Similarity and Dimensionality
Mark Solomonovich

I. Real life problems

It would not surprise anyone that mathematical methods are applied in such areas as physics and electrical engineering. Quite unexpectedly, mathematical methods can sometimes provide a solution to a problem in an area that does not seem to allow a rigorous formulation.

The following problems, in spite of non-rigorous formulations, do allow rigorous solutions and qualitative substantiation based on certain mathematical ideas called Similarity and Dimensionality.

Try to solve these problems. If you cannot propose a well-substantiated solution, read the article and then try again or check out the solutions at the end of this article.

Problem 1: A blue whale and a fin whale have almost the same lengths, 30 meters and 25 meters, respectively, for the largest representatives of the species. How would you explain that a fin whale weighs just half of the weight of a blue whale?

Problem 2: If you look in a supermarket at the price per pound of small and large grapefruits of the same kind, you will see that the large ones cost more per pound. Why do you think this is the case?

Problem 3: A camel can carry approximately the same weight as a mule and move with the same speed. Which pack animal is able to travel further without water?

Problem 4: How does the height of jump of an animal depend on its size?

II. Similarity in a one-dimensional world.

The first question is: who are these creatures that inhabit the line? These are segments, finite parts of straight lines.

One can say that any two segments are similar to each other. Really, they differ only in their lengths, and their shape is the same. That corresponds to our common sense notion of similarity.

Thus, our first experience with similarity can be summarized as follows:

Two objects are similar if one of them can be transformed into another by means of an expansion, contraction, or reflection.

For segments of lengths $S_1$ and $S_2$ one can find such a coefficient $\kappa$ that

$$S_1 = \kappa \cdot S_2.$$ 

For instance:

![Figure 1](image1.png)

This coefficient, $\kappa$, is called the magnification or similarity ratio.

III. Similarity in a two-dimensional world.

There are infinitely many different kinds of objects lying in the plane.

![Figure 2(a)](image2.png)
Let us examine possible criteria for similarity based on the above examples and using our common sense.

(a) Not all triangles are similar; yet some of them are (for example two equilateral triangles).

(b) Two rectangles are seldom similar, but two squares are similar and two rectangles with the same ratio between their sides are similar.

(c) All circles look similar.

(d) Two ovals (ellipses) are not similar unless the ratios of their dimensions (length divided by width) are the same.

(e) Polygons with the same number of sides are not usually similar. Yet, regular polygons with the same number of sides are similar.

(f) Dogs are not similar to each other. Yet, two terriers, small and big, look similar.

It is clear that symmetric objects of the same kind, such as equilateral triangles, or circles, or regular hexagons, are similar to each other. What about non-symmetric objects?

Two figures of the same shape (e.g., two terriers) but of different sizes look similar. Another example is that the same face on various size photo portraits looks similar (Figure 3).

It would be probably a good idea to describe similar two-dimensional figures obtained from one another by a transformation similar to photo enlargement. Figure 4 illustrates in a simplified manner, how a photo enlarger works. Each elementary component of the picture grows proportionally, evenly in all directions.

What about the most elementary components of any figure? Each segment of the larger picture is obtained from the corresponding segment of the smaller picture by means of an enlargement, and the magnitude of the enlargement is the same for all segments lying in the picture.

We can formalize this “definition” to make it more rigorous:

We say the figure $F$ is similar to figure $F'$ with ratio $\kappa$ if $F$ can be transformed onto $F'$ in such a way that for any pair of points $P$ and $M$ of figure $F$ and their images $P'$ and $M'$ we have $P'M' = \kappa \cdot PM$.

Since $\kappa = \frac{P'M'}{PM}$ is the same for any choice of points $P$ and $M$, where $P'$ and $M'$ are their corresponding images, in order to determine the ratio $\kappa$ for two similar figures, it is sufficient to consider just two such points, which can be chosen according to a certain rule. For example, in the case of a triangle we could consider two of its vertices or the endpoints of the height, and in the case of a circle just the endpoints of its diameter. We can think about the length $L$ of the segment $PM$ as a characteristic linear size of $F$ that can be compared with the corresponding linear size $L'$ of an other similar figure $F'$.
move the two pictures and the pictures will still be similar.

So, a similarity with magnification factor $\kappa$ has the following properties (see Figure 5):

1. transforms segments of length $L$ in segments of length $\kappa \cdot L$;
2. preserves angles;
3. transforms triangle with sides $a$, $b$, $c$ and angles $A$, $B$, $C$ onto a triangle with sides $\kappa \cdot a$, $\kappa \cdot b$, $\kappa \cdot c$ and angles $A$, $B$, $C$;
4. transforms a figure with area $S$ onto a figure with area $\kappa^2 \cdot S$.

Let us prove the last property (4). We start with triangles:

For polygons: we decompose them into similar triangles:

$$S' = S_{\Delta_1} + S_{\Delta_2} + \ldots + S_{\Delta_n} = \kappa^2 S_{\Delta_1} + \kappa^2 S_{\Delta_2} + \ldots + \kappa^2 S_{\Delta_n} = \kappa^2 S.$$

In simple words, Property (4) says that for similar figures:

- The area of a figure is proportional to square of its linear size.

In particular, this statement means that for all similar figures the ratio $S/L^2$, where $S$ is the area of a figure and $L$ its characteristic linear size, is always the same. We can also say that for similar figures the area is growing proportionally to their linear size.

This fact can be explained more precisely. The area of a figure is proportional to $L^2$, where $L$ is a (characteristic) linear size of the figure. That is, $S' = \kappa^2 S$, where $\kappa = L'/L$. This means that

$$S' = (L')^2 \cdot \frac{S}{L^2}.$$ 

As the ratio $S/L^2$ is the same for all similar figures, we observe that the area $S'$ is proportional to $(L')^2$. We will write $S' \propto (L')^2$ to say that $S'$ is proportional to $(L')^2$, i.e. the factor $S'/L^2$ is the same for all similar figures $F'$.

For example, if there are two similar rectangles, one with side of length $L$, the other with side of length $L'$, then the ration of their areas $S$ to $S'$ is

$$\frac{S}{S'} = \left(\frac{L}{L'}\right)^2.$$ 

**Question for discussion:** All circles are similar to each other. Then, can we assert, basing on the above stated corollary, the existence of the universal\footnote{universal means the same for all circles.} number $\pi$ such that

$$S_{\text{circle}} = \pi R^2?.$$ 

**IV. Similarity in a three-dimensional world**

One can define similar solids by means of similarity transformations.

Let us point out a property analogous to the property (4) of the previous section:

The volumes of similar figures are in the same proportion as the cubes of their corresponding linear sizes.

In other words:

$$V \propto L^3.$$ 

Now you are ready to attempt solving the applied problems in the beginning of the article.

**Problem 1.** The ratio of the linear sizes of the two whales is approximately $0.8$, therefore the ratio of their volumes and, hence, their weights is $(0.8)^3 \approx 0.5$. 

\footnote{universal means the same for all circles.}
Problem 2. The mass of a grapefruit is proportional to its radius cubed, whereas the amount of waste, the volume of the skin, is proportional to its radius squared (if we assume that the thickness of the skin is constant). Therefore, the ratio of useful volume to total volume of the fruit is proportional to its radius.

Problem 3. Let $L$ be characteristic linear size of the animal. The amount of water that the animal can store is proportional to $L^3$. Evaporation of water is proportional to the surface area of the animal, i.e. to $L^2$. Hence, the maximum time an animal keeps water is proportional to $L$, i.e. the bigger animal (the camel) can move longer without drinking.

Problem 4. Let $L$ denote a characteristic linear size (e.g. height or length of a leg) of an animal. The energy $E$ required to jump to height $H$ is proportional to the height $H$ and to the mass of the animal, i.e. $E \propto L^3 H$. The physical work done by the animal’s muscles is equal to $FL$, where $F$, the strength of the muscles, is proportional to $L^2$ (the greater the area of the cross-section of a muscle, the more fibers it contains). Therefore, the physical work is proportional to $L^2 \cdot L = L^3$. Energy balance then requires that

$$L^3 H \propto L^3.$$

Thus $H$ does not depend on $L$. (Observations shows that a jerboa and a kangaroo can jump to the same height.)

A somewhat advanced society has figured how to package basic knowledge in pill form.

A student, needing some learning, goes to the pharmacy and asks what kind of knowledge pills are available. The pharmacist says “Here’s a pill for English literature.” The student takes the pill and swallows it and has new knowledge about English literature!

“What else do you have?” - asks the student.

“Well, I have pills for art history, biology, and world history,” replies the pharmacist.

The student asks for these, and swallows them and has new knowledge about those subjects.

Then the student asks, “Do you have a pill for math?”

The pharmacist says “Wait just a moment”, and goes back into the storeroom and brings back a whopper of a pill and plunks it on the counter.

“I have to take that huge pill for math?” - inquires the student.

The pharmacist replied “Well, you know math always was a little hard to swallow.”

A mathematician is showing a new proof he came up with to a large group of peers. After he’s gone through most of it, one of the mathematicians says, “Wait! That’s not true. I have a counter-example!”

He replies, “That’s okay. I have two proofs.”

Stefan Chakerian

“What do you mean pies are square? Everyone knows pies are round!”

A guy decided to go to the brain transplant clinic to refreshen his supply of brains. The secretary informed him that they had three kinds of brains available at that time. Doctors’ brains were going for $20 per ounce and lawyers’ brains were getting $30 per ounce. And then there were mathematicians’ brains, which were currently fetching $1000 per ounce.

“1000 dollars an ounce!” he cried. “Why are they so expensive!”

“It takes more mathematicians to get an ounce of brains,” she explained.