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# ONE-DIMENSIONAL DYNAMICAL SYSTEMS AND BENFORD'S LAW

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Abstract. Near a stable fixed point at 0 or $\infty$, many real-valued dynamical systems follow Benford's law: under iteration of a map $T$ the proportion of values in $\{x, T(x), T^2(x), \ldots, T^n(x)\}$ with mantissa (base $b$) less than $t$ tends to $\log_b t$ for all $t$ in $[1, b)$ as $n \to \infty$, for all integer bases $b > 1$. In particular, the orbits under most power, exponential, and rational functions (or any successive combination thereof), follow Benford's law for almost all sufficiently large initial values. For linearly-dominated systems, convergence to Benford's distribution occurs for every $x$, but for essentially nonlinear systems, exceptional sets may exist. Extensions to nonautonomous dynamical systems are given, and the results are applied to show that many differential equations such as $\dot{x} = F(x)$, where $F$ is $C^2$ with $F(0) = 0 > F'(0)$, also follow Benford's law. Besides generalizing many well-known results for sequences such as $(n!)$ or the Fibonacci numbers, these findings supplement recent observations in physical experiments and numerical simulations of dynamical systems.

## 1. Introduction

Benford's law is the probability distribution for the mantissa with respect to base $b \in \mathbb{N} \setminus \{1\}$ given by $\mathbb{P}(\text{mantissa}_b \leq t) = \log_b t$ for all $t \in [1, b[$; the most well-known special case is that

$$\mathbb{P}\big(\text{first significant digit}_{10} = d\big) = \log_{10}\left(1 + d^{-1}\right), \qquad d = 1, \ldots, 9\,.$$

Although first discovered by Newcomb [N], this logarithmic law for significant digits gained popularity following the article by Benford [Ben], which contained extensive empirical evidence of the distribution in diverse tables of data. Since Benford's article, numerous examples of empirical data sets following Benford's law have been found in real-life data (e.g., physical constants, stock market indices, tax returns [H2, R]); in stochastic processes (e.g., sums and products of random variables [R, S]); and in many deterministic sequences (e.g., $(n!)$, $(a^n)$, and Fibonacci numbers [Ben, BD, D]). Very recently, data from certain physical experiments and numerical simulations arising in dynamical systems have also been found to follow Benford's

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law [SCD, TBL]. Of course, many data sets do *not* follow Benford's law—e.g., telephone numbers, uniform random variables, $(\log n)$—and one of the objectives of research has been to establish criteria for predicting when data will be Benford-distributed.

Benford's law has been characterized in [H1] as the only continuous mantissa distribution which is *base-invariant*. It is natural to require that a general pattern of mantissa distribution, if one exists at all, does not depend on the particular choice of the base. Base-invariance, however, has significant implications: For a sequence $(x_n)$ to follow Benford's law for all (or at least an unbounded set of) bases, every weak limit of $\frac{1}{n}\sum_{j=1}^{n}\delta_{x_j}$ on the extended real line $\mathbb{R}\cup\{\pm\infty\}$ must be a convex combination of point-masses at 0 and $\pm\infty$, respectively. Under the additional assumption of *stability*, the present article establishes a fairly complete theory of Benford's law for sequences generated by continuous one-dimensional dynamical systems. Wide classes of dynamical systems having a subset of $\{0,\pm\infty\}$ as an attractor are shown to produce Benford sequences in abundance. The approach to Benford's law via dynamical systems not only generalizes and unifies many earlier special results obtained by number-theoretical and other methods (e.g. [BD, D]), it also illustrates the simple yet universal mechanism underlying the generation of most of the known Benford sequences. Even though stable fixed points at zero or infinity constitute a highly specific (and simple) dynamical scenario, this setting is quite natural and general: an efficient experimentalist, observing convergence of numerical data, will record *differences* from the prospective limit, rather than the nearly indistinguishable raw data themselves. Thus, the results of this work also add to the explanation of the ubiquity of the Benford distribution in experimental data.

The organization of this article is as follows. Section 2 provides definitions and preliminary results, including the basic relationship between Benford sequences and uniform distribution mod 1, and a shadowing lemma. Section 3 contains the main results for linearly-dominated one-dimensional dynamical systems; for example, the orbits under $T: x\mapsto xe+x^2e^{-x}$ follow Benford's law for all sufficiently large $x$. Section 4 contains the corresponding results for essentially nonlinear one-dimensional systems; e.g., the orbits under $T: x\mapsto x^2+1$ are Benford for Lebesgue almost all $x$. Section 5 establishes analogous results for nonautonomous dynamical systems including both linearly-dominated and essentially nonlinear systems; e.g., applications of $T_1: x\mapsto x^2$ and $T_2: x\mapsto 3^x$ in any successive combination also yield Benford sequences. Finally, Section 6 demonstrates how the results for discrete-time systems immediately carry over to differential equations.

## 2. Preliminaries

Throughout, $b$ will always denote a natural number larger than one (called a *base*). Every positive real number $x$ can be written uniquely as $x = M_b(x)b^k$ with $M_b(x)\in[1,b[$ and the appropriate integer $k$. The function $M_b:\mathbb{R}^+\to[1,b[$ is called the (*base* $b$) *mantissa function*; for convenience set $M_b(0):=0$ for all $b$. For every real $x$, the numbers $\lfloor x\rfloor$ and $\lceil x\rceil$ denote the largest integer not larger, and the smallest integer not smaller than $x$, respectively. The number $\lfloor M_b(x)\rfloor\in\{1,\dots,b-1\}$ is called the *first significant digit* of $x$ (with respect to base $b$). For a given base $b$, $\log_b$ will denote the logarithm with respect to $b$, where, for ease of notation, $\log_b 0 := 0$ for all $b$; if used without a subscript, the log symbol denotes the natural logarithm.

**Definition 2.1.** A sequence $(x_n)_{n\in\mathbb{N}_0}$ of real numbers is a $b$-*Benford sequence* if

$$\lim_{n\to\infty} \frac{\#\{j \le n : M_b(|x_j|) \le t\}}{n} = \log_b t \qquad \text{for all } t \in [1, b[\,,$$

and it is called a *strict Benford sequence* (or simply a *Benford sequence*) if it is a $b$-Benford sequence for all $b \in \mathbb{N}\setminus\{1\}$.

The most basic tool in this paper is the following direct correspondence between Benford sequences and uniform distribution modulo one, which allows application of the powerful classical tools for uniform distribution of sequences (e.g. [DT, KN]).

**Proposition 2.2** ([D])**.** *A sequence $(x_n)_{n\in\mathbb{N}_0}$ of real numbers is a b-Benford sequence if and only if $(\log_b |x_n|)_{n\in\mathbb{N}_0}$ is uniformly distributed modulo one.*

Henceforth, the term *uniformly distributed modulo one* will be abbreviated as *u.d. mod* 1. An immediate consequence of Proposition 2.2 is

**Proposition 2.3.** *Let $(x_n)_{n\in\mathbb{N}_0}$ be a* (*b- or strict*) *Benford sequence. Then for all $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$ with $\alpha k \neq 0$, the sequence $(\alpha x_n^k)_{n\in\mathbb{N}_0}$ is also a* (*b- or strict, respectively*) *Benford sequence.*

This paper primarily studies Benford properties of sequences generated recursively by

$$x_n := T_n(x_{n-1})\,, \qquad n = 1, 2, \ldots\,, \tag{2.1}$$

where $(T_n)$ is a sequence of maps of the real line (or a part thereof) into itself. Relation (2.1) is interpreted as a *nonautonomous dynamical system*. For $n \in \mathbb{N}$, $T^n$ denotes the composition $T^n = T_n \circ T_{n-1} \circ \ldots \circ T_2 \circ T_1$, and $T^0 := id$. Sections 3 and 4 deal with *autonomous* systems, i.e., $T_n = T$ for all $n$, in which case $T^n$ stands for the $n$-fold composition of $T$ with itself. The symbol $O_T(x)$, called the *orbit* of $x$ under $T$, will denote the sequence generated by (2.1) subject to the initial condition $x_0 = x$; in the above notation, $O_T(x) = \big(T^n(x)\big)_{n\in\mathbb{N}_0}$. Note that this interpretation of the orbit as a *sequence* differs from the standard terminology in dynamical systems theory (e.g., [KH]) where the orbit of $x$ is the mere *set* $\{x_n : n \in \mathbb{N}_0\}$.

The next lemma, recorded here for ease of reference, lists several basic results concerning uniform distribution mod 1 of sequences, which via Proposition 2.2 will be used to determine Benford properties of sequences. As no reference to (i) is known to the authors, a proof is included; conclusion (ii) is Weyl's classical result [KN, Thm. 3.3]; and (iii) is Koksma's metric theorem [KN, Thm. 4.3].

**Lemma 2.4.** (i) *If $(x_n)_{n\in\mathbb{N}}$ is nondecreasing and u.d. mod* 1*, then the sequence $(x_n/\log n)_{n\ge 2}$ is unbounded.*

(ii) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers with $\Delta x_n = x_{n+1} - x_n \to \theta$ irrational, then $(x_n)_{n\in\mathbb{N}}$ is u.d. mod* 1*.*

(iii) *Suppose $f_n \in C^1[a,b]$, $n \in \mathbb{N}$. If $f'_m(x) - f'_n(x)$ is monotone in $x$, and $|f'_m(x) - f'_n(x)| \ge C > 0$ for all $m \neq n$, where $C$ does not depend on $x$, $m$, and $n$, then $\big(f_n(x)\big)_{n\in\mathbb{N}}$ is u.d. mod* 1 *for almost all $x$ in $[a,b]$.*

*Proof of* (i). Let $(x_n)$ be nondecreasing and suppose that $c_n := x_n/\log n$ were bounded. For each $\alpha > 1$ and $\varepsilon > 0$ the inequality $c_{\lfloor \alpha n\rfloor} - c_n \le \varepsilon/\log n$ then holds infinitely often; otherwise, there exist $\alpha_0 > 1$, $\varepsilon_0 > 0$ and $N \in \mathbb{N}$ such that $c_{\lfloor \alpha_0 n \rfloor} - c_n > \varepsilon_0 / \log n$ for all $n \geq N$. But then, setting $N_1 := N$ and $N_{j+1} := \lfloor \alpha_0 N_j \rfloor$ for all $j \in \mathbb{N}$,

$$c_{N_k} - c_{N_1} > \sum_{j=1}^{k-1} \frac{\varepsilon_0}{\log N_j} \geq \sum_{j=1}^{k-1} \frac{\varepsilon_0}{j \log \alpha_0 + \log N} \to \infty \quad \text{as } k \to \infty$$

which contradicts the boundedness of $(c_n)$. Next fix $\alpha > 1$ such that $C \log \alpha < \frac{1}{4}$ where $C$ is an upper bound on $(c_n)$, and consider a subsequence $(n_k)$ with $c_{\lfloor \alpha n_k \rfloor} - c_{n_k} \leq 1/(4 \log n_k)$ for all $k$. Then

$$\begin{aligned} x_{\lfloor \alpha n_k \rfloor} - x_{n_k} &= (c_{\lfloor \alpha n_k \rfloor} - c_{n_k}) \log \lfloor \alpha n_k \rfloor + c_{n_k} \log \frac{\lfloor \alpha n_k \rfloor}{n_k} \\ &\leq \frac{\log \lfloor \alpha n_k \rfloor}{4 \log n_k} + C \log \alpha < \frac{1}{2} \end{aligned}$$

for all sufficiently large $k$, which implies that $(x_n)$ cannot be u.d. mod 1. □

Another important tool in the proofs below are so-called *shadowing* arguments, where the iterates of one family of (nonlinear) maps are replaced by iterates of another (linear) family which are easier to analyze. More precisely, let $(S_j)_{j \in \mathbb{N}}$ denote a family of continuous maps of the real line into itself and assume that, for some $\xi > 0$ and $C > 0$, the growth condition

$$\sup_{|x| \geq \xi} \left| \frac{S_j(x)}{\beta_j} - x \right| \leq C$$

holds for all $j \in \mathbb{N}$ with appropriate numbers $\beta_j \in \mathbb{R} \backslash \{0\}$. For brevity, let $B_0 := 1$ and $B_n := \prod_{j=1}^{n} \beta_j$ for $n \geq 1$. The following shadowing result identifies a general condition which guarantees that for every sufficiently large $x$, there exists a point $\overline{x}$ such that $O_S(x)$ is close to the sequence $(B_n \overline{x})_{n \in \mathbb{N}_0}$. Recall that $S^j = S_j \circ S_{j-1} \circ \ldots \circ S_1$.

**Theorem 2.5.** *Suppose the maps $S_j : \mathbb{R} \to \mathbb{R}$ are continuous, and that*

$$\sup_{|x| \geq \xi} \left| \frac{S_j(x)}{\beta_j} - x \right| \leq C < \infty \quad \textit{for all } j \in \mathbb{N},$$

*where $(\beta_j)$, $\xi$, $C$ are real numbers with $\beta_j \neq 0$ and $\xi > 0$, $C > 0$. Then there exists $\eta \geq \xi$ satisfying:*

(i) *If $\sum_{j=0}^{\infty} |B_j|^{-1}$ converges, then $h(x) := \lim_{j \to \infty} B_j^{-1} S^j(x)$ exists for all $|x| \geq \eta$. Moreover, $h$ is a continuous function, and $\sup_{|x| \geq \eta} |h(x) - x| \leq C \sum_{j=0}^{\infty} |B_j|^{-1} < \infty$.*

(ii) *If $\underline{\lim}_{j \to \infty} |\beta_j| > 1$, then for each $x$ with $|x| \geq \eta$, there exists precisely one point $\overline{x}$ such that $\big(|B_n \overline{x} - S^n(x)|\big)_{n \in \mathbb{N}_0}$ is bounded; in fact, $\overline{x} = h(x)$, where $h$ is the function in* (i).

*Proof.* (i) Since $\sum_{j=0}^{\infty} |B_j|^{-1} < \infty$, the quantity

$$\eta := (\inf_{j \in \mathbb{N}_0} |B_j|)^{-1} \xi + C \sum_{j=0}^{\infty} |B_j|^{-1}$$

is finite and larger than $\xi$. It is easy to check that $|S^j(x)| \geq \xi$ for all $j \in \mathbb{N}_0$ whenever $|x| \geq \eta$. Thus

$$g(x) := x + \sum_{j=0}^{\infty} B_{j+1}^{-1}\big(S^{j+1}(x) - \beta_{j+1}S^j(x)\big) \tag{2.2}$$

defines a continuous function for $|x| \geq \eta$, which implies that

$$\left|g(x) - \frac{S^n(x)}{B_n}\right| \leq C\sum_{j=n}^{\infty}|B_j|^{-1} \tag{2.3}$$

for all $n \in \mathbb{N}_0$, $|x| \geq \eta$. Since the right-hand side of (2.3) tends to 0 as $n \to \infty$,

$$g(x) = \lim_{j\to\infty} B_j^{-1}S^j(x) = h(x)\,,$$

which establishes (i).

(ii) If $\beta := \underline{\lim}_{j\to\infty}|\beta_j| > 1$, then $|\beta_j| \geq \frac{1+\beta}{2} > 1$ for all $j \geq j_0$. Multiplying (2.3) by $B_n$ implies that for all $n \geq j_0$

$$|B_n h(x) - S^n(x)| \leq C\sum_{j=n}^{\infty}\left|\frac{B_n}{B_j}\right| \leq C\sum_{j=0}^{\infty}\left(\frac{2}{1+\beta}\right)^j = C\frac{\beta+1}{\beta-1} < \infty\,.$$

Since $|B_j| \to \infty$ as $j \to \infty$, the point $\overline{x} := h(x)$ is the only point with the desired property. ☐

The following corollary specializes Theorem 2.5 for the autonomous case, and helps explain the terminology *Shadowing Lemma*. (See, for instance, [KH, Thm. 18.1.2] for a version of the classical Shadowing Lemma for hyperbolic systems.)

**Corollary 2.6.** *Let $\beta$ be a real number with $|\beta| > 1$, and assume that the continuous map $S : \mathbb{R} \to \mathbb{R}$ satisfies $\sup_{|x|\geq\xi}|S(x) - \beta x| \leq C < \infty$ for some $\xi > 0$, $C > 0$. For each $x$ with $|x|$ sufficiently large, there exists precisely one point $\overline{x}$ such that the sequence $\big(|\beta^n\overline{x} - S^n(x)|\big)_{n\in\mathbb{N}_0}$ is bounded. The assignment $h : x \mapsto \overline{x}$ defines a continuous map with $h \circ S(x) = \beta h(x)$ for $|x|$ sufficiently large; moreover, $\overline{\lim}_{|x|\to\infty}|h(x) - x| \leq \frac{C}{|\beta|-1}$.*

*Proof.* With $S_j = S$ and $|\beta_j| = |\beta| > 1$ for all $j \in \mathbb{N}$, the assertions follow from Theorem 2.5, since $h(x) = \lim_{j\to\infty}\beta^{-j}S^j(x)$ in the autonomous case. ☐

## 3. Linearly-dominated systems

Throughout this section, $T$ will denote a $C^2$ map of the real line (or at least some neighborhood of the origin) into itself which has the origin as a stable attracting fixed point, so $T(0) = 0$ and $|\alpha| \leq 1$, where $\alpha := T'(0)$. The fixed point is called *weakly attracting*, (*regularly*) *attracting* or *super-attracting* depending on whether $|\alpha| = 1$, $0 < |\alpha| < 1$ or $\alpha = 0$, respectively. The present section considers the case $|\alpha| > 0$, where the linearization of $T$ at the origin, i.e., the map $x \mapsto \alpha x$, dominates the behavior (with respect to Benford's law) for all points near the fixed point, in a sense made precise below. Since $T$ has the origin as a fixed point,

$$T(x) = \alpha x\big(1 - f(x)\big) \tag{3.1}$$

where $f$ is $C^1$, and $f(0) = 0$. First, consider the case $0 < |\alpha| < 1$.

**Theorem 3.1.** *Suppose $T$, as given by* (3.1)*, is a $C^2$ map with* 0 *as a* (*regularly*) *attracting fixed point. Then the orbit $O_T(x)$ is a $b$-Benford sequence for all $x \neq 0$ sufficiently close to* 0 *if and only if $\log_b |\alpha|$ is irrational.*

*Proof.* Step 1: Suppose that (sufficiently close to the origin) $T$ is given by $T(x) = \alpha x$. Then, for $x \neq 0$, $\log_b |T^n(x)| = n \log_b |\alpha| + \log_b |x|$, which by Lemma 2.4(ii) is u.d. mod 1 if and only if $\log_b |\alpha|$ is irrational.

Step 2: Suppose that $T$ is a general $C^2$ map as in (3.1). The function $S$ with

$$S(x) := T\left(x^{-1}\right)^{-1} = \alpha^{-1}x + g(x)$$

is well defined and $C^1$ for sufficiently large $|x|$; here $g$ denotes the continuous function defined by

$$g(x) = \frac{xf(x^{-1})}{\alpha - \alpha f(x^{-1})} .$$

Clearly, $\overline{\lim}_{|x|\to\infty}|g(x)| = |\alpha|^{-1}\overline{\lim}_{x\to 0}\left|x^{-1}f(x)\right| = |\alpha|^{-1}|f'(0)| < \infty$. Corollary 2.6 yields a continuous function $h$ with $\overline{\lim}_{|x|\to\infty}|h(x) - x| < \infty$ and $h \circ S(x) = \alpha^{-1}h(x)$ for $|x|$ sufficiently large. Close to the origin, define a continuous map $H$ via $H(x) := h(x^{-1})^{-1}$, which satisfies $\overline{\lim}_{x\to 0}\left|H(x)^{-1} - x^{-1}\right| < \infty$, and also $H \circ T^n(x) = \alpha^n H(x)$ for all $n \in \mathbb{N}_0$. This implies

$$\log_b |T^n(x)| - \log_b |\alpha^n H(x)| \to 0 \qquad \text{as } n \to \infty ,$$

and therefore $O_T(x)$ is a $b$-Benford sequence precisely if $\left(\alpha^n H(x)\right)_{n\in\mathbb{N}_0}$ is. □

**Corollary 3.2.** *If $T$ is given by* (3.1)*, then the orbit $O_T(x)$ is a strict Benford sequence for all but countably many $\alpha \in \,]-1, 1[$, and all $x \neq 0$ sufficiently close to* 0*.*

*Remark* 3.3. (i) Although it is not crucial for the above argument, Corollary 2.6 and (2.2) provide an explicit expression for the function $H$ in the above proof, namely

$$\frac{x}{H(x)} = 1 + x\sum_{j=0}^{\infty} \frac{\alpha^j}{1 - f\circ T^j(x)} \cdot \frac{f\circ T^j(x)}{T^j(x)} = 1 + x\sum_{j=0}^{\infty} \alpha^{j+1}\frac{f\circ T^j(x)}{T^{j+1}(x)} ;$$

this formula displays a clear dynamical structure, and is valid for $|x|$ sufficiently small, in which case $H \circ T(x) = \alpha H(x)$.

(ii) The second part in the proof of Theorem 3.1 also works for an *analytic* map $T$ having the origin of the complex plane as a stable attracting fixed point, i.e., $T(0) = 0$ and $T'(0) = \lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$. In fact, via $H$, the application of Corollary 2.6 here provides a local *analytic* conjugacy between $T$ and its linearization $z \mapsto \lambda z$ at the origin. The existence of such a conjugacy is a well-known fact in dynamics, usually attributed to Poincaré [KH, Thm. 2.8.2].

(iii) Theorem 3.1 also follows immediately from Lemma 2.4(ii), in fact, even with the smoothness assumption on $T$ reduced from $C^2$ to $C^1$. The approach via Theorem 2.5 helps make the dynamical aspect of the assertion more transparent, and emphasizes the analogies to the essentially nonlinear case dealt with in the next section.

(iv) The hypothesis $T \in C^2$ in Theorem 3.1 may clearly be replaced by the weaker assumption that $T \in C^1$ and $\sup_{|x|\le 1} |x^{-1}f(x)| < \infty$, as is done in Theorem 5.1 below.

Next consider the weakly attracting case $|\alpha| = 1$. If $\alpha = -1$, then $T^2$ has 0 as a weakly attracting fixed point too, and $(T^2)'(0) = 1$. Since $T$ is given by (3.1) with $|\alpha| = 1$, the orbit $O_T(x)$ is a ($b$-) Benford sequence for all $x$ near the origin precisely if the same is true for $O_{T^2}(x)$. Without loss of generality, therefore, restrict the analysis to the positive semi-axis $x > 0$ and the case $\alpha = 1$, so

$$T(x) = x\bigl(1 - f(x)\bigr) \tag{3.2}$$

where $f$ is continuous with $0 = f(0) < f(x)$ for all sufficiently small $x > 0$. One might expect that because of the orbits' slow convergence to zero, the map $T$ will produce no Benford sequences at all. This turns out to be true under even weaker assumptions than adopted throughout this section (see, however, Example 3.5 below).

**Theorem 3.4.** *Let $T$ be a $C^{1+\varepsilon}$ map for some $\varepsilon > 0$, i.e., $T \in C^1$ and $T'$ satisfies a Hölder condition of order $\varepsilon$. Assume that $T$ has 0 as a weakly attracting fixed point. Then for every $b \in \mathbb{N}\backslash\{1\}$ the orbit $O_T(x)$ is not a $b$-Benford sequence for any $x$ sufficiently close to* 0.

*Proof.* Fix $\delta > 0$ such that $f(x) > 0$ for all $x \in ]0, \delta]$. Let $S_b$ be the map

$$S_b(y) := -\log_b T(b^{-y}) = y - \log_b\bigl(1 - f(b^{-y})\bigr),$$

which is well defined and continuous for sufficiently large $y$. For $x \le \delta$, clearly $O_T(x)$ is a $b$-Benford sequence precisely if $\bigl(S_b^n(-\log_b x)\bigr)_{n\in\mathbb{N}_0}$ is u.d. mod 1. The goal is to show that the latter sequence is *not* u.d. mod 1 for any $x \le \delta$. The Hölder assumption on $T$ implies that $f(x) \le f_\varepsilon(x)$ for some $\varepsilon > 0$ and all $x$ sufficiently close to 0, where

$$f_\varepsilon(x) := 1 - (1 + x^\varepsilon)^{-\varepsilon^{-1}}.$$

Step 1: Assume first that $f = f_\varepsilon$. Then $y_n := S_b^n(y_0)$ is monotonically increasing, and an elementary calculation shows that for all $n \in \mathbb{N}_0$,

$$y_n = \varepsilon^{-1}\log_b(n + b^{\varepsilon y_0}).$$

By Lemma 2.4(i) the sequence $(y_n)$ is not u.d. mod 1.

Step 2: Consider now the general case, i.e., $0 < f(x) \le f_\varepsilon(x)$ for all $x$ sufficiently close to 0. Denote by $\widetilde{S}_b$ the map

$$\widetilde{S}_b(y) := y - \log_b\bigl(1 - f_\varepsilon(b^{-y})\bigr)$$

and note that $-\log_b\bigl(1 - f_\varepsilon(b^{-y})\bigr)$ decreases monotonically to 0 as $y \to \infty$. Furthermore, $S_b(y) \le \widetilde{S}_b(y)$ for all sufficiently large $y$. Setting $g(y) := S_b(y) - y$, $h(h) := \widetilde{S}_b(y) - y$, clearly $0 < g \le h$. This implies

$$y_{n+1} - \widetilde{y}_{n+1} = y_n - \widetilde{y}_n + g(y_n) - h(\widetilde{y}_n) \le y_n - \widetilde{y}_n + h(y_n) - h(\widetilde{y}_n).$$

If $y_n > \widetilde{y}_n$, then $y_{n+1} - \widetilde{y}_{n+1} < y_n - \widetilde{y}_n$, since $h$ is decreasing. On the other hand, if $y_n \le \widetilde{y}_n$, then $y_{n+1} - \widetilde{y}_{n+1} \le h(y_n) - h(\widetilde{y}_n)$, which is bounded since $h(y) \to 0$ as $y \to \infty$. Therefore, the difference $y_n - \widetilde{y}_n$ is bounded from *above*, so by Step 1, there is a constant $D > 0$ such that for all $n$,

$$y_n \le D + \varepsilon^{-1}\log_b(n + b^{\varepsilon y_0}),$$

and again Lemma 2.4(i) shows that $(y_n)$ cannot be u.d. mod 1. For $x$ close to the origin, $O_T(x)$ is thus not a $b$-Benford sequence. □

**Example 3.5.** The condition $T \in C^{1+\varepsilon}$ in the above theorem cannot be weakened to $T \in C^1$. For example, by means of a direct calculation of $T^n$, and using the fact that $(\beta\sqrt{n})$ is u.d. mod 1 whenever $\beta \neq 0$ [KN, Exp. 2.7], it can be seen that the map $T$ with

$$T(x) := \begin{cases} e^{-\sqrt{1+(\log x)^2}} & \text{if } x > 0\,, \\ 0 & \text{if } x = 0\,, \end{cases}$$

is a $C^1$ map with 0 as a weakly attracting fixed point, and $O_T(x)$ is a strict Benford sequence for all $x$ sufficiently close to 0. On the other hand, using Lemma 2.4(i), it is easily verified that a similar definition

$$T(x) := \begin{cases} e^{-\sqrt{1+(\log \tau(x))^2}} & \text{if } 0 < x < e^{-1}\,, \\ 0 & \text{if } x = 0\,, \end{cases}$$

with $\tau(x) := e^{\sqrt{(\log x)^2-1}}+1$ yields a $C^1$ map which also has 0 as a weakly attracting fixed point, but $O_T(x)$ is not a $b$-Benford sequence for any $x \in ]0, e^{-1}]$ and any base $b$.

With regard to Benford's law, linearly dominated one-dimensional dynamics, i.e., $0 < |\alpha| \leq 1$, may be summarized as follows. In the attracting case ($0 < |\alpha| < 1$), typically all orbits are Benford sequences whereas for the weakly attracting case $|\alpha| = 1$, there are no $b$-Benford sequences whatsoever. This all-or-nothing type of statement should be compared to the metric (i.e., almost all or almost none) assertions in the next section. Notice also that together, Theorems 3.1 and 3.4 provide a complete classification with respect to Benford's law for $C^2$ maps.

**Proposition 3.6.** *Let $T$ be a $C^2$ map which has 0 as a stable attracting fixed point with $0 < |T'(0)| \leq 1$. Then $O_T(x)$ is a $b$-Benford sequence for all $x \neq 0$ sufficiently close to 0 if and only if $\log_b |T'(0)|$ is irrational. If $\log_b |T'(0)|$ is rational, then there are no $b$-Benford sequences at all near 0.*

By Remark 3.3(ii), an analogous statement holds for (complex) analytic maps [Ber2]. In view of Example 3.5 above, however, no such clear classification should be expected for $C^1$ maps.

## 4. Essentially nonlinear systems

Unlike the cases studied in Section 3, the dynamics near a super-attracting fixed point, i.e., for $\alpha = 0$, is essentially nonlinear. Rather than dealing with more general classes of maps, assume that $T \in C^\infty$ throughout this section (see, however, Remarks 4.2 and 4.6 below for maps with less smoothness). Again it proves useful to distinguish two quite different situations. If $T^{(p)}(0) \neq 0$ for some $p \in \mathbb{N}$, $p \geq 2$, then the fixed point 0 is called super-attracting *of finite order*; otherwise, 0 is referred to as a *flat* super-attracting fixed point.

With respect to Benford's law, the dynamics of $T$ near a super-attracting fixed point of finite order is governed by the first nonvanishing term in the Taylor expansion of $T$.

**Theorem 4.1.** *If $T$ is a $C^\infty$ map with 0 as a super-attracting fixed point of finite order, then $O_T(x)$ is a strict Benford sequence for almost all $x$ sufficiently close to 0, but there also exist uncountably many exceptional points, i.e., points $x$ for which $O_T(x)$ is not a Benford sequence.*

*Proof.* Rewrite $T$ as $T(x) = \gamma x^p\bigl(1 - f(x)\bigr)$, where $f \in C^\infty$ and $f(0) = 0$. Without loss of generality, $\gamma > 0$, because otherwise $T$ could be replaced by $-T$ (if $p$ is even) or $T^2$ (if $p$ is odd). Since rescaling $T$ as $x \mapsto \alpha^{-1}T(\alpha x)$ with $\alpha = \gamma^{-(p-1)^{-1}} > 0$ does not affect any of the asserted properties, it is in fact sufficient to solely study the case $\gamma = 1$. Again restrict attention to the half-axis $x > 0$.

Step 1: Suppose that (sufficiently close to the origin) $T$ is given by $T(x) = x^p$. For $x > 0$ thus $\log_b T^n(x) = p^n \log_b x$ for every $b$ and all $n \in \mathbb{N}_0$. By the Birkhoff Ergodic Theorem [KH, Thm. 4.1.2], applied to the ergodic map $y \mapsto py \pmod 1$, the orbit $O_T(x)$ is a $b$-Benford sequence for Lebesgue almost every point $x$ near 0. Since the countable union of sets of measure zero has measure zero, $O_T(x)$ is in fact a strict Benford sequence for almost every $x$.

Step 2: As in the proof of Theorem 3.1, reduce the case of a general map $T$ given by $T(x) = x^p\bigl(1 - f(x)\bigr)$ to the case in Step 1. Fix a base $b \in \mathbb{N}\setminus\{1\}$. The map $S_b$ with

$$S_b(y) := -\log_b T(b^{-y}) = py + g_b(y)$$

is well defined and smooth for sufficiently large $y$; here $g_b$ is the $C^\infty$ function given by

$$g_b(y) = -\log_b\bigl(1 - f(b^{-y})\bigr)$$

with $\lim_{y\to\infty} g_b(y) = 0$. By (a one-sided version of) Corollary 2.6, there exists a continuous function $h_b$ with $\lim_{y\to\infty} |h_b(y) - y| = 0$ such that $h_b \circ S_b(y) = ph_b(y)$ for all sufficiently large $y$. Using the chain rule and termwise differentiation of the formula

$$h_b(y) = y + \sum_{j=0}^{\infty} p^{-(j+1)} g_b \circ S_b^j(y)$$

it can be seen that for large $y$ the function $h_b$ is in fact $C^1$ with positive derivative. In particular, $h_b$ is a homeomorphism of some unbounded interval, and it maps sets of measure zero onto sets of measure zero. Since $\lim_{n\to\infty} |p^n h_b(y) - S_b^n(y)| = 0$, Step 1 implies that the sequence $\bigl(S_b^n(y)\bigr)$ is u.d. mod 1 for almost all points $y \ge \eta$, where the threshold $\eta$ can be chosen simultaneously for all $b$. For almost all $x$ sufficiently close to 0, the sequence $\bigl(-\log_b T^n(x)\bigr) = \bigl(S_b^n(-\log_b x)\bigr)$ is thus u.d. mod 1 for all $b$, so $O_T(x)$ is a Benford sequence.

To see that uncountably many exceptional points exist, let $\overline{y} \in ]0,1[$ denote a number which is not $p$-normal. Fix a base $b$ and take a sufficiently large integer $m$ such that $\overline{y} + m = h_b(y)$ for some $y$. With the above notation therefore

$$\lim_{n\to\infty} |S_b^n(y) - p^n(\overline{y} + m)| = 0\,.$$

Thus $\bigl(S_b^n(y)\bigr)_{n\in\mathbb{N}_0}$ is not u.d. mod 1, and $O_T(b^{-y})$ is not a $b$-Benford sequence. Since the set of real numbers which are not $p$-normal is uncountable [Ber1, Thm. 5.21] the exceptional set is uncountable. □

*Remark* 4.2. (i) As a by-product, the above application of Corollary 2.6 yields a continuous map $H$ with $H \circ T(x) = H(x)^p$ sufficiently close to 0. From the explicit formula for $\gamma = 1$,

$$H(x) = x \prod_{j=0}^{\infty} \bigl(1 - f \circ T^j(x)\bigr)^{p^{-(j+1)}}, \tag{4.1}$$

it is also evident that $\lim_{x\to 0} x^{-1} H(x) = 1$.

(ii) As for the linearly dominated case, the argument in the proof of Theorem 4.1 may be readily adapted to an *analytic* map $T$ having 0 as a super-attracting fixed point, i.e., $T(z) = \lambda z^p + \dots$, with $\lambda \in \mathbb{C}\setminus\{0\}$. Formula (4.1) then yields a local *analytic* conjugacy between $T$ and the map $z \mapsto z^p$. The existence of such a conjugacy is also well known (e.g. see [Bea, Thm. 6.10.1]).

(iii) Theorem 4.1 is in fact true for a wider class of maps. Let $T$ be a $C^1$ map and assume that $T(0) = 0$ is a stable attracting fixed point. This point is called *power-like* super-attracting if

$$T(x) = \gamma x^{1+\varepsilon}\big(1 - f(x)\big),$$

where $f$ denotes a $C^1$ function with $f(0) = 0$, and $\gamma, \varepsilon$ are positive real numbers. (Obviously, every super-attracting fixed point of finite order is power-like.) Analogously to the above arguments, one can show that the conclusion of Theorem 4.1 holds for maps which have 0 as a power-like super-attracting fixed point; the details of the case $\varepsilon \notin \mathbb{N}$ are left to the interested reader (cf. [DT] and the references therein for the relevant results from uniform distribution theory).

**Example 4.3.** For simplicity, all the results above were stated for fixed points at the origin. They also hold, *mutatis mutandis*, for fixed points at infinity. In particular, Theorem 4.1 may be used to show that for every rational map $R$ with $\lim_{|x|\to\infty} \big|x^{-1}R(x)\big| \to \infty$, the orbit $O_R(x)$ is a strict Benford sequence for almost all, but not all, $x$ with $|x|$ sufficiently large. Indeed, these maps have infinity as a super-attracting fixed point of finite order in the sense that

$$R(\infty) := \lim_{x\to\infty} R(x) = \infty\,, \quad R'(\infty) := \lim_{x\to\infty} \frac{x^2 R'(x)}{\big(R(x)\big)^2} = 0$$

and $R^{(j)}(\infty) = 0$ for $j = 1, 2, \dots, r-1$ but $R^{(r)}(\infty) \neq 0$, where $r \geq 2$ is the difference between the degrees of the numerator and denominator polynomial of $R$, respectively. Consequently, Theorem 4.1 applies to the $C^\infty$ map $T : x \mapsto R(x^{-1})^{-1}$ near its super-attracting fixed point at 0.

As a specific example, consider the function $Q : x \mapsto x^2 + 1$: For almost all $x \in \mathbb{R}$ the orbit $O_Q(x)$ is a Benford sequence, but there is also an uncountable set of exceptional points, which are easily found by means of an explicit formula analogous to (4.1). More precisely, for the continuous map $H$ defined as

$$H(x) := \sqrt{1+x^2}\prod_{j=1}^{\infty}\big(1 + (Q^j(x))^{-2}\big)^{2^{-(j+1)}} = \lim_{j\to\infty}\big(Q^j(x)\big)^{2^{-j}},$$

it is readily verified that $H \circ Q(x) = H(x)^2$ for all $x \in \mathbb{R}$, and $\lim_{|x|\to\infty} H(x)/|x| = 1$. Thus $O_Q(x)$ is a Benford sequence if and only if $\big(H(x)^{2^n}\big)_{n\in\mathbb{N}_0}$ is. In this simple example, an explicit formula for $H^{-1}$ is available, namely

$$H^{-1}(x) = \lim_{j\to\infty}\sqrt{\dots\sqrt{\sqrt{x^{2^j}-1}-1}\dots-1}\,,$$

which is valid for $x \geq H(0) \approx 1.2259$. In particular, for any base $b$, the orbit of $\xi_b := H^{-1}(b)$ is not a $b$-Benford sequence since the first significant digit of $Q^n(\xi_b)$ eventually always equals $b-1$, i.e., $\big\lfloor M_b\big(Q^n(\xi_b)\big)\big\rfloor = b-1$ for all but finitely many $n$. For example, in its decimal representation, the first significant digit of $Q^n(\xi_{10})$

with

$$\xi_{10} = H^{-1}(10) = 9.949623089593959412183321241093 26\ldots$$

equals 9 for all $n \in \mathbb{N}_0$.

Not surprisingly, the situation near a flat super-attracting fixed point can be more complicated, and an additional regularity condition will be imposed on $T$ (see also Example 4.7 below). Again restrict to $x > 0$, and assume that $T(x) > 0$ for all $x \in ]0, \delta]$ for some $\delta > 0$. Given $T$, define the function

$$\Phi_T(x) := \frac{xT'(x)}{T(x)} \qquad (0 < x \le \delta)\,.$$

**Theorem 4.4.** *Suppose $T$ has 0 as a super-attracting fixed point. If $\Phi_T$ is nonincreasing on $]0, \delta]$ and $\lim_{x\to 0} \Phi_T(x) > 1$, possibly infinite, then $O_T(x)$ is a strict Benford sequence for almost all $x \in ]0, \delta]$.*

*Proof.* Fix a base $b$, define $f_n(x) := \log_b T^n(x)$, and view $(f_n)$ as a sequence of real-valued $C^1$ functions on $]0, \delta]$. For any two natural numbers $m, n$,

$$\begin{aligned} f'_{m+n}(x) - f'_n(x) &= \frac{(T^n)'(x)}{T^n(x)\log b}\left(\frac{(T^m)' \circ T^n(x)}{T^m \circ T^n(x)} T^n(x) - 1\right) \\ &= \frac{\Phi_{T^n}(x)}{x \log b}\left(\Phi_{T^m} \circ T^n(x) - 1\right). \end{aligned}$$

Since $T'(x) > 0$ for sufficiently small $x \neq 0$, the map $T$ is monotone near 0, and so

$$\Phi_{T^n} = \Phi_T \circ T^{n-1} \cdot \Phi_T \circ T^{n-2} \cdot \ldots \cdot \Phi_T \circ T \cdot \Phi_T$$

is nonincreasing on $]0, \delta]$, and $\lim_{x\to 0} \Phi_{T^n} > 1$. Therefore, $f'_{m+n} - f'_n$ is monotone, and for some $0 < \theta < \delta$,

$$f'_{m+n}(x) - f'_n(x) \ge C > 0 \quad \text{for all } m, n \in \mathbb{N} \text{ and } x \in [T(\theta), \theta],$$

where $C$ depends neither on $m, n$ nor on $x$. Lemma 2.4(iii) implies that $\big(f_n(x)\big)_{n\in\mathbb{N}_0}$ is u.d. mod 1 for almost all $x$. □

**Example 4.5.** Theorem 4.4 applies to the (families of) $C^\infty$ maps

$$T(x) = \begin{cases} e^{-x^{-\gamma}} & \text{if } x > 0\,, \\ 0 & \text{if } x = 0\,, \end{cases} \quad \text{and} \quad T(x) = \begin{cases} e^{-|\log x|^{1+\gamma}} & \text{if } x > 0\,, \\ 0 & \text{if } x = 0\,, \end{cases}$$

with $\gamma > 0$, which both have 0 as a flat super-attracting fixed point.

*Remark* 4.6. (i) Clearly, flatness plays no role in the above proof, and the same technique also works for several classes of maps already covered by Theorem 4.1, e.g., for the map given by $T(x) = x^p - \gamma x^{p+1} + x^{p+2} f(x)$ with $\gamma > 0$, $p \ge 2$ and any $C^\infty$ function $f$.

(ii) Although the focus throughout this section is on smooth maps, note that the proof of Theorem 4.4 requires $T$ only to be $C^1$ (cf. Remark 4.2). Therefore, this theorem can also be used to show that for any $\gamma > 0$ and

$$T(x) = x^{1+\gamma} \qquad \text{or} \qquad T(x) = \begin{cases} -x^{1+\gamma}\log x & \text{if } x > 0\,, \\ 0 & \text{if } x = 0\,, \end{cases}$$

$O_T(x)$ is a strict Benford sequence for almost all $x > 0$ sufficiently close to 0.

(iii) The conditions on $\Phi_T$ in Theorem 4.4 are restrictive; they imply, for example, that $F(x) := -\log T(e^{-x})$ defines a convex function, and $\underline{\lim}_{x\to\infty} x^{-1}F(x) > 1$ (cf. Theorem 5.5 below).

**Example 4.7.** The conditions in Theorem 4.4 are certainly not sharp, but without any additional assumption, the conclusion of that theorem will generally be false, even if $T$ is strictly increasing. In fact, given any base $b$ and any map $T$ which has 0 as a flat super-attracting fixed point, it is possible to find a nondecreasing $C^\infty$ map $S$ with $0 < S(x) < T(x)$ for all $x \in ]0,\delta]$ such that for almost all $x \le \delta$, the orbit $O_S(x)$ is not a $b$-Benford sequence.

Given $T$ and $b$, a sketch of the construction of such a map $S$ is as follows. Consider the map given by $R_b(y) = -\log_b T(b^{-y})$, which is continuous and positive for $y \ge j_0$ for some $j_0 \in \mathbb{N}$. Furthermore, $\lim_{y\to\infty} R_b(y) = \infty$, and without loss of generality, assume that $R_b$ is nondecreasing. Define a sequence $(r_j)_{j\ge j_0-1}$ of natural numbers by

$$r_{j_0-1} := 1 \qquad \text{and} \qquad r_j := j - j_0 + r_{j-1} + \lceil \max_{|y-j|\le 1} R_b(y) \rceil \quad \text{for } j \ge j_0 \,.$$

Let $(J_j)$ and $(J_j^*)$ be the two sequences of mutually disjoint intervals

$$J_j := \left[ j - \frac{3}{8}, j + \frac{3}{8} \right] \quad \text{and} \quad J_j^* := \left[ j + \frac{7}{16}, j + \frac{9}{16} \right] \quad (j \ge j_0)$$

and let $\widetilde{R}_b$ be a nondecreasing $C^\infty$ function with $\widetilde{R}_b(y) = r_j$ for all $y \in J_j$, which increases linearly from $r_j + \frac{1}{16}$ to $r_{j+1} - \frac{1}{16}$ on $J_j^*$ for all $j \ge j_0$. It is easily checked that the orbit $O_{\widetilde{R}_b}(y)$ of almost every $y \ge j_0$ eventually consists of integers. Setting

$$S(x) := \begin{cases} b^{-\widetilde{R}_b(-\log_b x)} & \text{if } x > 0\,, \\ 0 & \text{if } x = 0\,, \end{cases}$$

it follows that $0 < S(x) < T(x)$ for $x$ sufficiently close to 0. By construction, $S$ is a nondecreasing $C^\infty$ map, and for almost all $x$ near 0, the orbit $O_S(x)$ is not a $b$-Benford sequence, since typically $\lim_{n\to\infty} \log_b S^n(x) = 0 \pmod 1$. (By slightly modifying $\widetilde{R}_b$ on the flat pieces $J_j$, one could even make $S$ strictly increasing.) So, no matter how attracting 0 is for $T$, there is always a smooth map $S$ such that 0 is even more attracting for $S$, but typical $S$-orbits near 0 are nevertheless not Benford sequences.

In summary, orbits near a super-attracting fixed point of finite order at 0 typically follow Benford's law. In the flat case this statement is also true provided that an additional, somewhat restrictive growth condition on the map is satisfied. The following complete analysis of super-attracting fixed points for *real-analytic* maps follows immediately from Theorem 4.1.

**Proposition 4.8.** *Let $T \not\equiv 0$ be a real-analytic map which has 0 as a super-attracting fixed point. Then $O_T(x)$ is a Benford sequence for almost all $x$ near 0, but there are also uncountably many exceptional points.*

The conclusion of the above proposition carries over to (complex) analytic maps [Ber2]. Obviously, Example 4.7 rules out such a simple over-all conclusion even for $C^\infty$ maps.

## 5. Nonautonomous systems

The purpose of this section is to establish general classes of nonautonomous real-valued dynamical systems which follow Benford's law. The main result, Theorem 5.5, demonstrates the robustness of the Benford behavior; for example, applying the maps $x^2$, $3^x$ or $x^x$ successively in any order will yield a Benford sequence for almost all sufficiently large initial points. As a complement to the previous sections, and in the spirit of many known Benford sequences such as $(n!)$, $(e^n)$ and the Fibonacci numbers $(F_n)$, the focus in this section will be on orbits which go to infinity. Of course, since $(x_n)$ is Benford if and only if $(x_n^{-1})$ is Benford (via Proposition 2.3), these results easily translate into results for an attractor at zero.

Recall that $T^n$ is the $n$-fold composition $T^n(x) = T_n \circ T_{n-1} \circ \ldots \circ T_1(x)$ of maps $T_n : \mathbb{R} \to \mathbb{R}$, and that $O_T(x)$ is the orbit of $x$, i.e., the sequence $(x, T_1(x), T_2 \circ T_1(x), \dots) = (x, T^1(x), T^2(x), \ldots)$; see (2.1). Unlike the cases in the previous sections, in the general nonautonomous setting, linear and nonlinear systems may overlap in the sense that the same orbit may be generated by a family of linear as well as nonlinear maps. For example, if $O_T(x)$ is the orbit of the nonautonomous system defined by $T_n(x) = 2^{2^{n-1}}x$, and $O_{\widehat{T}}(x)$ is the orbit for $\widehat{T}_n(x) = x^2$, then $O_T(2) = O_{\widehat{T}}(2)$, even though the maps defining $T$ are linear, and those for $\widehat{T}$ are quadratic. The measure-theoretic conclusions for linear versus nonlinear nonautonomous systems are analogous to those for the autonomous systems above, namely, orbits of linear systems are generally Benford for *all* initial points, but orbits of nonlinear systems are Benford only for *almost all* initial points.

Analogously to Theorem 3.1, first consider nonautonomous systems dominated by a linear term. Assuming that each $T_j$ is $C^1$ and has infinity as a fixed point with $T_j'(\infty) = \beta_j \neq 0$, the family $(T_j)$ may be rewritten as

$$T_j(x) = \beta_j x\bigl(1 - f_j(x)\bigr), \qquad j = 1, 2, \dots, \tag{5.1}$$

where the continuous functions $f_j$ satisfy $\lim_{|x|\to\infty} f_j(x) = 0$ for all $j \in \mathbb{N}$. Recall the abbreviations $B_0 = 1$ and $B_n := \prod_{j=1}^n \beta_j$ for $n \in \mathbb{N}$.

**Theorem 5.1.** *Let $(T_j)$ be given by (5.1), and suppose that $\sum_{j=1}^\infty |B_j|^{-1} < \infty$, and $\sup_{|x|\geq\xi} |xf_j(x)| \leq C < \infty$ for all $j \in \mathbb{N}$, for some constants $\xi > 0$, $C > 0$. Then for all $|x|$ sufficiently large, the orbit $O_T(x)$ is a $b$-Benford sequence precisely if $\bigl(B_n x\bigr)_{n\in\mathbb{N}_0}$ is.*

*Proof.* The assumptions imply that $\beta_j \neq 0$ for all $j \in \mathbb{N}$. Since

$$\sup_{|x|\geq\xi}\left|\frac{T_j(x)}{\beta_j} - x\right| = \sup_{|x|\geq\xi}|xf_j(x)| \leq C < \infty \quad \text{for all } j \in \mathbb{N},$$

Theorem 2.5 yields the existence of a function $h$ which is continuous for $|x|$ sufficiently large, and which satisfies

$$h(x) = \lim_{j\to\infty}\frac{T^j(x)}{B_j},$$

as well as $\overline{\lim}_{|x|\to\infty}|h(x) - x| < \infty$. Since $|B_j| \to \infty$, for $n \to \infty$ both $|T^n(x)| \to \infty$ and

$$\log_b|T^n(x)| - \log_b|B_n h(x)| \to 0\,.$$

For $|x|$ sufficiently large, therefore, $O_T(x)$ is a $b$-Benford sequence precisely if $\big(B_n h(x)\big)$ is, and since $h(x) \neq 0$ for $|x|$ sufficiently large, the sequence $(B_n x)$ is $b$-Benford, too. □

**Corollary 5.2.** *Let $(T_j)$ satisfy the hypotheses of Theorem 5.1. Then $O_T(x)$ is a $b$-Benford sequence for all $|x|$ sufficiently large if and only if $(\gamma_n)_{n\in\mathbb{N}_0}$ is u.d. mod 1, where*

$$\gamma_n := \log_b |B_n| = \sum_{j=1}^{n} \log_b |\beta_j| \,.$$

Theorem 5.1 and Corollary 5.2 can be rewritten in a form which makes the analogy to Theorem 3.1 even more transparent. In fact, the latter theorem is a special case of

**Theorem 5.3.** *Let $(T_j)$ be given by (5.1), and suppose that $\beta_j \neq 0$ and $\sum_{j=1}^{\infty} |B_j| < \infty$, and $\sup_{|x|\leq 1} \left|x^{-1} f_j(x)\right| \leq C < \infty$ for all $j \in \mathbb{N}$. Then $O_T(x)$ is a $b$-Benford sequence for all $x \neq 0$ sufficiently close to 0 if and only if $(\log_b |B_n|)_{n\in\mathbb{N}_0}$ is u.d. mod 1.*

*Proof.* The existence of the uniform bound $C$ on $\sup_{|x|\leq 1} \left|x^{-1} f_j(x)\right|$ implies that $T_j(x) \neq 0$ for all $j$ and $0 < |x| < \frac{1}{2} \min(C^{-1}, 1) =: \xi$. Setting

$$\eta := \xi \big(\max_{j\in\mathbb{N}_0} |B_j| + 2C\xi \sum_{j=0}^{\infty} |B_j|\big)^{-1} ,$$

clearly $0 < \eta < \xi$, and an argument analogous to that in Theorem 2.5 shows that $0 < |T^n(x)| < \xi$ for all $n$ and $0 < |x| < \eta$. Proceed as in the proof of Theorem 3.1, and define

$$S_j : x \mapsto T_j \left(x^{-1}\right)^{-1} = \beta_j^{-1} x\big(1 - g_j(x)\big), \quad j = 1, 2, \ldots ,$$

where the functions $g_j$ are continuous for $|x|$ sufficiently large, and

$$g_j(x) = -\frac{f_j(x^{-1})}{1 - f_j(x^{-1})} \,.$$

For sufficiently large constants $\zeta$ and $D$, $\sup_{|x|\geq\zeta} |x g_j(x)| \leq D < \infty$ for all $j$, so Theorem 5.1 applies to the family $(S_j)$. On the other hand, for $x$ sufficiently close to 0, the orbit $O_T(x)$ is a $b$-Benford sequence precisely if $O_S\left(x^{-1}\right)$ is. □

**Example 5.4.** Corollary 5.2 reduces the question whether (5.1) generates Benford sequences to a problem of uniform distribution. Using standard techniques from that theory (e.g. [KN]), it is straightforward to prove that for the classes of sequences $(\beta_j)$ listed below, $O_T(x)$ is a strict Benford sequence for all initial points $x$ close to infinity or 0, respectively. The proof of (i) uses [KN, Thm. 3.3]; (ii) uses [KN, Thm. 2.7] and Euler's summation formula; (iii) is analogous to [D]; and (iv) uses [KN, Thm. 3.2]. (Of course the respective uniform growth conditions on $(f_j)$ in Theorems 5.1 and 5.3 also have to be met.)

(i) $\lim_{j\to\infty} \beta_j = \beta_\infty$, where $\beta_\infty$ is not a rational power of any base, i.e.,

$$\beta_\infty \notin \left\{b^q : q \in \mathbb{Q},\, b \in \mathbb{N}\backslash\{1\}\right\}.$$

In particular, setting $\beta_1 = 1$ and $\beta_j := F_j/F_{j-1}$ ($j \geq 2$), where $F_j$ denotes the $j$-th Fibonacci number, leads to $\beta_\infty = \frac{1+\sqrt{5}}{2}$. Since $\log_b \beta_\infty$ is irrational for every base $b$, Theorem 5.3 and Lemma 2.4(ii) imply that the sequence of Fibonacci numbers is a strict Benford sequence (cf. [BD, KN]).

(ii) $\beta_j = r(j)$, where $r$ denotes any rational function with real coefficients such that $\beta_j$ is finite for all $j$, and such that $\lim_{j\to\infty} |r(j)| \in \{0, \infty\}$. Notice that this includes, as the special case $\beta_j = j$, the well-known fact [D] that the sequence $(n!)$ is strictly Benford.

(iii) $\beta_j = j^\gamma$, where $\gamma \neq 0$ is any nonzero real number. (This also includes the $(n!)$ case.)

(iv) $\beta_j = e^{\pm p(j)}$, where $p(j) = j^k + a_{k-1}j^{k-1} + \ldots + a_1 j + a_0$ with $k \in \mathbb{N}$ and $a_0, \ldots, a_{k-1} \in \mathbb{R}$, i.e., $p$ is a monic polynomial of degree $k$ with real coefficients.

Next, consider nonlinear nonautonomous dynamics. The following theorem is the main result of this section.

**Theorem 5.5.** *Suppose* $T_j : \mathbb{R}^+ \to \mathbb{R}^+$, $j \in \mathbb{N}$, *are such that*

(i) $\log T_j(e^x)$ *is convex in* $x$, *and*

(ii) $x^{-1} \log T_j(e^x)$ *is nondecreasing in* $x$, *and* $\geq \beta > 1$

*for all* $x \geq x_0$ *and all* $j \in \mathbb{N}$. *Then* $O_T(x)$ *is a strict Benford sequence for almost all sufficiently large* $x$.

The proof of Theorem 5.5 will use three lemmas, the first of which is a basic fractional-parts inequality which will be used to establish asymptotic independence and strong law convergence for random variables associated with $O_T(x)$. Let $\lambda$ denote Lebesgue measure, and for a convex function $f : \mathbb{R} \to \mathbb{R}$, let $f'^{+}(x)$ denote the right-hand derivative $\lim_{h\searrow 0} \frac{f(x+h)-f(x)}{h}$ of $f$ at $x$ (which exists everywhere, since $f$ is convex); similarly, $f'^{-}$ is the left-hand derivative.

**Lemma 5.6.** *Let* $f : [0,1] \to \mathbb{R}$ *be convex, nondecreasing and nonnegative. Then for all* $c \in ]0,1[$,

$$c - \frac{1}{f'^{+}(0)} \leq \lambda\bigl(\{x \in [0,1] : f(x) (\operatorname{mod} 1) \leq c\}\bigr) \leq c + \frac{2}{f'^{+}(0)} \,. \tag{5.2}$$

*Proof.* Fix $c \in ]0,1[$. If $f'^{+}(0) < 1$, the conclusion is trivial, so without loss of generality, assume that $f'^{+}(0) \geq 1$. Since all the terms in inequality (5.2) remain unchanged if $f(x)$ is replaced by $f(x) - \lfloor f(0) \rfloor$, further assume, without loss of generality, that $f(0) \in [0,1[$.

Set $s_0 = 0$, set $t_0 = 0$ if $f(0) > c$ and $= f^{-1}(c)$ otherwise, and for $k \in \mathbb{N}$, let $s_k = \min\{x : f(x) = k\}$ and $t_k = \min\{x : f(x) = k + c\}$, if such $x$'s exist, and $s_k = 1$, respectively $t_k = 1$, otherwise. By the convexity of $f$, $f'^{+}$ is nondecreasing, so by the definitions of $(s_k)$ and $(t_k)$, and the assumption that $f(0) \in [0,1[$, it is clear that

$$t_k - s_k \leq \frac{c}{f'^{+}(0)} \quad \text{and} \quad s_{k+1} - t_k \leq \frac{1-c}{f'^{+}(0)} \quad \text{for all} \quad k = 0, 1, 2, \ldots \,. \tag{5.3}$$

Since $f$ is convex and nondecreasing, and $s_k \leq t_k \leq s_{k+1}$ for all $k \in \mathbb{N}$, the definitions of $(s_k)$ and $(t_k)$ imply that

$$(1-c)(t_{k+1} - s_{k+1}) \leq c(s_{k+1} - t_k) \leq (1-c)(t_k - s_k) \quad \text{for all} \quad k \in \mathbb{N}.$$

Setting $a := \sum_{k=0}^{\infty}(t_k - s_k) = \lambda\bigl(\{x \in [0,1] : f(x)(\operatorname{mod} 1) \leq c\}\bigr)$ and analogously $b := \sum_{k=0}^{\infty}(s_{k+1} - t_k)$, this implies that

$$(1-c)\bigl(a - (t_0 - s_0) - (t_1 - s_1)\bigr) \leq c\bigl(b - (s_1 - t_0)\bigr) \leq (1-c)\bigl(a - (t_0 - s_0)\bigr),$$

so since $s_1 \geq t_0 \geq s_0$,

$$\left(\frac{1-c}{c}\right)\big(a-(t_0-s_0)-(t_1-s_1)\big) \leq b \leq \left(\frac{1-c}{c}\right) a + (s_1 - t_0). \tag{5.4}$$

Since $a+b=1$, $s_k \geq 0$, $t_k \geq 0$ and $c \in (0,1)$, (5.3) and (5.4) imply that

$$c - \frac{1}{f'^{+}(0)} \leq a \leq c + \frac{2}{f'^{+}(0)}\,.$$

□

**Lemma 5.7.** *Let $f, g : \mathbb{R}^+ \to \mathbb{R}^+$. If $\log f$, $\log g$ are convex, nondecreasing and nonnegative, then so are $f$, $g$ and $\log(f \circ g)$.*

*Proof.* Taking $h(x) = e^x$, $f = h \circ \log f$ is convex since an increasing convex function of a convex function is convex; $f$ is nondecreasing since $h$ is increasing and $\log f$ is nondecreasing, and $f$ is nonnegative (in fact $\geq 1$) since $\log f$ is nonnegative. Similarly, $g$ and $\log(f \circ g)$ are convex, nondecreasing and nonnegative. □

**Lemma 5.8** (Loéve [L, p. 154])**.** *Let $X_1, X_2, \ldots$ be mean-zero random variables such that, for some $0 < M < \infty$,*

(i) $|X_n| \leq M$, *and*

(ii) $\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{n=1}^{N} \sum_{m=1}^{N} E|X_n X_m| \leq M$.

*Then* $\frac{1}{n}\sum_{i=1}^{n} X_i \to 0$ *a.s.*

*Proof of* Theorem 5.5. Fix $\beta > 1$. By Proposition 2.2, $O_T(x)$ is Benford if and only if $\log_b T^n(x)$ is u.d. mod 1 for all $b \in \mathbb{N}\setminus\{1\}$. Since the set of such bases $b$ is countable, and $\log_b(\cdot)$ maps sets of measure zero into sets of measure zero, setting $S_n(x) = \log_b T^n(b^x)$, it suffices to show that for each $b \in \mathbb{N}\setminus\{1\}$, and all sufficiently large $j \in \mathbb{N}$,

$$\big(S_n(x)\big) \text{ is u.d.} \quad \text{mod } 1 \text{ for almost all } x \in [j-1, j]. \tag{5.5}$$

To see (5.5), fix $c \in ]0,1[$ and define the sequence of random variables $(Y_n)$ on the probability space $([j-1,j], \mathcal{B}, \lambda)$ by letting $Y_n$ be the composition of the indicator function of the set $\bigcup_{k=0}^{\infty}[k, k+c)$ with $S_n$, that is, $Y_n = 1$ if $S_n \mod 1 \leq c$, and $= 0$ otherwise. Since a random variable $X$ is uniformly distributed on $[j-1, j]$ if and only if $P(X \leq j-1+c) = c$ for all rational $c \in ]0,1[$, and since countable unions of sets of measure zero have measure zero themselves, to establish (5.5), it suffices to show

$$\frac{Y_1 + \cdots + Y_n}{n} \to c \quad \text{a.s. as } n \to \infty. \tag{5.6}$$

(Note that in general the $(Y_n)$ are neither independent nor identically distributed.)

By (i) and Lemma 5.7, $S_n$ is convex, nondecreasing and nonnegative for each $n \in \mathbb{N}$, and sufficiently large $x$. By (ii), $S_1(x) = \log_b T_1(b^x) \geq \beta \log_b b^x = \beta x$, so since $S_1$ is convex, without loss of generality (taking $j$ large), $S_1'^{+}(j-1) = m \geq \frac{1+\beta}{2} > 1$. By (ii) and the definitions of $T^n$ and $S_n$,

$$\frac{S_{n+1}(x)}{S_n(x)} = \frac{\log_b T^{n+1}(b^x)}{\log_b T^n(b^x)} = \frac{\log T_{n+1}\big(T^n(b^x)\big)}{\log T^n(b^x)} \geq \beta,$$

so by (ii), for $x \in ]j-1, j[$ and all $n, k \in \mathbb{N}$,

$$\frac{S_{n+k}(x)}{S_n(x)} \text{ is nondecreasing in } x, \text{ and } \geq \beta^k\,. \tag{5.7}$$

Since $S_n$ is nondecreasing and nonnegative, and $S_1(j-1) > 0$ and $S_n(x) > 0$ (without loss of generality, taking $j$ large) $S_{n+1}'^{+} = [(S_n)(S_{n+1}/S_n)]'^{+} \geq \beta S_n'^{+}$, so by the convexity of $S_{n+k}$,

$$S_{n+k}'^{+}(x) \geq \beta^k S_n'^{+}(x) \geq \beta^{n+k-1} S_1'^{+}(j-1) \geq m\beta^{n+k-1}, \tag{5.8}$$

for all $n, k \in \mathbb{N}$ and $x \in ]j-1, j[$. For each $n \in \mathbb{N}$, let $A_n = \{x \in [j-1, j] : S_n(x) \,(\text{mod } 1) \leq c\}$, so that $Y_n = \mathbf{1}_{A_n}$. By (5.8) and Lemma 5.6,

$$|\lambda(A_n) - c| \leq \frac{2}{m\beta^n} \qquad \text{for all } n \in \mathbb{N}. \tag{5.9}$$

Next it will be shown that

$$|E(Y_n Y_{n+k}) - EY_n EY_{n+k}| \leq \frac{8}{c\beta^k} \qquad \text{for all } \; n, k \in \mathbb{N}. \tag{5.10}$$

By definition of $S_n$, $A_n = \bigcup_{i=0}^{\infty}[s_i, t_i]$, where $s_0 = j-1$ and, for all $i \geq 0$, $s_{i+1} \geq t_i \geq s_i$ and $t_i \leq j$. Since $S_n$ is convex, $t_i - s_i \geq t_{i+1} - s_{i+1}$ for all $i \in \mathbb{N}$, and by the definition of $A_n$ and convexity of $S_n$ again,

$$S_n'^{+}(s_{i+1}) \geq S_n'^{-}(t_i) \geq \frac{c}{t_i - s_i} \qquad \text{for all } \; i \in \mathbb{N}.$$

Thus by (5.8), $S_{n+k}'^{+}(s_{i+1}) \geq \frac{c\beta^k}{t_i - s_i}$ for all $i \in \mathbb{N}$, so by Lemma 5.6,

$$|\lambda(A_{n+k} \cap [s_{i+1}, t_{i+1}]) - c(t_{i+1} - s_{i+1})| \leq \frac{2d_i}{c\beta^k} \qquad \text{for all } i \in \mathbb{N}. \tag{5.11}$$

By (5.8) and the convexity of $S_{n+k}$ again, Lemma 5.6 implies that $\lambda(A_{n+k} \cap [s_i, t_i]) - c(t_i - s_i) \leq \frac{2}{m\beta^{n+k-1}}$ for $i = 0, 1$. Thus, since $m > c > 0$,

$$|\lambda(A_{n+k} \cap A_n) - c\lambda(A_n)| \leq \sum_{i=1}^{\infty} \frac{2(t_i - s_i)}{c\beta^k} + \frac{4}{m\beta^{n+k-1}} \leq \frac{6}{c\beta^k}\,. \tag{5.12}$$

By (5.9), $|\lambda(A_{n+k}) - c| = |EY_{n+k} - c| \leq \frac{2}{m\beta^{n+k}}$, so by (5.9) and (5.12),

$$\begin{aligned} |E(Y_{n+k}Y_n) - EY_{n+k}EY_n| &= |\lambda(A_{n+k} \cap A_n) - \lambda(A_{n+k})\lambda(A_n)| \\ &\leq |\lambda(A_{n+k} \cap A_n) - c\lambda(A_n)| + |c\lambda(A_n) - \lambda(A_n)\lambda(A_{n+k})| \\ &\leq \frac{6}{c\beta^k} + \lambda(A_n)|c - \lambda(A_{n+k})| \leq \frac{6}{c\beta^k} + \frac{2}{m\beta^{n+k}} \leq \frac{8}{c\beta^k}, \end{aligned}$$

which proves (5.10), the key result for asymptotic independence of the $(Y_n)$.

For $n \in \mathbb{N}$, let $X_n = Y_n - EY_n$. By (5.10),

$$\sum_{n=1}^{N}\sum_{m=1}^{N} |E(X_n X_m)| \leq \sum_{n=1}^{N} E|X_n|^2 + 2N \sum_{k=1}^{N-1} \frac{8}{c\beta^k} \leq N\widehat{M},$$

where $\widehat{M} = 1 + \frac{16}{c}\sum_{k=1}^{\infty}\frac{1}{\beta^k} < \infty$, since $c > 0$ and $\beta > 1$. Thus,

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{n=1}^{N}\sum_{m=1}^{N} |E(X_n X_m)| \leq \widehat{M} \sum_{N=1}^{\infty} \frac{1}{N^2} < \infty,$$

so letting $M = \max\bigl(2, \widehat{M}\sum_{N=1}^{\infty}\frac{1}{N^2}\bigr)$, Lemma 5.8 implies that $\frac{1}{n}\sum_{i=1}^{n} X_i \to 0$ a.s. Hence $\frac{1}{n}\sum_{j=1}^{n}(Y_j - EY_j) \to 0$ a.s., and since $\beta > 1$, (5.9) implies that $EY_n \to c$ as $n \to \infty$, which proves (5.6). □

**Example 5.9.** The orbits of the following systems are Benford sequences for almost all sufficiently large initial points:

(i) $T_j(x) = x^{p_j}$ with $\inf_{i\in\mathbb{N}} p_i > 1$ (this generalizes the results for the autonomous system with $T(x) = x^p$, $p > 1$ in Section 4);

(ii) $T_j(x) = \begin{cases} x^2 & \text{if } j \text{ is even,} \\ 2^x & \text{if } j \text{ is odd;} \end{cases}$

(iii) $T_j(x) = j^x$.

(For instance, to show (i), $\log T_j(e^x) = \log e^{p_j x} = p_j x$ is convex, nondecreasing and nonnegative for $x > 0$, and $x^{-1} \log T_j(e^x) = p_j \geq \inf_{i\in\mathbb{N}} p_i =: \beta > 1$, so Theorem 5.5 applies.) The conclusion of Theorem 5.5 may fail if even a single function does not satisfy the hypotheses, as the next simple example shows.

**Example 5.10.** $T_1(x) = 2$, and $T_j(x) = \frac{1}{2}x^2$ for $j > 1$. Clearly, $T_j$ satisfies the hypotheses of Theorem 5.5 for all $x$ and all $j > 1$, but $O_T(x) \equiv (x, 2, 2, 2, \dots)$ is not $b$-Benford for any $b \in \mathbb{N}\setminus\{1\}$.

Since autonomous systems are a special case of nonautonomous systems, general nonlinear nonautonomous systems may also have exceptional sets, and the conclusion of Theorem 5.5 may clearly fail if $\beta = 1$; the autonomous system $T(x) = x$ satisfies hypothesis (i) of the theorem, but not (ii) because $x^{-1} \log T_j(e^x) \equiv 1$ (and clearly $O_T(x)$ is not a Benford sequence for *any* $x$).

Note that the conclusion of Theorem 5.5 does *not* include all the nonlinear autonomous systems in Section 4 above; e.g., for $T(x) = x^2 + 1$ nondecreasing in Theorem 5.5(ii) fails and for $T(x) = x^2 - 1$ convexity in (i) fails. On the other hand, that theorem does not require any of the differentiability assumptions of Sections 3 and 4, and applies, for example, to the autonomous system defined by the convex map $T$ with $T(x) = \sum_{i=0}^{\infty} \frac{x^{i+1}}{i!} \mathbf{1}_{[i,i+1[}(x)$ which is not differentiable at any $i \in \mathbb{N}$.

*Remark* 5.11. (i) Many real-valued recursive sequences of *higher order* such as the Fibonacci sequence (cf. [BD] or Example 5.4(i) above) are Benford sequences because they are multi-dimensional analogues of the results in this paper, and the interested reader is referred to [Ber2] for an analysis of Benford's law in multi-dimensional dynamical systems.

(ii) Given the ubiquity of Benford behavior, it might be interesting to develop tests for the statistical analysis of significant digits of numerical data, which could be used to detect hidden periodicity, or help determine the nature of attracting fixed points.

(iii) The authors do not know whether analogous general results hold without stability assumptions; for example, they know of no example of an unstable system whose orbits obey Benford's law for more than a finite number of bases.

## 6. Differential equations and Benford's law

This final section treats Benford properties of dynamical systems generated by ordinary differential equations

$$\dot{x} = F(x, t) \tag{6.1}$$

on the real line. Though analogous to the discrete-time case, the results are somewhat easier due to the dynamical simplicity of differential equations in one dimension, especially autonomous ones.

First the Benford property for real-valued functions is defined. Recall that $\mathbf{1}_A$ is the indicator function of $A \subseteq \mathbb{R}$.

**Definition 6.1.** A measurable real-valued function $f : [0, +\infty[ \to \mathbb{R}$ is a *b-Benford function* if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_{[1,t[} \circ M_b(|f(\tau)|)\, d\tau = \log_b t \qquad \text{for all } t \in [1, b[\,,$$

and it is called a *strict Benford function* (or simply a *Benford function*) if it is a $b$-Benford function for all $b \in \mathbb{N}\setminus\{1\}$.

As in the discrete-time case, there is a direct correspondence between Benford functions and continuous uniform distributions. (The term *continuously uniformly distributed modulo one* will henceforth be abbreviated as *c.u.d. mod* 1.) Recall that $\log_b 0 := 0$ for all bases $b$. The next result is the continuous analogue of Proposition 2.2.

**Proposition 6.2.** *A function $f$ is a b-Benford function if and only if* $(\log_b |f(t)|)_{t\geq 0}$ *is c.u.d. mod* 1.

The symbol $(\varphi_t x_0)_{t\geq 0}$ will denote the solution of (6.1), subject to the initial condition $x(0) = x_0$; conditions must be imposed on $F$ to assure that this solution exists and is unique. Use of the notation $(\varphi_t x_0)_{t\geq 0}$ will, however, imply that this solution is defined for all $t \geq 0$. First, assume that the function $F$ in (6.1) is $C^2$, and also that $F(0, t) = 0$ for all $t$, so

$$F(x, t) = -\alpha(t)x + xf(x, t) \tag{6.2}$$

where $\alpha$ and $f$ are $C^1$, and $f(0, t) \equiv 0$. For brevity, the quantities $A(t) := \int_0^t \alpha(\tau)\, d\tau$ refer to $\alpha(\cdot)$ in the initial value problem

$$\dot{x} = -\alpha(t)x + xf(x, t)\,, \quad x(0) = x_0\,. \tag{6.3}$$

The following theorem is a direct analogue of Theorem 5.3.

**Theorem 6.3.** *Suppose that* $\int_0^\infty e^{-A(\tau)} d\tau < \infty$ *and* $\sup_{\tau\geq 0} e^{-A(\tau)} < \infty$ *as well as* $\sup_{|x|\leq 1} \left|x^{-1} f(x, t)\right| \leq C < \infty$ *for all* $t \geq 0$. *Then for all* $x_0$ *sufficiently close to* 0 *there is a unique solution* $(\varphi_t x_0)_{t\geq 0}$ *of* (6.3)*, and this solution is a b-Benford function if and only if* $(A(t)/\log b)_{t\geq 0}$ *is c.u.d. mod* 1.

*Proof.* By standard local existence and uniqueness results (e.g. [A, Sec. 7]), solutions of (6.3) exist locally. The uniform bound $C$ on $\sup_{|x|\leq 1} \left|x^{-1} f(x, t)\right|$, and $\int_0^\infty e^{-A(\tau)} d\tau < \infty$ together imply that locally the origin is the only equilibrium in (6.3). Every nonconstant solution $x$ to (6.3) yields, via $x \mapsto x^{-1}$, a local solution to

$$\dot{x} = -x^2 F(x^{-1}, t) = \alpha(t)x + g(x, t)\,, \quad x(0) = x_0^{-1} \tag{6.4}$$

with $g(x, t) := -xf(x^{-1}, t)$. Since $g$ is uniformly bounded as $|x| \to \infty$, the solution of (6.4) exists globally, that is, for all $t \geq 0$, provided $|x_0|^{-1}$ is sufficiently large [A, Prop. 7.8]; denote this solution by $(\psi_t x_0^{-1})_{t\geq 0}$ and notice that it may be (implicitly) represented as

$$\psi_t x_0^{-1} = e^{A(t)} x_0^{-1} + \int_0^t e^{A(t)-A(\tau)} g(\psi_\tau x_0^{-1}, \tau)\, d\tau \,.$$

Clearly, $\varphi_t x_0 = (\psi_t x_0^{-1})^{-1}$ for all $t \geq 0$.

To apply a continuous-time shadowing argument, consider the map

$$h : y \mapsto y + \int_0^\infty e^{-A(\tau)} g(\psi_\tau y, \tau)\, d\tau$$

which is well defined and continuous for $|y|$ sufficiently large; furthermore, $h$ satisfies $\overline{\lim}_{y\to\infty} |h(y) - y| < \infty$. From

$$e^{A(t)} h(y) - \psi_t y = \int_t^\infty e^{A(t)-A(\tau)} g(\psi_\tau y, \tau)\, d\tau$$

for all $t \geq 0$, it follows that for sufficiently large $y$

$$\left| 1 - \frac{\psi_t y}{e^{A(t)} h(y)} \right| \leq \frac{C}{|h(y)|} \int_t^\infty e^{-A(\tau)}\, d\tau,$$

where the right-hand side tends to zero as $t \to \infty$. For $|x_0|$ sufficiently small, the solution $(\varphi_t x_0)_{t\geq 0}$ of (6.3) is therefore a $b$-Benford function if and only if $(A(t)/\log b)_{t\geq 0}$ is c.u.d. mod 1. □

*Remark* 6.4. The condition $\int_0^\infty e^{-A(\tau)} d\tau < \infty$ may be considered the continuous-time analogue of $\sum_j |B_j| < \infty$ in Theorem 5.3. While the latter automatically implies $\sup_j |B_j| < \infty$, the condition $\sup_{\tau\geq 0} e^{-A(\tau)} < \infty$ has to be added here.

**Corollary 6.5.** *Let $F$ be a $C^2$ function with $F(0) = 0$ and $F'(0) > 0$. Then for every $x_0 \neq 0$ sufficiently close to 0, the solution of the initial value problem*

$$\dot{x} = -F(x)\,, \quad x(0) = x_0$$

*is a Benford function.*

**Example 6.6.** Clearly, Theorem 6.3 could be formulated in a "reciprocal" version analogous to Theorem 5.1 with infinity (instead of 0) as an attractor; details are left to the interested reader. As in Example 5.4, the extensive knowledge about continuous uniform distribution ([KN, Sec. 1.9]) yields the following classes of functions $\alpha$ to which Theorem 6.3 applies:

(i) $\alpha = \alpha(t)$, with $\int_0^\infty |\alpha(\tau) - \alpha_\infty|\, d\tau < \infty$ for some $\alpha_\infty > 0$;
(ii) $\alpha = p(t)$, where $p$ is any real polynomial with $\lim_{t\to\infty} p(t) = +\infty$;
(iii) $\alpha(t) = t^{\gamma-1}$ with $\gamma > 0$;
(iv) $\alpha(t) = e^{p(t)}$, where $p$ is any real polynomial with $\lim_{t\to\infty} p(t) = +\infty$.

The rest of this section focuses on *autonomous* differential equations

$$\dot{x} = -F(x) \tag{6.5}$$

where $F$ denotes a $C^2$ function with $F(0) = 0$ and $xF(x) > 0$ for all $x \neq 0$ sufficiently close to 0. If the origin is to be a stable equilibrium, $\alpha := F'(0)$ must be nonnegative. The case $\alpha > 0$ has already been dealt with in Corollary 6.5. If $\alpha = 0$, then the solutions of (6.5) tend to 0 rather slowly, and they may fail to yield Benford functions. Since solutions cannot cross the origin, the analysis may be restricted to the positive half-axis $x > 0$. As in Theorem 3.4, no Benford functions appear as solutions of (6.5) under even slightly weaker smoothness assumptions on $F$.

**Theorem 6.7.** *Let $F$ be $C^{1+\varepsilon}$ for some $\varepsilon > 0$, and assume that $F(0) = 0$, $F'(0) = 0$ and $F(x) > 0$ for $x \in ]0, \delta]$ for some $\delta > 0$. Then no solution $(\varphi_t x_0)_{t\geq 0}$ of (6.5) with $x(0) = x_0 \leq \delta$ is a $b$-Benford function for any $b \in \mathbb{N}\backslash\{1\}$.*

*Proof.* Solutions of (6.5), together with the initial condition $x(0) = x_0$, exist locally. Within finite time, these solutions can reach neither values larger than $\delta$ nor, by the divergence of $\int_0^\delta F(x)^{-1}dx$, the origin, so they are in fact defined for all $t \geq 0$ [A, Thm. 7.6]. The function $y$ with $y(t) := -\log_b(\varphi_t x_0)$ solves the autonomous ordinary differential equation

$$\dot{y} = \frac{F(b^{-y})b^y}{\log b} =: G_b(y)\,. \tag{6.6}$$

Obviously, $G_b$ is a positive $C^2$ function with $\lim_{y\to\infty} G_b(y) = 0$. Thus the solution $(\psi_t y_0)_{t\geq 0}$ of (6.6), together with $y(0) = y_0$ sufficiently large, exists for all $t \geq 0$ and is unique. It will now be shown that $(\psi_t y_0)_{t\geq 0}$ is *not* c.u.d. mod 1.

The assumptions on $F$ imply that $G_b(y) \leq Cb^{-\varepsilon y}/\log b$ for all sufficiently large $y$, for some $\varepsilon > 0$. But then

$$t = \int_{y_0}^{\psi_t y_0} \frac{dy}{G_b(y)} \geq C^{-1}\int_{y_0}^{\psi_t y_0} b^{\varepsilon y}\log b\, dy = (C\varepsilon)^{-1}(b^{\varepsilon\psi_t y_0} - b^{\varepsilon y_0})$$

for all $t \geq 0$, and thus

$$\psi_t y_0 \leq \varepsilon^{-1}\log_b(C\varepsilon t + b^{\varepsilon y_0}).$$

If $(\psi_t y_0)_{t\geq 0}$ were c.u.d. mod 1, then by [KN, Thm. 9.7] the sequence $(\psi_n y_0)_{n\in\mathbb{N}_0}$ would be u.d. mod 1, which contradicts Lemma 2.4(i). □

**Example 6.8.** Analogously to Example 3.5, note that the assumption $F \in C^{1+\varepsilon}$ in the above theorem cannot be weakened to $F \in C^1$. Indeed, with the $C^1$ function

$$F(x) := \begin{cases} -\frac{x}{2\log x} & \text{if } 0 < x < 1\,, \\ 0 & \text{if } x = 0\,, \end{cases}$$

the solution $(\varphi_t x_0)_{t\geq 0}$ of (6.5) with $x(0) = x_0$ is a Benford function for all $x_0 \in ]0, 1[$.

Finally, consider solutions of (6.5) for functions $F$ with $F(0) = 0$ and $F(x) > 0$ for $x > 0$, but with $\lim_{x\to 0} x^{-1}F(x) = \infty$. Obviously, such functions are not differentiable at the origin.

**Theorem 6.9.** *Let $F$ be continuous on $[0, \delta]$ and continuously differentiable on $]0, \delta[$ for some $\delta > 0$. Suppose that $F(0) = 0$, that $x \mapsto x^{-1}F(x)$ is nonincreasing with $\lim_{x\to 0} x^{-1}F(x) = \infty$, and that $\int_0^\delta F(x)^{-1}dx$ diverges. Then for all $x_0$ sufficiently close to 0, there exists a unique solution of*

$$\dot{x} = -F(x)\,, \quad x(0) = x_0\,,$$

*and this solution is a Benford function.*

*Proof.* Unique solutions exist locally for $x_0 \in ]0,\delta]$; since $\int_0^\delta F(x)^{-1}dx = \infty$, they are defined for all $t \geq 0$, and $\lim_{t\to\infty} \varphi_t x_0 = 0$ for all $x_0$ sufficiently close to 0. Fix a base $b$ and define the $C^1$ function $g_b$ on $[0,+\infty[$ by

$$g_b(t) := -\log_b(\varphi_{e^t-1}x_0)\,.$$

Clearly, $g_b(t) \to \infty$ monotonically as $t \to \infty$. Since

$$\frac{d}{dt}g_b(t) = (\log b)^{-1} b^{g_b(t)} F(b^{-g_b(t)})e^t \;>\; 0$$

and $\frac{d}{dt}g_b$ is increasing, the function $g_b$ is convex and increasing. Furthermore, $\lim_{t\to\infty} t^{-1}g_b(t) = \infty$, which by [KN, Thm. 9.5] implies that

$$\big(g(\log(t+1))\big)_{t\geq 0} = \big(-\log_b \varphi_t x_0\big)_{t\geq 0}$$

is c.u.d. mod 1. Thus $(\varphi_t x_0)_{t\geq 0}$ is a $b$-Benford function, and since $b$ was arbitrary, it is strictly Benford. □

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