"Quickie" Inequalities

Murray S. Klamkin†

"Quickie" problems first appeared in the March 1980 issue of Mathematics Magazine. They were originated by the late Charles W. Trigg, a prolific problem proposer and solver who was then the Problem Editor. Many of the first good Quickie proposals were due to the late Leo Moser (who incidentally was a member of the University of Alberta Mathematics Department and subsequently its chairman). These Quickie problems are even now still a popular part of the journal. Also Quickies have proliferated to the problem sections of Crux Mathematicorum, Math Horizons, SIAM Review and Mathematical Intelligencer (unfortunately, no longer in the latter two journals).

Trigg noted that some problems will be solved by laborious methods but with proper insight[1] may be disposed of with dispatch. Hence the name "Quickie".

The probability that two random numbers are equal is zero. It follows that there are more inequalities than equations. Consequently, the study of inequalities are important throughout mathematics. In past issues of π *in the Sky*, December 2001, September 2002, Professor Hrimiuc has provided some good notes on inequalities and we shall be referring to some of them.

Here we illustrate 16 Quickie inequalities and after each one we include for the interested reader an exercise that can be solved in a related manner.

Our first example will set the stage for our Quickie Inequalities.

1. There have been very many derivations published giving the formulas for the distance from a point to a line and a plane. Here is a Quickie derivation for the distance from the point (h,k,l) to the plane $ax+by+cz+d=0$ in $\mathbb{E}^3$. Here we want to find the minimum value of $[(x-h)^2+(y-k)^2+(z-l)^2]^{1/2}$ where (x,y,z) is a point of the given plane. By Cauchy's Inequality,

$$[(x-h)^2+(y-k)^2+(z-\ell)^2]^{1/2}[a^2+b^2+c^2]^{1/2} \geq |a(x-h)+b(y-k)+c(z-\ell)|$$

or

$$\min[(x-h)^2+(y-k)^2+(z-\ell)^2]^{1/2} = |ah+bk+c+d|/[a^2+b^2+c^2]^{1/2}.$$

Exercise. Determine the distance from the point (h,k) to the line $ax+by+c=0$.

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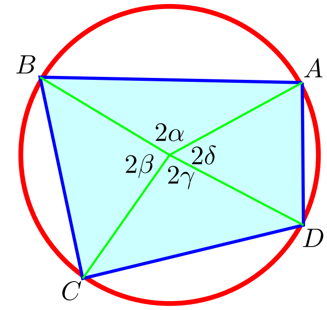
[1] and appropriate knowledge-MSK

2. KöMaL problem F. 3097. A convex quadrilateral $ABCD$ is inscribed in a unit circle. Its sides satisfy the inequality $AB \cdot BC \cdot CD \cdot DA \geq 4$. Prove that $ABCD$ is a square.



Let the angles subtended by the four sides from the center be 2α, 2β, 2γ, and 2δ (see figure above). Then $AB = 2\sin\alpha$, $BC = 2\sin\beta$, $CD = 2\sin\gamma$ and $CD = 2\sin\delta$ where $\alpha+\beta+\gamma+\delta=\pi$, $\pi > \alpha,\ \beta,\ \gamma,\ \delta > 0$. Since $\ln(\sin x)$ is concave,

$$\ln(\sin\alpha)+\ln(\sin\beta)+\ln(\sin\gamma)+\ln(\sin\delta) \leq 4\ln\left(\sin\frac{\pi}{4}\right)$$

or $AB \cdot BC \cdot CD \cdot DA \leq 4$. Hence the product is exactly 4 and $\alpha=\beta=\gamma=\delta=\frac{\pi}{4}$ so $ABCD$ is a square.

Exercise. Of all convex n-gons inscribed in a unit circle, determine the maximum of the product of its n sides.

3. KöMaL problem F. 3238. Prove that the inequality

$$\sqrt{a^2+(1-b)^2} + \sqrt{b^2+(1-c)^2} + \sqrt{c^2+(1-a)^2} \geq \frac{3\sqrt{2}}{2}$$

holds for arbitrary real numbers a,b,c.

By Minkowski's Inequality, the sum of the three radicals is grater or equal than $\sqrt{(a+b+c)^2+(3-a-b-c)^2}$. Then by the power mean inequality or else letting $a+b+c=x$, the expression under the radical is $2(x-3/2)^2+9/2$, so the minimum value is $\frac{3\sqrt{3}}{2}$.

Exercise. Determine the minimum value of

$$\{x^3+(c-y)^3+a^3\}^{1/3}+\{y^3+b^3+(d-x)^3\}^{1/3},$$

where a, b, c, d are given positive numbers and $x,y \geq 0$.

4. Determine the maximum and minimum z coordinates of the surface

$$5x^2+10y^2+2z^2+10xy-2yz+2zx-8z=0 \ \text{ in } \ \mathbb{E}^3.$$

One method would be to use Lagrange Multipliers. Another more elementary method would be to use discriminants of quadratic equations since if $z=h$ is the maximum, the intersection of the plane $z=h$ with the quadric must be a single point. Even simpler is to express the quadric that is an ellipsoid as a sum of squares, i.e., $(2x+y)^2+(x-y+z)^2+(z-4)^2=16$. Hence $\max z=8$ and $\min z=0$.

Exercise. Determine the maximum value of y^2 and z^2 where x,y,z are real and satisfy

$$(y-z)^2+(z-x)^2+(x-y)^2+x^2=a^2.$$

5. Let $a_r=(b_r+b_{r+1}+b_{r+2})/b_{r+1}$ where $b_1,\ b_2,\ \ldots,\ b_n>0$ and $b_{r+n}=b_r$. Determine the minimum value of

$$\sqrt[3]{a_1}+\sqrt[3]{a_2}+\cdots+\sqrt[3]{a_n}.$$

Even more generally, let

$$x_j = x_{1j} + x_{2j} + \cdots + x_{mj}, \quad j = 1, 2, \cdots, n,$$

where all $x_{ij} > 0$ and $\prod_{j=1}^{n} x_{ij} = P_i^n$, $i = 1, 2, \cdots, m$. Then

$$S \equiv \sqrt[r]{x_1} + \sqrt[r]{x_2} + \cdots + \sqrt[r]{x_n} \ \geq \ n\sqrt[r]{P_1 + P_2 + \cdots + P_m}.$$

We first use the Arithmetic–Geometric Mean Inequality to get

$$S \geq n(x_1 x_2 \cdots x_n)^{1/rn}.$$

Then applying Holder's Inequality we are done. There is equality if and only if $x_{ij} = x_{jk}$ for all i, j, k.

The given inequality corresponds to the special case where $r = m = 3$, $P_1 = P_2 = P_3 = 1$, so that the minimum value is $n\sqrt[3]{3}$.

The inequalities here are extensions of problem #M1277, Kvant, 1991, which was to show that

$$\sum_{i=1}^{n} \{a_i + a_{i+1})/a_{i+2}\}^{1/2} \geq n\sqrt{2}.$$

Exercise. Gy. 2887, KöMal. The positive numbers $a_1, a_2, \ldots, a_n$ add up to 1. Prove the following inequality:

$$(1 + 1/a_1)(1 + 1/a_2) \cdots (1 + 1/a_n) \geq (n + 1)^n.$$

6. If a, b, c are sides of a triangle ABC and R_1, R_2, R_3 are the distances from a point P in plane of ABC to the respective vertices A, B, C. Prove that

$$aR_1^2 + bR_2^2 + cR_3^2 \geq abc.$$

This is a polar moment of inertia inequality and is a special case of the more general inequality

$$(x\vec{\mathbf{A}} + y\vec{\mathbf{B}} + z\vec{\mathbf{C}})^2 \geq 0,$$

where $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, $\vec{\mathbf{C}}$ are vectors from P to the respective vertices A, B, C. Expanding out the square, we get

$$x^2R_1^2 + y^2R_2^2 + z^2R_3^2 + 2yz\vec{\mathbf{B}} \cdot \vec{\mathbf{C}} + 2zx\vec{\mathbf{C}} \cdot \vec{\mathbf{A}} + 2xy\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}.$$

Since $2\vec{\mathbf{B}} \cdot \vec{\mathbf{C}} = R_2^2 + R_3^2 - a^2$, etc., the general polar moment of inertia inequality reduces to

$$(x + y + z)(xR_1^2 + yR_2^2 + zR_3^2) \geq yza^2 + zxb^2 + xyc^2.$$

Many triangle inequalities are special cases since x, y, z are arbitrary real numbers. In particular by letting $(x, y, z) = (a, b, c)$, we get our starting inequality. Letting P be the circumcenter and $x = y = z$, we get $9R^2 \geq a^2 + b^2 + c^2$ or equivalently $\sin^2 A + \sin^2 B + \sin^2 C \leq \frac{9}{4}$.

Exercise. Prove that

$$aR_2R_3 + bR_3R_1 + cR_1R_2 \geq abc.$$

7. Prove the identity

$$u(v - w)^5 + u^5(v - w) + v(w - u)^5 + v^5(w - u)$$
$$+w(u - v)^5 + w^5(u - v) = -10uvw(u - v)(v - w)(w - u),$$

and from this obtain the triangle inequality

$$aR_1(a^4 + R_1^4) + bR_2(b^4 + R_2^4) + cR_3(c^4 + R_3^4) \geq 10abcR_1R_2R_3$$

(with the same notation as in Problem 6).

The identity is a 6th degree polynomial. The left hand side vanishes for $u = 0$, $v = 0$, $w = 0$, $u = v$, $v = w$, and $w = u$. Hence the right hand side equals $kuvw(u - v)(v - w)(w - u)$, where k is a constant. On comparing the coefficients of uv^3w^2 on both sides, $k = -1$.

Now, let u, v, w denote complex numbers representing the vectors from the point P to the respective vertices A, B, C. Taking the absolute values of the both sides of the identity and using the triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$, we obtain the desired triangle inequality.

Exercise. Referring to Problem 6, prove that

$$aR_1R_1' + bR_2R_2' + cR_3R_3' \geq abc, \text{ where } R_1', R_2', R_3'$$

are the distances from another point Q to the respective vertices A, B, C.

8. Determine the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the constraint $x^2 + y^2 + z^2 + 2xyz = 1$.

Since it is known that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1$ is a triangle identity, we let $x = \cos\alpha$, $y = \cos\beta$, and $z = \cos\gamma$ where $\alpha + \beta + \gamma = \pi$ and $\pi \geq \alpha$, β, $\gamma \geq 0$. Clearly the maximum of $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$ is 3 and is taken on for $(x, y, z) = (1, 1, -1)$ and permutations thereof. For the minimum (using the above),

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 3 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma) \geq \frac{3}{4}.$$

Exercise. Determine the maximum of

$$\left\{\sum_{i=1}^{n} x_i\right\}\left\{\sum_{j=1}^{n} \sqrt{a_i^2 - x_i^2}\right\},$$

where $a_i \geq x_i \geq 0$.

9. Problem # 2, Final Round 21st Austrian Mathematical Olympiad. Show that for all natural numbers $n > 2$,

$$\sqrt{2\sqrt[3]{3}\sqrt[4]{4}\cdots\sqrt[n]{n}} < 2.$$

Here we get a better upper bound. If P denotes the left hand side, then

$$\ln P = \frac{\ln 2}{2!} + \frac{\ln 3}{3!} + \cdots + \frac{\ln n}{n!}.$$

Since $\dfrac{\ln x}{x}$ is a decreasing function for $x \geq e$,

$$\ln P < \frac{\ln 2}{2!} + \frac{\ln 3}{3}\left\{\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots\right\} = \frac{\ln 2}{2!} + \frac{\ln 3}{3}(e-2) \approx 1.7592.$$

Exercise. Determine a good lower bound for P.

10. Prove that for any distinct real numbers a, b,

$$\frac{e^b - e^a}{b-a} > e^{\frac{b+a}{2}}.$$

This is a special case of the following result due to J. Hadamard [1]: *If a function f is differentiable, and its derivative is an increasing function on a closed interval $[r,s]$, then for all x_1, $x_2 \in (r,s)$ $(x_1 \neq x_2)$, then*

$$\int_{x_1}^{x_2} \frac{f(x)dx}{x_2 - x_1} > f\left\{\frac{x_2 + x_1}{2}\right\}.$$

Letting $f(x) = e^x$, we get the desired result.

Exercise. Prove that

$$e^{b^2} - e^{a^2} > (b^2 - a^2)e^{\frac{(b+a)^2}{4}}.$$

11. Prove that

$$\cosh(y-z) + \cosh(z-x) + \cosh(x-y) \geq \cosh x + \cosh y + \cosh z$$

where x, y, z are real numbers whose sum is 0.

Since $\cosh x = \cosh(y+z)$, etc., the inequality can be rewritten as

(i) $\quad \sinh y \sinh z + \sinh z \sinh x + \sinh x \sinh y \leq 0.$

Since (i) is obviously valid if at least one of $x, y, z = 0$, we can assume that $xyz \neq 0$ and $x, y > 0$. Since $z = -(x+y)$, (i) becomes $\operatorname{csch} x + \operatorname{csch} y \geq \operatorname{csch}(x+y)$ for all $x, y > 0$. This follows immediately since $\operatorname{csch} t$ is a decreasing function for all $t > 0$.

Exercise. Prove that

$$\frac{v}{w} + \frac{w}{v} + \frac{w}{u} + \frac{u}{w} + \frac{u}{v} + \frac{v}{u} \geq u + \frac{1}{u} + v + \frac{1}{v} + w + \frac{1}{w}$$

where $u, v, w > 0$ and $uvw = 1$.

12. It is known and elementary that in a triangle, the longest median is the one to the shortest side and the shortest median is the one to the longest side. Determine whether or not the longest median of a tetrahedron is the one to the smallest area face and the shortest median is the one to the largest area face.

Let the sides of tetrahedron $PABC$ be given by $PA = a$, $PB = b$, $PC = c$, $PC = d$, $CA = e$, and $AB = f$. The median m_p from P is given by $\frac{|\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}}|}{3}$ where $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, $\vec{\mathbf{C}}$ are vectors from P to A, B, C respectively. Then

$$9m_p^2 = |\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}}|^2 = 3(a^2 + b^2 + c^2) - (d^2 + e^2 + f^2)$$

and similar formulas for the other medians. It now follows that $9m_a^2 - 9m_b^2 = 4(a^2 + f^2) - 4(b^2 + e^2)$. It is now possible to have $m_a = m_b$ with their respective face areas unequal, so that the longest median is not one to the smallest face area. The valid analogy is that the longest median is the one to the face for which the sum of the squares of its edges is the smallest, and the shortest median is the one to the face for which the sum of the squares of its edges is the largest.

Exercise. Prove that the four medians of a tetrahedron are possible sides of a quadrilateral.

13. a, b, c, d are positive numbers such that $a^5 + b^5 + c^5 + d^5 = e^5$. Can $a^n + b^n + c^n + d^n = e^n$ for any number $n > 5$?

Let $S_t = x_1^t + x_2^t + \dots + x_n^t$ where the $x_i \geq 0$. A known result [2] is that the sum $\mathbb{S}_t$ of order t, defined by $\mathbb{S}_t = (S_t)^{1/t}$ decreases steadily from $\min x_i$ to 0 as t increases from $-\infty$ to 0^-, and decreases steadily from ∞ to $\max x_i$ as t increases from 0^+ to $+\infty$. Consequently, there is no such n.

Exercise. Prove that $\mathbb{S}_T \leq \sum_{i=1}^{n} \alpha_1 \mathbb{S}_{t_i}$ for arbitrary $t_i > 0$ and for $\alpha_1 > 0$, $\sum_{i=1}^{n} \alpha_1 = 1$ and $T = \sum_{i=1}^{n} \alpha_1 t_i$.

14. Prove that

$$\frac{x^{t+1}}{y^t} + \frac{y^{t+1}}{z^t} + \frac{z^{t+1}}{x^t} \geq x + y + z$$

where x, y, $z > 0$ and $t \geq 0$.

Let

$$F(t) = \frac{\left[y\left(\frac{x}{y}\right)^{t+1} + z\left(\frac{y}{z}\right)^{t+1} + x\left(\frac{z}{x}\right)^{t+1}\right]^{\frac{1}{t+1}}}{[x+y+z]^{\frac{1}{t+1}}}.$$

Then by the Power Mean Inequality, $F(t) \geq F(0) = 1$.

Exercise. Prove more generally that

$$\frac{x^{t+1}}{a^t} + \frac{y^{t+1}}{b^t} + \frac{z^{t+1}}{c^t} \geq \frac{(x+y+z)^{t+1}}{(a+b+c)^t},$$

where x, y, z, a, b, $c > 0$ and $t \geq 0$.

15. Determine the maximum value of

$$S = 3(a^3 + b^2 + c) - 2(bc + ca + ab),$$

where $1 \geq a$, b, $c \geq 0$.

Here, $S \leq 3(a+b+c) - 2(bc + ca + ab)$. Since this latter expression is linear in each of a, b, c, its maximum value is taken on for a, b, $c = 0$ or 1. Hence the maximum is $6 - 2 = 4$.

Exercise. Determine the maximum value of

$$S = 4(a^4 + b^4 + c^4 + d^4) - (a^2bc + b^2cd + c^2da + d^2ab) - (a^2b + b^2c + c^2d + d^2a),$$

where $1 \geq a, b, c, d \geq 0$.

16. Determine the maximum and minimum values of

$$\sin A + \sin B + \sin C + \sin D + \sin E + \sin F,$$

where $A + B + C + D + E + F = 2\pi$ and $\frac{\pi}{2} \geq A, B, C, D, E, F \geq 0$.

Here we get a quick solution by applying Karamata's Inequality [3]. If two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ having n components, a_i and b_i, are arranged in non-increasing magnitude are such that

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i, \quad k = 1, 2, \ldots, n-1,$$

and

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i,$$

we say that $\vec{\mathbf{A}}$ majorizes $\vec{\mathbf{B}}$ and write $\vec{\mathbf{A}} \succ \vec{\mathbf{B}}$. We then have for a convex function $F(x)$ that

$$F(a_1) + F(a_2) + \cdots + F(a_n) \geq F(b_1) + F(b_2) + \cdots + F(b_n).$$

If $F(x)$ is concave, the inequality is reversed.

Since $\sin x$ is concave in $[0, \pi/2]$, and

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, 0\right) \succ (A, B, C, D, E, F) \succ \left(\frac{2\pi}{6}, \frac{2\pi}{6}, \frac{2\pi}{6}, \frac{2\pi}{6}, \frac{2\pi}{6}, \frac{2\pi}{6}\right).$$

The maximum value is $6 \sin \frac{\pi}{3}$ or $\sqrt[3]{3}$ and the minimum value is $4 \sin \frac{\pi}{2}$ or 4.

Exercise. Determine the extreme values of $a^5 + b^5 + c^5 + d^5 + e^5 + f^5$ given that a, b, c, d, e, f, are distinct positive integers with sum 36.

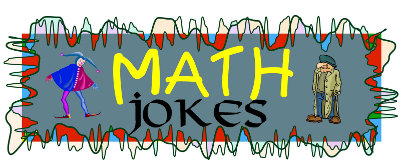
References:

1. D.S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970, p. 14.

2. E.F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, New York, 1965, p. 18.

3. A.W. Marshall and I. Olkin, *Inedqualities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

Murray S. Klamkin has a long and distinguished career in both industry and academia. He is known primarily as a problem solver, editing the problem corners of many journals over the years. He put this talent to good use in leading the USA team in the IMO, chairing the USAMO committee and authoring several books on mathematics competitions. He is particularly fond of triangle inequalities and spherical geometry. (Andy Liu)



Mother to her daughter: "Why does the tablecloth you just put on the table have the word 'truth' written on it?"

Daughter: "Because I want to turn the table into a truth table!"

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