracting from the right.

It remains to verify (2.6) for $f\in C^k(I)$. To this end, assume first that $k<\infty$ and $f(x^*)=f'(x^*)=\cdots=f^{(k)}(x^*)=0$. In this case, since f is C^k, the left-hand side in (2.4) with $j=k$ tends to a finite limit as $x\searrow x^*$. Consequently,

$$\lim_{\varepsilon\searrow 0}\int_\varepsilon^\delta \frac{k-1-kh(t)}{1-h(t)}\frac{dt}{t}<+\infty. \tag{2.7}$$

If $h(0)<1-k^{-1}$, then the integrand in (2.7) would eventually be positive near $t=0$, which clearly is impossible. Therefore $h(0)\geq 1-k^{-1}$. Since $h(0)\leq 1$ by the same argument,

$$N_f'(x^*)=h(0)\in[1-k^{-1},1]\subset\Delta_k.$$

If $k=\infty$ and $f^{(j)}(x^*)=0$ for all $j\in\mathbb{N}$, then similar reasoning shows that $N_f'(x^*)\in\bigcap_{j\in\mathbb{N}}[1-j^{-1},1]=\{1\}\subset\Delta_\infty$.

Finally assume that $f(x^*)=f'(x^*)=\cdots=f^{(j)}(x^*)=0$ yet $f^{(j+1)}(x^*)\neq 0$ for some j with $0\leq j<k$. The same argument as before with k replaced by j shows that $N_f'(x^*)\in[1-(j+1)^{-1},1]$. If $h(0)>1-(j+1)^{-1}$, then (2.4) with j replaced by $j+1$ would imply that $\lim_{x\searrow x^*}(x-x^*)^{-(j+1)}f(x)=0$, which contradicts $f^{(j+1)}(x^*)\neq 0$. Thus $N_f'(x^*)=h(0)=1-(j+1)^{-1}\in\Delta_\infty\subset\Delta_k$. ☐

EXAMPLE 2.7. Lemma 2.6 is best possible in the following sense: For every $k\in\mathbb{N}_\infty$ and $\delta\in\Delta_k$ there exists a C^k function f with $N_f\in C^1$ having a single fixed point x^* such that $N_f'(x^*)=\delta$. For $k\in\mathbb{N}$ and $\delta\in\Delta_k\backslash\{1\}$ let $\gamma=(1-\delta)^{-1}$ and consider the function

$$f(x)=\begin{cases} x^\gamma\left(1+(2k+4)^{-1}x^{(1+\gamma)(1+k)}\sin(x^{-\gamma})\right) & \text{if } 0<|x|<1,\\ 0 & \text{if } x=0,\end{cases}$$

where, for non-integer γ, each argument x has to be replaced by $|x|$. Taking $I=]-1,1[$, it is readily checked that $f\in C^k(I)$ and $N_f\in C^1(I)$. Moreover, $x^*=0$ is the only fixed point of N_f in I, and $N_f'(x^*)=1-\gamma^{-1}=\delta$. For $\delta=1$, an example is provided by the C^k function

$$f(x)=e^{-1/|x|}+\frac{1}{2}e^{-(k+4)/|x|}\sin\left(e^{1/|x|}\right),$$

for which N_f is C^1, has $x^*=0$ as its only fixed point, and $N_f'(x^*)=1$. Simple examples in the case $k=\infty$ are $f(x)=x^\gamma$ for $\delta<1$, and $f(x)=\exp(-|x|^{-1})$ for $\delta=1$, respectively.

An important special case for which Lemma 2.6 can be strengthened is the case of a root of finite multiplicity. Recall that $x^* \in I$ is a root of $f \in C^k(I)$ of multiplicity $j \in \mathbb{N}$ if $f(x) = (x - x^*)^j g(x)$ for all $x \in I$, where $g \in C^k(I)$ and $g(x^*) \neq 0$.

LEMMA 2.8. *Let x^* be a root of $f \in C^k(I)$ of finite multiplicity j. Then, for some open interval $J \subset I$ containing x^*, $N_f \in C^{k-1}(J)$, and $N_f'(x^*) = 1 - j^{-1}$; in particular,* $\mathrm{Fix}[N_f] \cap J = \{x^*\}$ *is attracting.*

PROOF. Since $f(x) = (x - x^*)^j g(x)$ for some $g \in C^k$ with $g(x^*) \neq 0$,

$$N_f(x) - x^* = (x - x^*)\frac{(j-1)g(x) + (x - x^*)g'(x)}{jg(x) + (x - x^*)g'(x)} = (x - x^*)h(x), \tag{2.8}$$

where h is C^{k-1} on some open interval $J \subset I$ containing x^*, and

$$N_f'(x^*) = h(x^*) = 1 - j^{-1}.$$

Thus, for J chosen sufficiently small, $\mathrm{Fix}[N_f] \cap J = \{x^*\}$, and the fixed point x^* clearly is attracting. □

3. Main theorem

Lemma 2.6 contains necessary conditions for a map to be Newton. In general it is too much to expect that every $T \in C^1(I)$ whose fixed point set is attracting and satisfies (2.6) would be a Newton map associated with some $f \in C^k(I)$.

EXAMPLE 3.1. Let $I =]-1, 1[$ and consider the map

$$T(x) = \begin{cases} x/\log|x| & \text{if } 0 < |x| < 1, \\ 0 & \text{if } x = 0, \end{cases}$$

which has $x^* = 0$ as its only and attracting fixed point and, with $T'(x^*) := 0$, is C^1 on I. Obviously $T'(x^*) \in \Delta_k$ for all $k \in \mathbb{N}_\infty$. Suppose that $N_f = T$ for some $f \in C^k(I)$. Then, with some nonzero constant C,

$$f(x) = Cx(1 - \log x), \quad \text{for all } x : 0 < x < 1.$$

Clearly, this function cannot be extended to even a *differentiable* function on I. Thus $N_f \neq T$ for every $f \in C^k(I)$. The fact that in this example T is barely C^1 is not important, as it is easy to find similar examples with T showing any *finite* degree of differentiability: For every $l \in \mathbb{N}$ (and $k \in \mathbb{N}_\infty$) there exist maps $T \in C^l(I)$ such that $T'(\mathrm{Fix}[T]) = \{\delta\}$ with $\delta \in \Delta_k$, yet $N_f \neq T$ for all $f \in C^k(I)$.

Example 3.1 shows that there is no hope for a converse of Lemma 2.6 to hold, even if N_f is assumed to be more regular than C^1. However, the situation is much clearer for smooth maps, that is, for $l = \infty$. In this case, the converse of Lemma 2.6 does actually hold, that is, the stated conditions are also sufficient.

THEOREM 3.2. *Let $k \in \mathbb{N}_\infty$, and suppose $T \in C^\infty(I)$. Then T is a Newton map, associated with $f \in C^k(I)$, if and only if* Fix$[T]$ *either is empty or an attracting (possibly one-point) interval, and*

$$T'(\mathrm{Fix}[T]) = \{\delta\}, \quad \textit{for some } \delta \in \Delta_k. \tag{3.1}$$

Moreover, the function f is uniquely determined up to a multiplicative constant if either $\delta \in \{0, 1/2, 1/3, \ldots, 1 - k^{-1}\}\backslash\{1\}$ or the set $I \backslash$ Fix$[T]$ *is connected.*

PROOF. If T is a Newton map then, by Lemma 2.6, Fix$[T]$ is an attracting interval (which may be empty or one-point), and (3.1) holds. Thus only the converse statement and the uniqueness assertion have yet to be proved. To this end, three cases will be distinguished; throughout let $g(x) := x - T(x)$.

Case 1. Assume that $\mathrm{Fix}[T] = \emptyset$. Then g is nonvanishing and C^∞ on I, and so is

$$f(x) = \exp\left(\int_\xi^x \frac{dt}{g(t)}\right), \quad \text{for all } x \in I,$$

where ξ is any point in I. Since g is C^∞ and does not vanish on I, the solution f of the first-order ODE $f'/f = 1/g$, or equivalently, $N_f = T$, is unique up to multiplication by a constant.

Case 2. Assume that $x^* \in \mathrm{Fix}[T]$ and $T'(x^*) = \delta$ with $\delta \in \Delta_k\backslash\{1\}$. Clearly this implies that $\mathrm{Fix}[T] = \{x^*\}$, and T can be written as

$$T(x) = x^* + \delta(x - x^*) + (1 - \delta)(x - x^*)^2 h(x),$$

with a uniquely determined $h \in C^\infty$. Note that $(x - x^*)h(x) \neq 1$ for all $x \in I$. Let $\gamma = (1 - \delta)^{-1}$, pick points $x^-, x^+ \in I$ with $x^- < x^* < x^+$, and define $f : I \to \mathbb{R}$ by

$$f(x) := \begin{cases} c^+(x^+ - x^*)^\gamma \exp\left(-\int_x^{x^+} dt/g(t)\right) & \text{if } x > x^*, \\ 0 & \text{if } x = x^*, \\ c^-(x^* - x^-)^\gamma \exp\left(\int_{x^-}^x dt/g(t)\right) & \text{if } x < x^*; \end{cases} \tag{3.2}$$

here c^+, c^- are nonzero real constants. Since x^* is the only fixed point of T in I it follows that $f \in C^\infty(I\backslash\{x^*\})$, and $N_f = T$. By using the identity

$$(x - x^*)^\gamma = (x^+ - x^*)^\gamma \exp\left(-\gamma \int_x^{x^+} \frac{dt}{t - x^*}\right), \quad \text{for all } x > x^*, \tag{3.3}$$

a short computation yields

$$(x-x^*)^{-\gamma} f(x) = c^+ \exp\left(-\gamma \int_x^{x^+} \frac{h(t)\,dt}{1-(t-x^*)h(t)}\right), \quad \text{for all } x > x^*.$$

An analogous computation for $x < x^*$ yields

$$(x^*-x)^{-\gamma} f(x) = c^- \exp\left(\gamma \int_{x^-}^{x} \frac{h(t)\,dt}{1-(t-x^*)h(t)}\right), \quad \text{for all } x < x^*.$$

Since the integrand $h(t)/(1-(t-x^*)h(t))$ is C^∞ on I, both one-sided limits for $|x-x^*|^{-\gamma} f(x)$, as x approaches x^*, are finite and nonzero. If $\delta = 1 - j^{-1}$ for some $1 \le j \le k$ then, for f to be C^j on I, these two one-sided limits have to be equal or, equivalently,

$$c^- = (-1)^j c^+ \exp\left(-j \int_{x^-}^{x^+} \frac{h(t)\,dt}{1-(t-x^*)h(t)}\right)$$

must hold. In the latter case, for all $x \in I$,

$$f(x) = c^+(x-x^*)^j \exp\left(-j \int_x^{x^+} \frac{h(t)\,dt}{1-(t-x^*)h(t)}\right),$$

which shows $f \in C^k(I)$. Since the two-parameter family defined in (3.2) contains all solutions of $N_f = T$ on $x < x^*$ and $x > x^*$ separately, the solution of $N_f = T$ is unique up to multiplication by a nonzero constant if $\delta \in \{0, 1/2, 1/3, \ldots, 1-k^{-1}\}\backslash\{1\}$.

If, on the other hand, $\delta > 1 - k^{-1}$, and correspondingly $\gamma > k$, then $f \in C^k(I)$ for any choice of the constants c^+, c^-, and $f(x^*) = f'(x^*) = \cdots = f^{(k)}(x^*) = 0$.

Case 3. Assume that $T'(\mathrm{Fix}[T]) = \{1\}$. If $\mathrm{Fix}[T] = I$, then trivially T is the Newton map associated with $f \equiv 1$. Without loss of generality, therefore, assume that x^* is the right boundary point of $\mathrm{Fix}[T]$. In this case

$$T(x) = x - (x-x^*)^2 h(x),$$

where $h \in C^\infty(I)$ and $h(x) > 0$ whenever $x > x^*$, and $h(x) = 0$ for all $x \in \mathrm{Fix}[T]$; in particular, therefore, $h(x^*) = 0$. As before, pick $x^+ \in I$ with $x^+ > x^*$ and, analogously to (3.2), let

$$f^+(x) := \begin{cases} \exp\left(-\int_x^{x^+} \frac{dt}{g(t)}\right) & \text{if } x > x^*, \\ 0 & \text{if } x \le x^*. \end{cases}$$

Using (3.3), with γ replaced by j, and recalling that $g(t) = (t-x^*)^2 h(t)$, it follows that $\lim_{x \searrow x^*} (x-x^*)^{-j} f^+(x) = 0$ for all $j \in \mathbb{N}$. Thus $f^+ \in C^\infty(I)$ and $N_{f^+}(x) = T(x)$

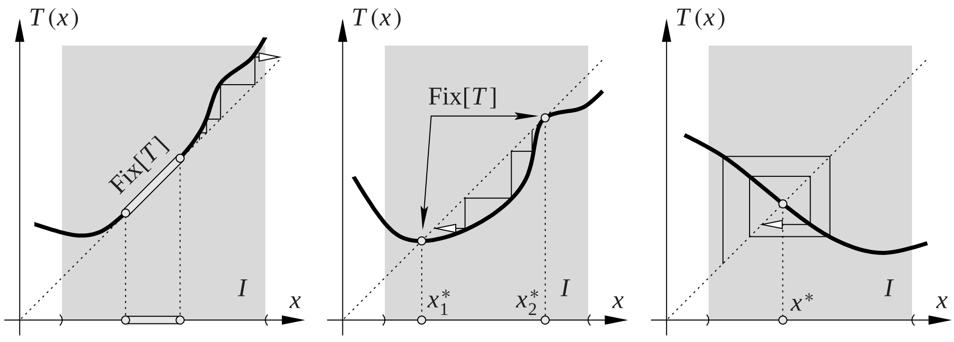

FIGURE 2. Three C^∞ maps T which are not Newton maps associated with any C^k function on the interval I because Fix[T] is not attracting (left), Fix[T] is not an interval (middle), and $T'(x^*) \notin \Delta_k$ for any $k \in \mathbb{N}_\infty$, respectively.

whenever $x > x^*$ or $x \in \text{Fix}[T]$. If Fix[T] has a left boundary point in I as well, then define f^- in a "mirrored" manner and let $f = c^+ f^+ + c^- f^-$ with nonzero constants c^+, c^-. Clearly, $f \in C^\infty(I)$ and $N_f = T$ for any choice of c^+, c^-.

The assertion concerning uniqueness up to multiplication by a constant is now obvious from the three cases detailed above. □

COROLLARY 3.3. *Suppose $T \in C^\infty(I)$. Then T is a Newton map, associated with $f \in C^\infty(I)$, if and only if* Fix[T] *is either empty or an attracting (possibly one-point) interval, and*

$$T'(\text{Fix}[T]) = \{1 - j^{-1}\}, \quad \textit{for some} \;\; j \in \mathbb{N}_\infty. \tag{3.4}$$

Moreover, f is uniquely determined up to a multiplicative constant unless $j = \infty$ in (3.4) and the set $I \setminus$ Fix[T] *is not connected.*

The next corollary requires T to be not only C^∞ but even real-analytic. Recall that a map is *real-analytic* if it can be represented by its Taylor's series in a neighbourhood of every point in its domain. Real-analytic Newton maps are especially easy to characterise. Although analyticity is a strong assumption indeed, the class of real-analytic functions is of great historical [5, 9] and practical relevance, as it contains, for example, all rational and trigonometric functions and compositions thereof [1, 4]. If f is real-analytic then so is N_f, provided the latter map is continuous [1, 2].

COROLLARY 3.4. *Let T be real-analytic on I, and $T(x) \not\equiv x$. Then T is a Newton map, associated with a real-analytic function f, if and only if T has at most one fixed point in I, and, in case a fixed point x^* exists, $T'(x^*) = 1 - j^{-1}$ for some $j \in \mathbb{N}$. Moreover, f is unique up to multiplication by a constant.*

EXAMPLE 3.5. For $f(x) = \exp(-x)$ and $f_j(x) = x^j$, $j \in \mathbb{N}$, clearly $N_f(x) = x+1$ and $N_{f_j}(x) = (1 - j^{-1})x$, respectively. Thus all cases referred to in Corollary 3.4 can actually occur.

EXAMPLE 3.6. The much-studied logistic map $F_\mu(x) = \mu x(1 - x)$ is a Newton map associated with a real-analytic function on $I =]0, 1[$ if and only if $\mu \in M$, with $M :=] - \infty, 1] \cup \{1 + j^{-1} : j \in \mathbb{N}\}$. Indeed, $F_\mu = N_{f_\mu}$ with functions

$$f_\mu(x) = \left(\frac{x}{\mu x + 1 - \mu}\right)^{(1-\mu)^{-1}} \qquad \text{for } \mu \neq 1,$$

and $f_1(x) = \exp(-x^{-1})$. Note that while f_μ is real-analytic on I for all $\mu \in M$, it is only in the trivial case $\mu = 0$ that f_μ could be extended to a real-analytic function such that $N_{f_\mu}(x) = F_\mu(x)$ for all $x \in \mathbb{R}$. Consequently, F_μ is not a Newton map on $\mathbb{R}$ unless $\mu = 0$.

EXAMPLE 3.7. It must be emphasised that Theorem 3.2 and Corollaries 3.3 and 3.4 do not force the set $\mathrm{Fix}[T]$ of a C^∞ or real-analytic Newton map T to attract *all* points in I. In fact, the map T may at the same time exhibit some *stable* dynamical feature other than a fixed point. For a simple concrete example consider the (real-analytic) function

$$f(x) = x\,\frac{3 + x^2}{1 + x^2},$$

for which the associated Newton map

$$N_f(x) = -\frac{4x^3}{3 + x^4}$$

has the stable (in fact, super-attracting) 2-periodic orbit $\{\sqrt{3}, -\sqrt{3}\}$.

REMARK. It is well known that if f is a *rational* function (that is, a quotient of two polynomials) then N_f can be extended uniquely to (and studied appropriately as) a smooth function $\overline{N}_f$ on $\overline{\mathbb{R}}$, the one-point compactification of $\mathbb{R}$. Though finite, $\mathrm{Fix}[\overline{N}_f]$ generally contains more than one point [2, 3]. Corollary 3.4, however, clearly still applies to $\mathrm{Fix}[\overline{N}_f] \cap I$ for every interval I on which f is real-analytic.

The above results about Newton maps have an immediate bearing on the distribution of the floating-point fractions of the iterates $x_n = N_f^n(x_0)$, that is, on the numerical data generated by NM. (See [7] for an account on the relevance of fraction parts distributions for practical computations.) In particular, this distribution depends significantly on the analytic properties of N_f discussed in this note; the interested reader is referred to [4] for details.

Acknowledgements

The first author was partially supported by a HUMBOLDT research fellowship. He wishes to thank the Johann Wolfgang von Goethe–Universität Frankfurt for its hospitality. The second author was partially supported by a National Security Agency Grant. He is grateful to the University of Leiden and the Universität Bonn, for invitations to visit in April 2005, during which time some of this research was carried out, and to the California Polytechnic State University, for its hospitality and its support as Research Scholar in Residence 2005–2006.

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