Math Strategies

Inequalities for Convex Functions (Part I)

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1. Convex functions.

Convex functions are powerful tools for proving a large class of inequalities. They provide an elegant and unified treatment of the most important classical inequalities.

A real-valued function on an interval $I$ is called convex if
\[ f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \]
for every $x, y \in I$ and $\lambda \in [0, 1]$; it is called strictly convex if
\[ f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y) \]
for every $x, y \in I$, $x \neq y$ and $\lambda \in (0, 1)$.

Notice: $f$ is called concave (strictly concave) on $I$ if $-f$ is convex (strictly convex) on $I$.

The geometrical meaning of convexity is clear: $f$ is strictly convex if and only if for every two points $P = (x, f(x))$ and $Q = (y, f(y))$ on the graph of $f$, the point $R = (z, f(z))$ lies below the segment $PQ$ for every $z$ between $x$ and $y$.

How to recognize a convex function without the graph? We can use directly, but the following criterion is often very useful:

**Test for Convexity:** Let $f$ be a twice differentiable function on $I$. Then
- $f$ is convex on $I$ if $f''(x) \geq 0$ for every $x \in I$.
- $f$ is strictly convex on $I$ if $f''(x) > 0$ for every $x$ in the interior of $I$.

Remark: If $f$ is a continuous function on $I$, then it can be proved that $f$ is convex if and only if for all $x_1, x_2 \in I$
\[ f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}; \]
and it is strictly convex if and only if
\[ f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2} \]
for all $x_1, x_2 \in I$, $x_1 \neq x_2$ (see [1]).

Here are some basic examples of strictly convex functions:
(i) $f(x) = x^{2n}$, $x \in \mathbb{R}$ and $n$ is a positive integer;
(ii) $f(x) = x^p$, $x \geq 0$, $p > 1$;
(iii) $f(x) = \frac{1}{(x+a)^p}$, $x > -a$, $p > 0$;
(iv) $f(x) = \tan x$, $x \in [0, \frac{\pi}{2})$;
(v) $f(x) = e^x$, $x \in \mathbb{R}$.

The following are examples of strictly concave functions:
(i) $f(x) = \sin x$, $x \in [0, \pi]$;
(ii) $f(x) = \cos x$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$;
(iii) $f(x) = \ln x$, $x \in (0, \infty)$;
(iv) $f(x) = x^p$, $x \geq 0$, $p \in (0, 1)$.

Notice:
1. The linear function $f(x) = ax + b$, $x \in \mathbb{R}$ is convex and also concave.
2. The sum of two convex (concave) functions is a convex (concave) function.

2. Jensen’s Inequality.

Jensen’s inequality is an extension of (i). It was named after the Danish mathematician who proved it in 1905.

**Jensen’s Inequality:** Let $f : I \to \mathbb{R}$ be a convex function. Let $x_1, \ldots, x_n \in I$ and $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$. Then
\[ f(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n). \]

**Proof:** Let’s use mathematical induction. The inequality is true for $n = 1$. Now assume that it is true for $n = k$, and let’s show that it remains true for $n = k + 1$.
Let $x_1, \ldots, x_k, x_{k+1} \in I$ and let $\lambda_1, \ldots, \lambda_k, \lambda_{k+1} \geq 0$ with $\lambda_1 + \lambda_2 + \ldots + \lambda_k + \lambda_{k+1} = 1$. At least one of $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ must be less than 1 (otherwise the inequality is trivial). Without loss of generality, let $\lambda_{k+1} < 1$ and $u = \frac{\lambda_1}{1-\lambda_{k+1}} x_1 + \ldots + \frac{\lambda_k}{1-\lambda_{k+1}} x_k$. We have
\[ \frac{\lambda_1}{1-\lambda_{k+1}} + \ldots + \frac{\lambda_k}{1-\lambda_{k+1}} = 1, \]
and also
\[ \lambda_1 x_1 + \ldots + \lambda_k x_k + \lambda_{k+1} x_{k+1} = (1 - \lambda_{k+1}) u + \lambda_{k+1} x_{k+1}. \]
Now, since $f$ is convex,
\[ f((1-\lambda_{k+1})u + \lambda_{k+1} x_{k+1}) \leq (1-\lambda_{k+1}) f(u) + \lambda_{k+1} f(x_{k+1}) \]
and, by our induction hypothesis,
\[ f(u) \leq \frac{\lambda_1}{1-\lambda_{k+1}} f(x_1) + \ldots + \frac{\lambda_k}{1-\lambda_{k+1}} f(x_k). \]

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Hence, combining the above two inequalities, we get:
\[ f(\lambda_1 x_1 + \ldots + \lambda_{k+1} x_{k+1}) \leq \lambda_1 f(x_1) + \ldots + \lambda_{k+1} f(x_{k+1}) \]
Thus, the inequality is established for \( n = k + 1 \), and therefore, by mathematical induction, it holds for any positive integer \( n \).

Remarks:
1. For strictly convex functions, the inequality in (3) holds if and only if \( x_1 = x_2 = \ldots = x_n \). Use mathematical induction to prove it.
2. If \( \lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{1}{n} \), then (3) becomes
\[ f \left( \frac{x_1 + \ldots + x_n}{n} \right) \leq \frac{f(x_1) + \ldots + f(x_n)}{n} \]
3. If \( f \) is a concave function, then (3) and (4) read as
\[ f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \geq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \ldots + \lambda_n f(x_n) \]
and
\[ f \left( \frac{x_1 + \ldots + x_n}{n} \right) \geq \frac{f(x_1) + \ldots + f(x_n)}{n} \]
Jensen’s Inequality has variety of applications. It can be used to prove many of the most important classical inequalities.

**Weighted AM–GM Inequality:**
Let \( x_1, \ldots, x_n \geq 0 \), \( \lambda_1, \ldots, \lambda_n > 0 \) such that \( \lambda_1 + \ldots + \lambda_n = 1 \). Then
\[ \lambda_1 x_1 + \ldots + \lambda_n x_n \geq x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} \]
The equality holds if and only if \( x_1 = x_2 = \ldots = x_n \).

**Proof:** We may assume that \( x_1, \ldots, x_n > 0 \). Let \( f(x) = \ln x, x \in (0, \infty) \). Since \( f \) is strictly concave on \((0, \infty)\), by using (3) we get:
\[ \ln(\lambda_1 x_1 + \ldots + \lambda_n x_n) \geq \lambda_1 \ln x_1 + \ldots + \lambda_n \ln x_n, \]
or, equivalently, \( \ln(\lambda_1 x_1 + \ldots + \lambda_n x_n) \geq \ln x_1^{\lambda_1} \ldots x_n^{\lambda_n} \), and hence,
\[ \lambda_1 x_1 + \ldots + \lambda_n x_n \geq x_1^{\lambda_1} \ldots x_n^{\lambda_n} \]
(since \( f(x) = \ln x \) is a strictly increasing function).

By taking \( \lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{1}{n} \) in (3), we obtain:

**AM–GM Inequality:**
If \( x_1, \ldots, x_n \geq 0 \), then
\[ \frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}, \]
with equality if and only if \( x_1 = x_2 = \ldots = x_n \).

Let \( x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n > 0 \) be such that \( \lambda_1 + \ldots + \lambda_n = 1 \). For each \( t \in \mathbb{R}, t \neq 0 \), the weighted mean \( M_t \) of order \( t \) is defined as
\[ M_t = \left( \frac{\lambda_1 x_1^t + \lambda_2 x_2^t + \ldots + \lambda_n x_n^t}{n} \right)^{\frac{1}{t}}. \]
Some particular situations are significant:
\[ M_1 = \frac{\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n}{n} \]
is called the weighted arithmetic mean (WAM); and
\[ M_{-1} = \frac{\lambda_1 x_1^{-1} + \lambda_2 x_2^{-1} + \ldots + \lambda_n x_n^{-1}}{n} \]
is called the weighted harmonic mean (WHM),
\[ M_2 = \sqrt[2]{\frac{\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2}{n}} \]
is called the weighted root mean square (WRMS).

It can be shown by using l’Hôpital’s Rule that
\[ \lim_{t \to 0} M_t = x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} \]
So, if we denote \( M_0 = \lim_{t \to 0} M_t \), we see that
\[ M_0 = x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} \]
which is called the weighted geometric mean (WGM).

**Power Mean Inequality:**
Let \( x_1, x_2, \ldots, x_n, \lambda_1, \ldots, \lambda_n > 0 \) be such that \( \lambda_1 + \ldots + \lambda_n = 1 \). If \( t \) and \( s \) are non-zero real numbers such that \( s < t \), then
\[ \left( \frac{\lambda_1 x_1^s + \ldots + \lambda_n x_n^s}{n} \right)^{\frac{1}{s}} \leq \left( \frac{\lambda_1 x_1^t + \ldots + \lambda_n x_n^t}{n} \right)^{\frac{1}{t}}. \]

**Proof:** If \( 0 < s < t \) or \( s < 0 < t \), the inequality (7) is obtained by applying Jensen’s Inequality (6) to the strictly convex function \( f(x) = x^s \). Indeed, if \( a_1, a_2, \ldots, a_n, \lambda_1, \ldots, \lambda_n > 0 \) and \( \lambda_1 + \ldots + \lambda_n = 1 \), then
\[ \left( \frac{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n}{n} \right)^{\frac{1}{s}} \leq \frac{\lambda_1 a_1^s + \lambda_2 a_2^s + \ldots + \lambda_n a_n^s}{n}. \]
By choosing \( a_1 = x_1^s, \ldots, a_n = x_n^s \), we immediately obtain (6).
If \( s < t < 0 \), then \( 0 < -t < -s \), and by applying (2) for \( \frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n} \), we get

\[
\left[ \frac{\lambda_1(\frac{1}{x_1})^{-s} + \cdots + \lambda_n(\frac{1}{x_n})^{-s}}{n} \right] \frac{1}{s} \leq \left[ \frac{\lambda_1(\frac{1}{x_1})^{-t} + \cdots + \lambda_n(\frac{1}{x_n})^{-t}}{n} \right] \frac{1}{t},
\]

which can be rewritten as (3).

**Remark:** If \( t < s < 0 \), then \( M_t \leq M_0 \leq M_s \). Also, we have the following classical inequality:

\[
M_{-\infty} \leq M_{-1} \leq M_0 \leq M_1 \leq M_2 \leq M_{\infty}.
\]

**Hölder’s Inequality:** If \( p, q > 1 \) are real numbers such that \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( a_1, \ldots, a_n, b_1, \ldots, b_n \) are real (complex) numbers, then

\[
\sum_{k=1}^{n} |a_k||b_k| \leq \left( \sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n} |b_k|^q \right)^{\frac{1}{q}},
\]

**Proof:** We may assume that \( |a_k| > 0, k = 1, \ldots, n \). (Why?) The function \( f(x) = x^q \) is strictly convex on \((0, \infty)\), hence, by Jensen’s Inequality,

\[
\left( \sum_{k=1}^{n} \lambda_k x_k \right)^q \leq \sum_{k=1}^{n} \lambda_k x_k^q,
\]

where \( x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n > 0 \) and \( \lambda_1 + \cdots + \lambda_n = 1 \). Let \( A = \sum_{k=1}^{n} |a_k|^p \). By choosing \( \lambda_k = \frac{1}{A} |a_k|^p \) and \( x_k = \frac{1}{\lambda} |a_k||b_k| \) in the above inequality, we obtain (3).

**Remarks:**

1. In (3), the equality holds if and only if \( x_1 = x_2 = \ldots = x_n \). That is,

\[
\frac{|a_1|^p}{|b_1|^q} = \frac{|a_2|^p}{|b_2|^q} = \cdots = \frac{|a_n|^p}{|b_n|^q}.
\]

Notice that this chain of equalities is taught in the following way: if a certain \( b_k = 0 \), then we should have \( a_k = 0 \).

2. If \( p = q = 2 \), Hölder’s Inequality is just **Cauchy’s Inequality:**

\[
\left( \sum_{k=1}^{n} |a_k||b_k| \right)^2 \leq \left( \sum_{k=1}^{n} |a_k|^2 \right) \left( \sum_{k=1}^{n} |b_k|^2 \right).
\]

The equality occurs when

\[
\frac{|a_1|}{|b_1|} = \frac{|a_2|}{|b_2|} = \cdots = \frac{|a_n|}{|b_n|}.
\]

**Minkowski’s Triangle Inequality:** If \( p > 1 \) and \( a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \geq 0 \), then

\[
\left( \sum_{k=1}^{n} (a_k + b_k)^p \right)^{\frac{1}{p}} \leq \left( \sum_{k=1}^{n} a_k^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{n} b_k^p \right)^{\frac{1}{p}}.
\]

**Proof:** We may assume \( a_k > 0, k = 1, \ldots, n \). (Why?) The function \( f(x) = (1 + x^p)^{\frac{1}{p}}, x \in (0, \infty) \), is strictly concave since \( f''(x) = \frac{1-p}{p} \left( 1 + x^p \right)^{\frac{1}{p}} - x^{\frac{1}{p}-2} < 0 \). By Jensen’s Inequality,

\[
1 + \left( \sum_{k=1}^{n} \lambda_k x_k \right)^{\frac{1}{p}} \geq \sum_{k=1}^{n} \lambda_k \left( 1 + x_k^{\frac{1}{p}} \right),
\]

where \( x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n > 0 \) and \( \lambda_1 + \cdots + \lambda_n = 1 \). Let \( A = \sum_{k=1}^{n} a_k^p \). By taking \( \lambda_k = \frac{a_k^p}{A} \) and \( x_k = \frac{b_k^p}{A} \) for \( k = 1, \ldots, n \) in the above inequality, we obtain (3).

**Remarks:**

1. The equality in (3) occurs if and only if \( \frac{b_1}{a_1} = \frac{b_2}{a_2} = \cdots = \frac{b_n}{a_n} \).

2. If \( p = 2 \) we get the so-called Triangle Inequality:

\[
\sqrt{\sum_{k=1}^{n} (a_k + b_k)^2} \leq \sqrt{\sum_{k=1}^{n} a_k^2} + \sqrt{\sum_{k=1}^{n} b_k^2}.
\]

**Example 1.** If \( a, b \geq 0 \) and \( a + b = 2 \), then

\[
(1 + \sqrt[5]{a})^5 + (1 + \sqrt[5]{b})^5 \leq 2^6.
\]

**Solution:** Since \( f(x) = (1 + \sqrt[5]{x})^5 \) is strictly concave on \([0, \infty)\), by using Jensen’s Inequality (3) we get

\[
2 \left( 1 + \sqrt{\frac{a + b}{2}} \right)^5 \geq (1 + \sqrt[5]{a})^5 + (1 + \sqrt[5]{b})^5.
\]

By substituting \( a + b = 2 \), we get the required inequality. The equality occurs when \( a = b = 1 \).

**Example 2.** If \( a, b, c > 0 \), then

\[
a^a \cdot b^b \cdot c^c \geq \left( \frac{a + b + c}{3} \right)^{a+b+c}.
\]

**Solution:** The above inequality is equivalent to

\[
\ln(a^a \cdot b^b \cdot c^c) \geq \ln \left( \frac{a + b + c}{3} \right)^{a+b+c},
\]

where \( a^a \cdot b^b \cdot c^c = \exp(a \ln a + b \ln b + c \ln c) \). By using the convexity of \( \ln(x) \) on \((0, \infty)\), we have

\[
\ln(a^a \cdot b^b \cdot c^c) \geq \ln \left( \frac{a + b + c}{3} \right)^{a+b+c}.
\]
or

\[ a \ln a + b \ln b + c \ln c \geq (a + b + c) \ln \left( \frac{a + b + c}{3} \right). \]

Let \( f(x) = x \ln x, x \in (0, \infty) \). Since \( f''(x) = \frac{1}{x} > 0 \), the function \( f \) is strictly convex on \((0, \infty)\). Now, the above inequality follows from (3).

**Example 3.** If \( a, b, c > 0 \) then

\[ \frac{a}{a + 3b + 3c} + \frac{b}{3a + b + 3c} + \frac{c}{3a + 3b + c} \geq \frac{3}{7}. \]

**Solution:** Let \( s \) be a positive number and \( f(x) = \frac{x}{s-x} = \frac{x}{s-x} - 1, x \in (0, s) \). The function \( f \) is strictly convex since \( f''(x) = \frac{2x}{(s-x)^2} > 0 \). We get:

\[ \frac{2a}{s-2a} + \frac{2b}{s-2b} + \frac{2c}{s-2c} \geq \frac{3}{s} \frac{2(a + b + 2c)}{(2a + 2b + 2c)}, \]

or

\[ \frac{a}{s-2a} + \frac{b}{s-2b} + \frac{c}{s-2c} \geq \frac{3(a + b + c)}{3s}. \]

If we take \( s = 3(a + b + c) \), the required inequality follows.

**Example 4.** If \( a_1, a_2, \ldots, a_n \geq 1 \), then

\[ \sum_{k=1}^{n} \frac{1}{1 + a_k} \geq \frac{n}{1 + \sqrt[n]{a_1 a_2 \ldots a_n}}. \]

**Solution:** Let \( f(x) = \frac{1}{\sqrt[x]{x}}, x \in [0, \infty) \). The function \( f \) is strictly convex since \( f''(x) = \frac{e^x}{(x+1)^3} > 0 \) on \((0, \infty)\). Using (3), we get

\[ \sum_{k=1}^{n} \frac{1}{1 + e^{x_k}} \geq \frac{n}{1 + e^{\sum_{k=1}^{n} x_k}}. \]

By taking \( x_k = \ln a_k, k = 1, \ldots, n \), we obtain the required inequality.

**Example 5.** For a triangle with angles \( \alpha, \beta \) and \( \gamma \), the following inequalities hold:

- \( \sin \alpha + \sin \beta + \sin \gamma \leq \frac{3\sqrt{3}}{2} \);
- \( \sqrt{\sin \alpha} + \sqrt{\sin \beta} + \sqrt{\sin \gamma} \leq \frac{3\sqrt{3}}{2} \);
- \( \sin \alpha \cdot \sin \beta \cdot \sin \gamma \leq \frac{3\sqrt{3}}{8} \);  
- \( \cos \alpha \cdot \cos \beta \cdot \cos \gamma \leq \frac{1}{8} \);
- \( \sec \frac{\alpha}{2} + \sec \frac{\beta}{2} + \sec \frac{\gamma}{2} \geq 2\sqrt{3} \).

**Example 6.** Let \( a_1, \ldots, a_n, \lambda_1, \ldots, \lambda_n > 0 \) and \( \lambda_1 + \ldots + \lambda_n = 1 \). If \( a_1^{\lambda_1} \ldots a_n^{\lambda_n} = 1 \), then

\[ a_1 + a_2 + \ldots + a_n \geq \frac{1}{\lambda_1^{\lambda_1} \ldots \lambda_n^{\lambda_n}}. \]

The equality occurs if and only if \( a_k = \frac{\lambda_k}{\lambda_1^{\lambda_1} \ldots \lambda_n^{\lambda_n}} \) for \( k = 1, \ldots, n \).

**Solution:** By using the Weighted AM–GM Inequality, we have

\[ a_1 + \ldots + a_n = \lambda_1 \left( \frac{a_1}{\lambda_1} \right) + \ldots + \lambda_n \left( \frac{a_n}{\lambda_n} \right) \geq \left( \frac{a_1}{\lambda_1} \right)^{\lambda_1} \ldots \left( \frac{a_n}{\lambda_n} \right)^{\lambda_n} = \frac{1}{\lambda_1^{\lambda_1} \ldots \lambda_n^{\lambda_n}}. \]

The equality occurs if and only if

\[ \frac{a_1}{\lambda_1} = \frac{a_2}{\lambda_2} = \ldots = \frac{a_n}{\lambda_n}. \]

in which case the constraint \( a_1^{\lambda_1} \ldots a_n^{\lambda_n} = 1 \) leads to \( a_k = \frac{\lambda_k}{\lambda_1^{\lambda_1} \ldots \lambda_n^{\lambda_n}} \) for \( k = 1, \ldots, n \).

**Example 7.** If \( a_1, \ldots, a_n > 0 \) and \( a_1 a_2 \ldots a_n = 1 \), then

\[ a_1 + \sqrt{a_2} + \ldots + \sqrt[n]{a_n} \geq \frac{n+1}{2}. \]

**Hint:** Use the Weighted AM–GM Inequality.

**Example 8.** If \( a, b, c > 0 \), then

\[ \frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2} \geq \left( \frac{a + b + c}{3} \right)^3. \]

(ii) If \( a_1, a_2, \ldots, a_n > 0 \) and \( k > p \geq 0 \), then

\[ \frac{a_1^k + \ldots + a_n^k}{a_1^p + \ldots + a_n^p} \geq \left( \frac{a_1 + \ldots + a_n}{n} \right)^{k-p}. \]

**Solution:** (i) A particular case of (ii).

(ii) Let \( M_k = \left( \frac{a_1^k + \ldots + a_n^k}{n} \right)^{\frac{1}{k}} \). Then, by using the Power Mean Inequality (3), we get

\[ a_1^k + a_2^k + \ldots + a_n^k = nM_k^k = nM_k^k M_k^{k-p} \geq nM_p^p M_k^{k-p} \text{ by (7)} \]

\[ = (a_1^p + \ldots + a_n^p) \cdot \left( \frac{a_1^k + \ldots + a_n^k}{n} \right)^{k-p}. \]
Example 9. If \(a_1, a_2, \ldots, a_n > 0\), then

(i) \[a_1^{n+1} + \ldots + a_n^{n+1} \geq a_1 a_2 \ldots a_n (a_1 + \ldots + a_n)\],

(ii) \[a_1^{n-1} + \ldots + a_n^{n-1} \geq a_1 \ldots a_n \left(\frac{1}{a_1} + \ldots + \frac{1}{a_n}\right)\].

Solution: (i)

\[a_1^{n+1} + \ldots + a_n^{n+1} = nM_{n+1} = nM_n M_{n+1} \geq nM_0 M_1 = a_1 \ldots a_n (a_1 + \ldots + a_n)\].

(ii) See (i).

Example 10. If \(a, b, c, x, y, z, n > 0\), and

\[(a^n + b^n + c^n)^{n+1} = x^n + y^n + z^n,\]

then

\[\frac{a^{n+1}}{x} + \frac{b^{n+1}}{y} + \frac{c^{n+1}}{z} \geq 1.\]

Solution:

\[a^n + b^n + c^n = \left(\frac{a^{n+1}}{x} + \frac{b^{n+1}}{y} + \frac{c^{n+1}}{z}\right)^{n/n+1} \cdot (x^n + y^n + z^n)^{1/n+1},\]

and the required inequality follows.

Example 11. If \(a_1, \ldots, a_n, b_1, \ldots, b_n > 0\), then

\[(a_1 + \ldots + a_n)^{n+1} \leq \frac{a_1^{n+1}}{b_1} + \ldots + \frac{a_n^{n+1}}{b_n}.\]

Solution: By using Hölder’s Inequality with \(p = \frac{n+1}{n}\) and \(q = \frac{p-1}{n} = n+1\), we get

\[a_1 + \ldots + a_n = \left(\frac{a_1^{n+1}}{b_1^{p-1}} \cdot b_1^{n/n} + \ldots + \frac{a_n^{n+1}}{b_n^{p-1}} \cdot b_n^{n/n}\right)^{n/n+1} \cdot (b_1 + \ldots + b_n)^{1/n+1},\]

so

\[(a_1 + \ldots + a_n)^{n+1} \leq \left(\frac{a_1^{n+1}}{b_1^{p-1}} + \ldots + \frac{a_n^{n+1}}{b_n^{p-1}}\right) (b_1 + \ldots + b_n)^{n},\]

from which we get the required inequality.

REFERENCE.