



## The Banach–Tarski Paradox or What Mathematics and Miracles Have in Common

by Volker Runde

*As he went ashore he saw a great throng; and he had compassion on them, and healed their sick. When it was evening, the disciples came to him and said: “This is a lonely place, and the day is now over; send the crowds away to go into the villages and buy food for themselves.” Jesus said: “They need not go away; you give them something to eat.” They said to him: “We have only five loaves here and two fish.” And he said: “Bring them here to me.” Then he ordered the crowds to sit down on the grass; and taking the five loaves and the two fish he looked up to heaven, and blessed, and broke and gave the loaves to the disciples, and the disciples gave them to the crowds. And they all ate and were satisfied. And they took up twelve baskets full of the broken pieces left over. And those who ate were about five thousand men, besides women and children.*

Mt 14:14–21

Why does an article that is supposed to be about mathematics start with the feeding of the five thousand?

In the 1920s two Polish mathematicians—Stefan Banach and Alfred Tarski—proved a mathematical theorem that sounds a lot like the feeding of the five thousand. In their honor, it is called the *Banach–Tarski paradox*<sup>†</sup>. Consequences of the Banach–Tarski paradox are, for example:

An orange can be chopped into a finite number of chunks, and these chunks can then be put together again to yield *two* oranges, each of which has the same size as the one that just went into pieces.

Another, even more bizarre consequence is:

A pea can be split into a finite number of pieces, and these pieces can then be reassembled to yield a solid ball whose diameter is larger than the distance of the Earth from the sun.

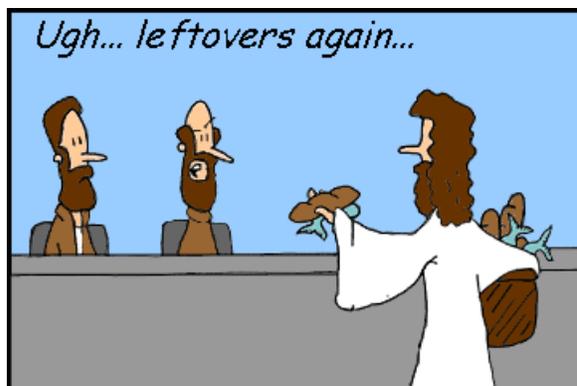
More generally, whenever you have a three-dimensional body (with a few strings attached), you can obtain any other such body by breaking the first into pieces and reassembling the parts. To turn five loaves and two fish into enough food to feed a crowd of more than five thousand then appears to be a minor exercise.

If you have read this far, your attitude will presumably be one of the following:

- Your belief in the absolute truth of mathematical theorems is so strong that it makes you swallow the Banach–Tarski paradox.

<sup>†</sup> The theorem is proved in the article: S. BANACH and A. TARSKI, *Sur la décomposition des ensembles de points en parts respectivement congruents*. *Fund. Math.* **6** (1924), 244–277.

- You are a staunch skeptic, so that you neither take the feeding of the five thousand nor the Banach–Tarski paradox at face value.



Copyright Gospel Films, Inc. <http://www.reverendfun.com>

If you fall into the first category, there is probably little incentive for you to read any further. Otherwise, I guess, your attitude is best described as follows: You may believe in the story of the feeding of the five thousand but not take it literally, and if you hear of a mathematical theorem whose consequences are obviously nonsense, you tend to think that the theorem is wrong.

Take an orange, a sharp knife and a chopping block. Chop the orange into pieces, and try to form two globes of approximately the same size out of the orange chunks. If the chunks are small enough, each of these two globes will bear reasonable resemblance to a ball, but, of course, each has a volume that is only about half of that of the original orange. Perhaps you just didn't chop up the orange in the right way? Give it another try. The result will be the same. You can try your luck on hundreds of oranges: you will produce tons of orange pulp, but no corroboration of the Banach–Tarski paradox. Doesn't this show that the Banach–Tarski paradox is wrong?

The Banach–Tarski paradox is a so-called existence theorem: there is a way of splitting up a pea such that the pieces can be reassembled into, say, a life-sized statue of Stefan Banach. The fact that you haven't succeeded in finding such a way doesn't mean that it doesn't exist—you just might not have found it yet. Let me clarify with an example from elementary arithmetic. A positive integer  $p$  is called *prime* if 1 and  $p$  itself are its only divisors; for example, 2, 3, and 23 are prime, whereas  $4 = 2 \cdot 2$  and  $243 = 3 \cdot 81$  aren't. The ancient Greeks knew that every positive integer has a *prime factorization*: if  $n$  is a positive integer, then there are prime numbers  $p_1, \dots, p_k$  such that  $n = p_1 \cdot \dots \cdot p_k$ . For small  $n$ , such a prime factorization is easy to find:  $6 = 2 \cdot 3$ ,  $243 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3$ , and  $6785 = 5 \cdot 23 \cdot 59$ , for example. There is essentially only one way of finding a prime factorization—trying. Already finding the prime factorization of 6785—armed only with pencil and paper—would have taken you some time. And now think of a large number, I mean, *really large*:

7380563434803675764348389657688547618099805.

This is a perfectly nice positive integer, and the theorem tells you that it has a prime factorization, but—please!—don't waste hours, days or even years of your

life trying to find it. You might think: what were computers invented for? It is easy to write a little program that produces the prime factorization of an arbitrary positive integer (and it may even produce one of 7380563434803675764348389657688547618099805 in a reasonable amount of time). However, the average time such a program would take to find the prime factorization of an integer  $n$  goes up dramatically as  $n$  gets large: for sufficiently large  $n$ , even the fastest super-computer available today would —on average—take longer to find the prime factorization of  $n$  than the age of the universe.

So, although a prime factorization of a positive integer always exists, it may be impossibly hard to find. In fact, this is a good thing—it is at the heart of the public key codes that make credit card transactions on the internet safe, for example. Now, think again of the Banach–Tarski paradox. Just because you couldn’t put it to work in your kitchen (just as you couldn’t find the prime factorization of some large integer), this doesn’t mean that the theorem is false (or that this particular integer doesn’t have a prime factorization).

Let’s try to refute the Banach–Tarski paradox with the only tool that works in mathematics: pure thought. What makes the Banach–Tarski paradox defy common sense is that, apparently, the volume of something increases out of nowhere. You certainly know a number of formulae to calculate the volumes of particular three-dimensional bodies. For example, if  $C$  is a cube whose edges have length  $l$ , then its volume  $V(C)$  is  $l^3$ ; if  $B$  is a ball with radius  $r$ , then its volume  $V(B)$  is  $\frac{4}{3}\pi r^3$ . But what’s the volume of an arbitrary three-dimensional body? No matter how the volume of a concrete body is calculated, the following are certainly true about the volumes of arbitrary, three-dimensional bodies:

- If the body  $\tilde{B}$  is obtained from the body  $B$  simply by moving  $B$  in three-dimensional space, then  $V(\tilde{B}) = V(B)$ ;
- If  $B_1, \dots, B_n$  are bodies in three-dimensional space, then the volume of their union is less than or equal to the sum of their volumes, i.e.,

$$V(B_1 \cup \dots \cup B_n) \leq V(B_1) + \dots + V(B_n);$$

- If  $B_1, \dots, B_n$  are bodies in three-dimensional space such that any two of them have no point in common, then the volume of their union is equal to the sum of their volumes, i.e.;

$$V(B_1 \cup \dots \cup B_n) = V(B_1) + \dots + V(B_n).$$

So, let  $B$  be an arbitrary three-dimensional body, and let  $B_1, \dots, B_n$  be subsets of  $B$  such that any two of them have no point in common and  $B = B_1 \cup \dots \cup B_n$ . Now, move each  $B_j$  in three-dimensional space, and obtain  $\tilde{B}_1, \dots, \tilde{B}_n$ . Finally, put the  $\tilde{B}_j$  together and obtain another body  $\tilde{B} = \tilde{B}_1 \cup \dots \cup \tilde{B}_n$ . Then we have for the volumes of  $B$  and  $\tilde{B}$ :

$$\begin{aligned} V(B) &= V(B_1 \cup \dots \cup B_n) \\ &= V(B_1) + \dots + V(B_n), && \text{by (iii),} \\ &= V(\tilde{B}_1) + \dots + V(\tilde{B}_n), && \text{by (i),} \end{aligned}$$

$$\begin{aligned} &\geq V(\tilde{B}_1 \cup \dots \cup \tilde{B}_n), && \text{by (ii),} \\ &= V(\tilde{B}). \end{aligned}$$

This means that the volume of  $\tilde{B}$  must be less than or equal to the volume of  $B$ —it can’t be larger. Banach and Tarski were wrong! Really?

Our refutation of Banach–Tarski seems to be picture perfect. All we needed were three very basic properties of the volume of three-dimensional bodies. But was this really all? Behind our argument, there was a hidden assumption—every three-dimensional body has a volume. If we give up that assumption, our argument suddenly collapses. If only one of the bodies  $B_j$  has no volume, our whole chain of (in)equalities makes no longer sense. But why shouldn’t every three-dimensional body have a volume? Isn’t that obvious? What is indeed true is that every orange chunk you can possibly produce on your chopping block has a volume. For this reason, you will never be able to use the Banach–Tarski paradox to reduce your food bill. A consequence of the Banach–Tarski paradox is therefore that there is a way of chopping up an orange so that you can form, say, a gigantic pumpkin out of the pieces—but you will never be able to do that yourself using a knife. What kind of twisted logic can make anybody put up with that?

Perhaps, you are more willing to put up with the *axiom of choice*:

If you have a family of non-empty sets  $S$ , then there is a way to choose an element  $x$  from each set  $S$  in that family.

That sounds plausible, doesn’t it? Just think of a finite number of non-empty sets  $S_1, \dots, S_n$ : Pick  $x_1$  from  $S_1$ , then proceed to  $S_2$ , and finally take  $x_n$  from  $S_n$ . What does the axiom of choice have to do with the Banach–Tarski paradox? As it turns out, a whole lot: *If the axiom of choice is true, then the Banach–Tarski paradox can be derived from it and, in particular, there must be three-dimensional bodies without volume.* So, the answer to the question of whether the Banach–Tarski paradox is true depends on whether the axiom of choice is true.

Certainly, the axiom of choice works for a finite number of non-empty sets  $S_1, \dots, S_n$ . Now think of an infinite sequence  $S_1, S_2, \dots$  of non-empty sets. Again, pick  $x_1$  from  $S_1$ , then  $x_2$  from  $S_2$ , and just continue. You’ll never come to an end, but eventually you’ll produce some element  $x_n$  from each  $S_n$ . So, the axiom of choice is true in this case, too. But what if we have a truly arbitrary family of sets? What if we have to deal with the family of all non-empty subsets of the real line? It can be shown that this family of sets can’t be written as a sequence of sets. How do we pick a real number from each set? There is no algorithm that enables us to pick one element from one set, a second element from another set and, eventually, to pick an element from every set in the family. Nevertheless, the axiom of choice still seems plausible—each set  $S$  in our family is non-empty and therefore contains some element  $x$ —why shouldn’t there be a way of choosing a particular element from each such set?

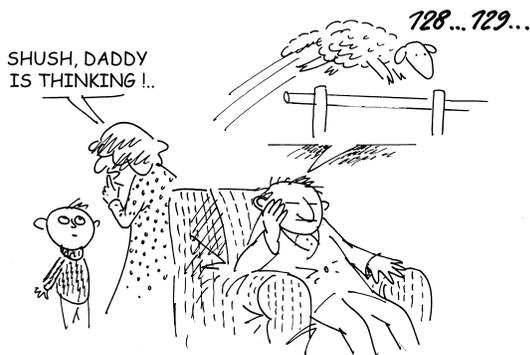
On the other hand, accepting the axiom of choice implies strange phenomena like the Banach–Tarski paradox. If it’s true, we have to put up with the mysterious duplication of oranges. If it’s false, then why? Please, don’t try to prove or to refute the axiom of choice—you won’t

succeed either way. The axiom of choice is beyond proof or refutation. We can *suppose* that it's true, or we can *suppose* that it's false. In other words, we have to *believe* in it or leave it. Most mathematicians these days are believers in the axiom of choice for a simple reason—with the axiom of choice, they can prove useful theorems, most of which are much less baffling than the Banach–Tarski paradox.

Are you disappointed? Instead of elevating the feeding of five thousand from a matter of belief to a consequence of a bullet-proof mathematical theorem, the Banach–Tarski paradox demands that you accept another article of faith—the axiom of choice—before you can take the theorem for granted. After all, the Banach–Tarski paradox is not that much removed from the feeding of the five thousand...



## Math Jokes



©Copyright 2000 by Zbigniew Jujka

Most prime numbers are even. **Proof:** Pick up any math text and look for a prime number. The first one you find will probably be even.

Never say “ $n$  factorial”; simply scream “ $n$ ” at the top of your lungs.

(Mark David Biesiada)

Some engineers are trying to measure the height of a flag pole. They have only a measuring tape and are quite frustrated trying to keep the tape along the pole; it falls down all the time. A mathematician comes along and asks what they are doing. They explain it to him. “Well, that’s easy...” He pulls the pole out of the ground, lays it down and measures it easily. After he has left, one of the engineers says, “That’s so typical of these mathematicians! What we need is the height - and he gives us the length!”

A mathematician, an engineer, and a computer scientist are vacationing together. They are riding in a car, enjoying the countryside, when suddenly the engine stops working. The mathematician says, “We drove past a gas station a few minutes ago. Someone should go back and ask for help.”

The engineer says, “I should have a look at the engine. Perhaps I can fix it.”

The computer scientist says, “Why don’t we just open the doors, slam them shut and see if everything works again?”

Two men are sitting in the basket of a balloon. For hours, they have been drifting through a thick layer of clouds, and they have lost their orientation completely. Suddenly, the clouds part, and the two men see the top of a mountain with a man standing on it. “Hey! Can you tell us where we are?!” The man doesn’t reply. The minutes pass as the balloon drifts past the mountain. When the balloon is about to be swallowed again by the clouds, the man on the mountain shouts: “You’re in a balloon!”

“That must have been a mathematician.”

“Why?”

“He thought long and thoroughly about what to say. What he eventually said was irrefutably correct. And it was of no use whatsoever...”

In the old days of the cold war, when it was very hard for Westerners visit the Soviet Union, a British mathematician traveled to Moscow to speak in the seminar of a famous Russian professor. He started his talk by writing a theorem on the board. When went to prove it, the professor interrupted him: “This theorem is clear!” The speaker was, of course, annoyed, but managed to conceal it. He continued his talk with a second theorem, but, again, when he went to start with the proof, he was interrupted by his host: “This theorem is also clear!” With a stern face, he wrote a third theorem on the board and asked, “Is this theorem clear, too?!” His host nodded. The visitor grinned and said, “This theorem—is false...”



©Copyright 2000 by Zbigniew Jujka

Why did the calculus student have so much trouble making Kool-Aid? Because he couldn’t figure out how to get a litre of water into the little package.