A Generalization of Synthetic Division

Rohitha Goonatilake†

I. Introduction

In this article, we consider a procedure for division of polynomials. This is an alternative to a previously known process called long division of polynomials that involves the coefficients of polynomials. As we know, synthetic division works only for a divisor of the form \( x - k \). In [1], Donnell showed that it can also be extended to a divisor of the form \( x^n - k \) for \( n \geq 2 \). The purpose of this article is to extend this method to any polynomial divisor with a unit leading coefficient. This procedure has many applications; it is particularly important in factoring and finding the zeros of a polynomial. Several preliminary topics discussed in this article stem from [2].

Definition 1. Division Algorithm

If \( p(x) \) and \( d(x) \) are polynomials such that \( d(x) \neq 0 \), and the degree of \( d(x) \) is less than or equal to the degree of \( p(x) \), then there exists unique polynomials \( q(x) \) and \( r(x) \) such that

\[
p(x) = d(x) \cdot q(x) + r(x),
\]

where \( r(x) = 0 \) or the degree of \( r(x) \) is less than the degree of \( d(x) \). If the remainder \( r(x) \) is zero, then we say that \( d(x) \) divides evenly into \( p(x) \). In this setting, \( p(x) \), \( d(x) \), \( q(x) \), and \( r(x) \) are respectively called dividend, divisor, quotient, and remainder.

Remark 1. The Division Algorithm can also be written as

\[
\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.
\]

The rational expression \( p(x)/d(x) \) is called improper because the degree of \( p(x) \) is greater than or equal to the degree of \( d(x) \). On the other hand, the rational expression \( r(x)/d(x) \) is called proper because the degree of \( r(x) \) is less than the degree of \( d(x) \). It is also assumed that \( p(x) \) and \( d(x) \) have no common factors.

II. Horner’s Method

Horner’s Method is a method of writing a polynomial in a nested manner. It gives us a method for evaluating polynomials that is very useful with a calculator. Consider the polynomial,

\[
p(x) = 3x^3 + 8x^2 + 5x - 7.
\]

Synthetic division by \((x - k)\) yields the following:

<table>
<thead>
<tr>
<th>( k )</th>
<th>3</th>
<th>8</th>
<th>5</th>
<th>(-7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-7)</td>
<td>3</td>
<td>3k + 8</td>
<td>(3k + 8)k + 5</td>
<td>(3k + 8)k + 5</td>
</tr>
</tbody>
</table>

Hence, by the remainder theorem, we know that \( p(k) = [(3k + 8)k + 5]k - 7 \). In terms of \( x \), we can write

\[
p(x) = 3x^3 + 8x^2 + 5x - 7 = [(3x + 8)x + 5]x - 7.
\]

This is called Horner’s method of writing a polynomial. It can be applied to any polynomial by successively factoring out \( x \) from each nonconstant term, as demonstrated in the following example.

\[
p(x) = 5x^4 - 3x^3 + x^2 - 8x + 7 = (5x^3 - 3x^2 - x - 8)x + 7 \quad \text{Factor } x \text{ from first four terms}
\]

\[
= (5x^2 - 3x + 1)x - 8x + 7 \quad \text{Factor } x \text{ from first three terms}
\]

\[
= [(5x - 3)x + 1]x - 8 \quad \text{Factor } x \text{ from first two terms}
\]

Before continuing the discussion of this topic any further, let us describe nested multiplication in a formal setting (so that it can be translated into a tableau), for a general polynomial \( p(x) \) of degree \( m \) in Newton’s form. It might be

\[
p(x) = a_0 + a_1[(x - x_0)] + a_2[(x - x_0)(x - x_1)] + \cdots + a_m[(x - x_0)(x - x_1) \cdots (x - x_{m - 1})].
\]

This can be written succinctly as

\[
p(x) = a_0 + \sum_{i=1}^{m} a_i \prod_{j=0}^{i-1} (x - x_j),
\]

where the standard product notation has been used. The nested form of \( p(x) \) is

\[
p(x) = a_0 + (x - x_0)\left\{ a_1 + (x - x_1)\left\{ a_2 + \cdots + (x - x_{m-1})a_m \right\} \right\} \cdots \\
= \left\{ \cdots \left\{ a_m(x - x_{m-1}) + a_{m-1}(x - x_{m-2}) + a_{m-2} \right\} \cdots + a_1 \right\} (x - x_0) + a_0.
\]

Thus, the polynomial \( p(x) \) considered before, with all the \( x_j \)'s equal to zero, takes the nested form

\[
p(x) = \left\{ \cdots \left\{ a_mx + a_{m-1}x + a_{m-2} \right\} \cdots + a_1 \right\} x + a_0,
\]

with appropriate choices of \( x_{m-1}, x_{m-2}, \cdots, x_0 \) for a polynomial of degree \( m \). For the tableau to be described in the next section, we write the divisor \( d(x) \) of degree \( m \) with leading coefficient one in nested form as

\[
d(x) = \left\{ \cdots \left\{ [(x - k_1)x - k_2]x - k_3 \right\} x - \cdots - k_{m-1} \right\} x - k_m.
\]

III. Synthetic Division

As illustrated above, there is a nice shortcut for long division of \( p(x) \) by polynomials of the form \( x - k \). The shortcut is called synthetic division and it involves the coefficients of the polynomial and \( k \). The essential steps of this division tableau

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are performed by using only the coefficients. By the remainder theorem, we know that the remainder \( r(x) \big|_{x=k} = p(k) \).

Donnell [1] considered division by a polynomial of the form \( x^n - k \) for \( n \geq 2 \). In the extension, we propose synthetic division of polynomials for any polynomial divisor. We insist that the leading coefficient of the divisor polynomial \( d(x) \) be 1. In the event that the leading coefficient is different from 1, we divide both dividend \( p(x) \) and divisor \( d(x) \) by the leading coefficient of the divisor as required by this division table.

It is understood that any divisions under consideration has this normalization and that the polynomials are written with descending powers of \( x \). The latter is often referred to as standard form. We now describe the pattern for synthetic division of a given polynomial by any polynomial divisor using a carefully chosen set of worked examples. A proof of the general statement of this method is not attempted in this article due to the notational difficulties it may cause. In fact, such a proof is not within the scope of this article.

Suppose \( p(x) \) is a polynomial of degree \( n \), which is to be divided by a polynomial divisor \( d(x) \) of degree \( m \), where \( 1 \leq m \leq n \). This results in a quotient \( q(x) \) of degree \( n - m \) and a remainder \( r(x) \) of degree \( m - 1 \) or less. The key steps of the procedure are explained below. Some discussions are very brief as we assume the reader is already familiar with the basic steps found in synthetic division and those of [2].

**Step 1:** The \( n+1 \) coefficients of \( p(x) \) are arranged in order of descending powers of \( x \) in the top row of the division tableau. Zeros are used to replace any missing coefficients of the expansion.

**Step 2:** \( k_1 \) is chosen from the nested form. First it is placed outside to the left of the left extremum column. And for any \( k_j \) for \( j \geq 2 \), they are placed outside to the left of corresponding rows after the sum is computed for each and every column as \( j \) increases. Next, place \( j \) number of dashes (number of dashes correspond to the subscript of \( k \)) directly under first \( j \) number of coefficients of \( p(x) \). For \( k_1 \), bring the first coefficient to be the first sum in the next row. The dashes are considered to have value 0 in computing this sum. Next, place \( j+1 \) dashes directly under the first \( j+1 \) coefficients of \( p(x) \). Bring down the first \( j+1 \) coefficients to be the first \( j+1 \) sums in its row.

**Step 3:** The next step is to multiply each of these \( j+1 \) for \( j \geq 1 \) sum by \( k_{j+2} \), placing the result of each multipliers in the next of \( j+1 \), for \( j \geq 1 \) positions to the right. Omit any product that would go beyond last column of the tableau.

**Step 4:** Add the next \( m \) (or possibly fewer) coefficients to these products and place sum in the next row. Continue this procedure until the bottom row is filled with \( n+1 \) numbers after all divisions are performed in this fashion for all \( k_j \) for \( m \geq j \geq 1 \). This can be easily understood by looking at the given set of examples.

**Step 5:** Finally, the first \( n-m+1 \) numbers on the last row from the left are chosen to be the coefficients of quotient polynomial \( q(x) \). The next \( m-1 \) are related to coefficients of remainder polynomial \( r(x) \). Before they are finally accepted as the coefficients of the remainder polynomial, they have to be adjusted. The numbers that appear in the boxes are the coefficients so obtained for the actual remainder polynomial.

**Step 6:** The number appearing next to the \( \rightarrow \) is the product of the leading coefficients of the dividend and all the \( k_j \)s for \( j \geq 1 \) in nested form of divisor \( d(x) \). This is only done under limited situations as given in the next step. The sign \( \downarrow \) means that the number is simply brought down to be the coefficient of the remainder. This is due to the fact that the sum is in question equals 0. The signs \( \downarrow \) and \( \rightarrow \) are used in accordance with the table given in Step 7 with priority given to \( \downarrow \), whenever both occur.

**Step 7:** Depending on the number of \( k_j \)s for \( m \geq j \geq 1 \), in the last row of the tableau, addends (of sums) in each row are subtracted from the number of the level of tableau. If the number of addends is 1, this subtraction is not carried out. If \( j = 2 \), this is done using the sum of the addend picked up from the top that amounts to \( j - 1 \). As \( j \) increases from 2 to 3, the sum of two addends is subtracted from the entries of the last row. This step is carried out for every column from the right. These numbers as well as their negated sums are underlined for easy referencing.

**Example 1.** Divide \( 2x^4 + 4x^3 - 5x^2 + 3x - 2 \) by \( x^2 + 2x - 3 \).

We write the divisor in a nested manner. Thus Horner’s method applied to the divisor gives

\[
x^2 + 2x - 3 = x(x + 2) - 3
\]

Note \( k_1 = -2 \) and \( k_2 = 3 \).

\[
\begin{array}{cccccc}
-2 & 2 & 4 & -5 & 3 & -2 \\
- & -4 & 0 & 10 & -26 \\
3 & 2 & 0 & -5 & 13 & -28 \\
- & - & 6 & 0 & 3 & \\
2 & 0 & 1 & 13 & -25 & \\
\rightarrow & -12 & 26 & & \\
\end{array}
\]

The numbers above the braces give the coefficients of the quotient and those in boxes give the coefficients of the remainder polynomial. Now we have

<table>
<thead>
<tr>
<th>Example</th>
<th>( j ) for any products dropped</th>
<th>( j ) - no. of products dropped at level ( j )</th>
<th>( \rightarrow ) or ( \downarrow )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2 - 1 = 1</td>
<td>( \rightarrow )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2 - 0 = 2</td>
<td>( \downarrow )</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2 - 1 = 1</td>
<td>( \rightarrow )</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2 - 1 = 1</td>
<td>( \rightarrow )</td>
</tr>
<tr>
<td>5</td>
<td>2, 3</td>
<td>2 - 1 = 1, 3 - 1 = 2</td>
<td>use ( \downarrow ) and not ( \rightarrow )</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>2 - 0 = 2</td>
<td>( \downarrow )</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>2 - 0 = 2</td>
<td>( \downarrow )</td>
</tr>
<tr>
<td>8</td>
<td>2, 3</td>
<td>2 - 1 = 1, 3 - 1 = 2</td>
<td>use ( \downarrow ) and not ( \rightarrow )</td>
</tr>
<tr>
<td>Remark 2</td>
<td>4</td>
<td>4 - 2 = 2</td>
<td>( \downarrow )</td>
</tr>
</tbody>
</table>
\[ \frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 0x + 1 + \frac{x + 1}{x^2 + 2x - 3}. \]

**Example 2.** Divide \( x^3 - 1 \) by \( x^2 + x + 1 \).

We write the divisor in a nested manner. Thus Horner’s method applied to the divisor gives
\[
x^2 + x + 1 = x(x + 1) + 1 \Rightarrow
\]
Note that \( k_1 = -1 \) and \( k_2 = -1 \).

\[
\begin{array}{cccc}
-1 & 1 & 0 & 0 \\
-1 & 1 & -1 & 1 \\
1 & -1 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 0 & 0 \\
\end{array}
\]

Now, similar to the previous example, we have
\[
x^3 - 1 = x - 1 + \frac{0x + 0}{x^2 + x + 1}.
\]

**Example 3.** Divide \( x^4 + 3x^2 + 1 \) by \( x^2 - 2x + 3 \).

We write the divisor in a nested manner. Thus Horner’s method applied to the divisor gives
\[
x^2 - 2x + 3 = x(x - 2) + 3 \Rightarrow
\]
Note that \( k_1 = 2 \) and \( k_2 = -3 \).

\[
\begin{array}{cccc}
2 & 1 & 0 & 3 \\
-2 & 4 & 14 & 28 \\
-3 & 2 & 7 & 14 & 29 \\
\end{array}
\]

\[
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 2 & 4 & 8 & 17 \\
\end{array}
\]

Now, as before, we have
\[
x^4 + 3x^2 + 1 = x^2 + 2x + 4 + \frac{2x - 11}{x^2 - 2x + 3}.
\]

**Example 4.** Divide \( x^4 + x^3 - x^2 + 2x \) by \( x^2 + 2x \).

We write the divisor in a nested manner. Thus, Horner’s method applied to the divisor gives
\[
x^2 + 2x = x(x + 2) + 0 \Rightarrow
\]
Note that \( k_1 = -2 \) and \( k_2 = 0 \).

\[
\begin{array}{cccc}
-2 & 1 & 1 & -1 \\
-2 & 2 & -2 & 0 \\
0 & 1 & -1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & -1 & 1 & 0 \\
\end{array}
\]

Now, as before, we have
\[
x^4 + x^3 - x^2 + 2x = x^2 - x + 1 + \frac{0x + 0}{x^2 + 2x}.
\]

**Example 5.** Divide \( x^4 + 3x^3 - 5x^2 + 6x + 10 \) by \( x^3 + x^2 + x + 2 \).

We write the divisor in a nested manner. Thus Horner’s method applied to the divisor gives
\[
x^3 + x^2 + x + 2 = ((x + 1)x + 1)x + 2 \Rightarrow
\]
Note that \( k_1 = -1, k_2 = -1 \) and \( k_3 = -2 \).

\[
\begin{array}{cccc}
-1 & 1 & 3 & -5 \\
-1 & 2 & -7 & 13 \\
-2 & 1 & -8 & 11 \\
\end{array}
\]

\[
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 2 & -8 & 9 \\
\end{array}
\]

Now, as before, we have
\[
x^4 + 3x^3 - 5x^2 + 6x + 10 = x + 2 + \frac{-8x^2 + 2x + 6}{x^3 + x^2 + x + 2}.
\]

**Example 6.** Divide \( 2x^3 - 4x^2 - 15x + 5 \) by \( (x - 1)^2 \).

We write the divisor in a nested manner. Thus Horner’s method applied to the divisor gives
\[
x^2 - 2x + 1 = (x - 2)x - 1 \Rightarrow
\]
Note that \( k_1 = 2 \) and \( k_2 = -1 \).

\[
\begin{array}{cccc}
2 & 2 & -4 & -15 \\
-4 & 0 & -30 \\
-1 & 2 & 0 & -15 \\
\end{array}
\]

\[
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
2 & 0 & 0 & -17 \\
\end{array}
\]

Now, as before, we have
\[
2x^3 - 4x^2 - 15x + 5 = 2x + 0 + \frac{-17x + 5}{(x - 1)^2}.
\]

**Example 7.** Divide \( x^3 \) by \( x^2 - x - 1 \).

We write the divisor in a nested manner. Thus Horner’s method applied to the divisor gives
\[
x^2 - x - 1 = (x - 1)x - 1 \Rightarrow
\]
Note that \( k_1 = 1 \) and \( k_2 = 1 \).

\[
\begin{array}{cccc}
1 & 1 & 0 & 0 \\
-1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 2 & 2 \\
\end{array}
\]

Now, as before, we have
\[
x^3 = x + 1 + \frac{2x + 1}{x^2 - x - 1}.
\]
Example 8. Divide $x^4$ by $(x - 1)^3$.

We write the divisor in a nested manner. Thus Horner’s method applied to the divisor gives

$$(x - 1)^3 = x^3 - 3x^2 + 3x - 1 = ((x - 3)x + 3)x - 1 \Rightarrow$$

Note that $k_1 = 3$, $k_2 = -3$ and $k_3 = 1$.

<table>
<thead>
<tr>
<th>3</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>81</td>
</tr>
<tr>
<td>-3</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>81</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>18</td>
<td>63</td>
</tr>
<tr>
<td>=</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>19</td>
<td>66</td>
</tr>
<tr>
<td>↓</td>
<td>-27</td>
<td>-63</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now, as before, we have

$$x^4 \div (x - 1)^3 = x + 3 + \frac{6x^2 - 8x + 3}{(x - 1)^3}.$$ 

This tableau easily works to obtain coefficients of the quotient polynomial $q(x)$ and requires a set of additional steps prior to identifying the coefficients of remainder polynomial $r(x)$.

Remark 2. As we see below, the divisor of the form $x^n - k$, where $n \geq 2$, will require fewer steps and easily reduce to the techniques and related tableaus depicted in [2]. Hence, the results of this article generalize the standard synthetic division and its easy extension found in [2]. For example, let us apply our techniques to divide $x^3 + 4x^2 + 3x - 2x + 8$ by $x^4 - 1$.

By Horner’s method, we have $k_j = 0$ for $4 > j \geq 1$ and $k_4 = 1$. The division tableau is:

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>0</th>
<th>4</th>
<th>3</th>
<th>-2</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>-2</td>
<td>8</td>
</tr>
<tr>
<td>-0</td>
<td>-0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>-2</td>
<td>8</td>
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<tr>
<td>-0</td>
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<td></td>
<td></td>
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<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>-2</td>
<td>8</td>
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<tr>
<td>-0</td>
<td>-0</td>
<td>-0</td>
<td>1</td>
<td>0</td>
<td></td>
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</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>-1</td>
<td>8</td>
</tr>
<tr>
<td>↓</td>
<td>-0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As a result of all $k_j = 0$ for $m > j \geq 1$, the first three steps are not necessary and the number next to sign $\rightarrow$, (if any) is 0. But in this case the only missing entries in the last row are replaced by $|$ as indicated in the table. For $k_4 = 1$ and $k_j = 0$ for $3 \geq j \geq 1$, this reduces to

<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
<th>0</th>
<th>4</th>
<th>3</th>
<th>-2</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0</td>
<td>-0</td>
<td>-0</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>-1</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

Thus, by identifying coefficients of the quotient and the remainder polynomials, we have

$$\frac{x^5 + 4x^3 + 3x^2 - 2x + 8}{x^4 - 1} = x + 0 + \frac{4x^3 + 3x^2 - x + 8}{x^4 - 1}.$$