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INTEGRAL GROUP RINGS WITH TRIVIAL CENTRAL UNITS II#

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All groups G with trivial central units in $\mathbb{Z}G$ are classified. This was done for finitegroups by Ritter and Sehgal [Ritter, J., Sehgal, S. K. (1990). Integral group rings with trivial central units. Proc. Amer. Math. Soc. 108:327–329].

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1. INTRODUCTION

Let $\mathcal{U} = \mathcal{U}(\mathbb{Z}G)$ be the unit group of the integral group ring $\mathbb{Z}G$. It is a classical result of G. Higman (Sehgal, 1978, p. 57) that if G is torsion then $\mathcal{U}$ is trivial if and only if G is abelian of exponent 2, 3, 4 or 6 or $G = K_8 \times E$, the product of the quaternion group K_8 of order 8 and an elementary abelian 2-group E. Since triviality of $\mathcal{U}$ for torsion free groups G is an open problem it is quite difficult to extend this result to arbitrary groups. However, if a mild condition is imposed on G a classification can be seen from the results in Sehgal (1978) as we show in Sec. 5.

In 1990, Ritter and Sehgal classified finite groups G so that the central units of $\mathbb{Z}G$ are trivial, namely of the form $\pm g$, where g is an element of the centre of G. A classification of arbitrary groups G with trivial central units in $\mathbb{Z}G$ was asked for in Sehgal (1993, Problem 26, p. 301). Recently, Parmenter (1999) renewed this call. We give below this classification in terms of the condition of Ritter and Sehgal (1990).

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Theorem. *The central units of $\mathbb{Z}G$ are trivial if and only if every finite normal subgroup A of G satisfies (RS) in G:*

$$\text{For all } a \in A \text{ and } j \in \mathbb{Z} \text{ with } (j, |a|) = 1 \text{ we have } a^j \sim_G a^\varepsilon, \quad \varepsilon = \pm 1. \tag{RS}$$

2. PRELIMINARIES

Our notation is standard as in Sehgal (1993). For any ring R (group B) we shall denote by $\mathfrak{Z}(R)(\mathfrak{Z}(B))$ its centre. In particular, we shall simply write $\mathfrak{Z}$ for $\mathfrak{Z}(\mathcal{U}\mathbb{Z}G)$. A subgroup of $\mathcal{U}$ is called trivial if it consists of trivial units only. We recall

Theorem 1 (Ritter and Sehgal). *If G is a finite group then $\mathfrak{Z} = \mathfrak{Z}(\mathcal{U}\mathbb{Z}G)$ consists of trivial units if and only if G satisfies*

$$\text{For every integer } j \text{ with } (j, |G|) = 1 \text{ and every } g \in G \text{ we have } g^j \sim g^\varepsilon, \quad \varepsilon = \pm 1. \tag{RS$_1$}$$

Lemma 2. *For a finite group G the following two conditions are equivalent:*

$$(j, |G|) = 1 \Rightarrow \forall g \in G, \quad g^j \sim g^\varepsilon, \quad \varepsilon = \pm 1, \tag{RS$_1$}$$

$$\forall g \in G (j, |g|) = 1 \Rightarrow g^j \sim g^\varepsilon, \quad \varepsilon = \pm 1. \tag{RS}$$

Proof. Since there are more integers j such that $(j, |g|) = 1$ than those such that $(j, |G|) = 1$ it follows that (RS) is stronger. Thus (RS) $\Rightarrow$ (RS)$_1$, we have only to prove (RS)$_1$ $\Rightarrow$ (RS). Then we are given $g \in G$, $|g| = m$, $(j, m) = 1$ We have to show that $g^j \sim g^\epsilon$. Let us write $|G| = n = m_1 s$ with m_1 an m-number and $(s, m) = 1$. Suppose that $j = s_1 l$, s_1 an s-number, $(l, n) = 1$. We write $s = s's''$ where s' an s_1-number an s'' an s_1'-number and set $j' = j + m_1 s'$. We have

$$(j', m_1) = 1 \text{ as } (j, m) = 1$$
$$(j', s') = 1 \text{ as } (s_1, m_1 s'') = 1 \qquad (\text{Remember } m | m_1)$$
$$(j', s'') = 1 \text{ as } (s'', s_1) = 1.$$

then $(j', s) = 1$ and $(j', m) = 1$, consequently, $(j', n) = 1$. Here, $g^{j'} = g^{j + m_1 s''} = g^j$. By (RS)$_1$ $g^j = g^{j'} \sim g^\epsilon$ as required. □

Example 3. $m = |g| = 2^3 5^4$, $n = |G| = 2^5 3^4 5^6 7^2 11$, $m_1 = 2^5 5^6$, $j' = 3^2 \cdot 11 \cdot 13$, $s = 3^4 7^2 11$, $s' = 3^4 \cdot 11$, $j' = 3^2 \cdot 11 \cdot 13 + 2^5 \cdot 5^6 \cdot 7^2$, $s'' = 7^2$.

We need the next two results which are crucial for our purposes.

The first result was proved for nilpotent groups in Jespers et al. (1996) and extended to arbitrary groups in Polcino Milies and Sehgal (1999).

Lemma 4. *Let G be any group. Then any central unit $\mu \in \mathcal{U}(\mathbb{Z}G)$ can be written as $\mu = gw, g \in G, w \in \mathbb{Z}T, gw = wg$ where T is the torsion subgroup of $\phi(G)$ the FC-subgroup of G.*

Proof. This is Theorem 1 of Polcino Milies and Sehgal (1999). □

Lemma 5. *Let μ be a central unit of $\mathbb{Z}G$ such that μ^n is trivial for some $n \in \mathbb{Z}$. Then μ is trivial.*

Proof. This is Lemma 2.5 of Polcino Milies and Sehgal (1999). □

3. OBSERVATIONS FOR FINITE GROUPS G

Suppose that G is a finite group.

Let L be a splitting field of $\mathbb{Q}G$. Express $\mathbb{Q}G$ as a direct sum

$$\mathbb{Q}G = \sum^{\oplus} S$$

of simple rings S. Let χ be an absolutely irreducible character of G and D the corresponding representation. Then there exists a unique S so that $D(S) \neq 0$. There is an embedding (see Huppert, 1967, p. 544):

$$D : S \longrightarrow L_{n \times n}$$

For a class sum C_g of $g \in G$, we have

$$D(C_g) = \frac{h}{n}\chi(g)I_n$$

with $h = |C_g|$, $n = \deg D$. This induces an isomorphism $\mathfrak{Z}(S) \simeq \mathbb{Q}(\chi) = \mathbb{Q}\{\chi(g) : g \in G\}$. We have

$$\mathfrak{Z}(\mathbb{Q}(G)) \simeq \sum_{\chi}^{\oplus} \mathbb{Q}(\chi)$$

with the map $C_g \to \sum_\chi \frac{h}{n}\chi(g)$, $h = h_\chi$, $n = n_\chi$. For a normal subgroup A of G restricting to the subring $\mathfrak{Z}(\mathbb{Q}G) \cap \mathbb{Q}A$, we have an injection

$$\mathfrak{Z}(\mathbb{Q}G) \cap \mathbb{Q}A \hookrightarrow \sum_{\chi}^{\oplus} \mathbb{Q}(\chi(A))$$

where $\mathbb{Q}(\chi(A)) = \mathbb{Q}\{\chi(a) : a \in A\}$. The typical projection

$$\pi : \mathfrak{Z}(\mathbb{Q}G) \cap \mathbb{Q}A \longrightarrow \mathbb{Q}(\chi(A))$$

is onto. We have

Lemma 6. *If $\mathbb{Z}A \cap \mathfrak{Z}$ is trivial then $\mathbb{Q}(\chi(A))$ is $\mathbb{Q}$ or imaginary quadratic.*

Proof. Let us write $K = \mathbb{Q}(\chi(A))$ and O_K for its ring of integers. Then $\pi(\mathbb{Z}A \cap \mathfrak{Z}(\mathbb{Q}G))$ is an order in O_K. If O_K has an element of infinite order then so does $\pi(\mathbb{Z}A \cap \mathfrak{Z}(\mathbb{Q}G))$ and therefore also $\mathbb{Z}A \cap \mathfrak{Z}(\mathbb{Q}G)$. This is a contradiction proving the lemma. □

Lemma 7. *If symmetric elements of $\mathfrak{Z} \cap \mathbb{Z}A$ are trivial then so are all elements of $\mathfrak{Z} \cap \mathbb{Z}A$.*

Proof. Let $u \in \mathbb{Z}A$, $u \in \mathfrak{Z}$. Thus $u^* \in \mathbb{Z}A \cap \mathfrak{Z}$. Further $(uu^*)^* = uu^*$ is a symmetric element of $\mathfrak{Z} \cap \mathbb{Z}A$. Thus by assumption we have $uu^* = z \in A$. Writing $u = \sum u_g g$, we have

$$uu^* = \left(\sum u_g^2\right) \cdot 1 + \cdots = z \in A.$$

It follows that $z = 1$ and $\sum u_g^2 = 1$. Thus $u = \pm g$, $g \in A$ as required. □

Theorem 8. *Let A be a normal subgroup of a finite group G. Then*

$$A \text{ satisfies (RS) in } G \Rightarrow \mathfrak{Z} \cap \mathbb{Z}A \text{ consists of trivial units.}$$

Proof. Suppose u is a non trivial central unit in $\mathbb{Z}A$. We may assume u is symmetric. Then

$$u = \sum q(a + a^{-1}), \qquad q \in \mathbb{Q}$$

(we may have $a = a^{-1}$ and in this case $q = 1/2$). Let σ be an automorphism of $\mathbb{Q}(\chi(A))$. Extend σ to $\mathbb{Q}(\chi)$ and then to a cyclotomic field $\mathbb{Q}(\zeta)$ containing $\mathbb{Q}(\chi)$. If we express $\chi(a) = \sum \xi$, a sum of roots of unity, then

$$\begin{aligned} \chi^\sigma(a) &= \sum \xi^j = \chi(a^j) \quad \text{with } (j, |G|) = 1 \\ &= \chi(a^\varepsilon), \ \varepsilon = \pm 1 \quad \text{by (RS)} \\ &= \chi(a) \quad \text{or} \quad \overline{\chi}(a). \end{aligned}$$

Then from $\chi(u) = \sum q(\chi(a) + \chi(a^{-1}))$, we conclude that $\chi^\sigma(u) = \chi(u)$ for all σ. It follows that $\chi(u) \in \mathbb{Q}$. Then it follows that the symmetric units in $\mathfrak{Z} \cap \mathbb{Z}A$ are contained in $\sum^\oplus \mathbb{Q}$. Remember that $\mathfrak{Z} \cap \mathbb{Z}A$ is an order in $\mathbb{Q}A \cap \mathfrak{Z}(\mathbb{Q}G)$ which is a direct sum of fields $\sum^\oplus K$. This has a unique maximal order $\sum^\oplus O_K$. Therefore, the symmetric units of $\mathfrak{Z} \cap \mathbb{Z}A$ are contained in $\sum^\oplus \mathbb{Z}$. Thus $u^2 = 1$. This contradicts the assumption that u is non trivial, proving the theorem. □

Theorem 9. *Suppose that A is a normal subgroup of a finite group G. Then $\mathfrak{Z} \cap \mathbb{Z}A$ trivial $\Rightarrow$ A satisfied (RS) in G.*

Proof. Let $a \in A$ be such that $(j, |a|) = 1$. Then we know by Lemma 6 that $\mathbb{Q}(\chi(A))$ is $\mathbb{Q}$ or imaginary quadratic. Observe that j is an automorphism of $\mathbb{Q}(\chi(a))$ which extends to an automorphism σ of $\mathbb{Q}(\chi(A))$. Then σ is identity or $^-$. We have for all χ:

$$\chi^\sigma(a) = \chi(a^j) = \chi(a) \quad \text{or} \quad \overline{\chi}(a)$$
$$\chi^\sigma(a^j) + \chi(a^{-j}) = \chi(a) + \chi(a^{-1}).$$

Define functions $T_g : \mathrm{Irr}(G) \to \mathbb{C} by \chi \to \chi(g)$. These functions, one for each conjugacy class, are linearly independent. We have

$$T_{a^j} + T_{a^{-j}} = T_a + T_{a^{-1}}$$

so that $T_{a^j} = T_a or T_{a^{-1}}$. Thus $\chi(a^j) = \chi(a)$ or $\chi(a^{-1})$ for all χ. Thus $a^j \sim a$ or $a^j \sim a^{-1}$ as desired. □

4. PROOF OF THEOREM

We try to reduce the problem to finite groups G. We begin with

Lemma 10. *Let A be a finite normal subgroup of G. Then there exists a finite group H containing A as a normal subgroup such that $\mathfrak{Z} \cap \mathbb{Z}A = \mathfrak{Z}(\mathcal{U}\mathbb{Z}H) \cap \mathbb{Z}A$.*

Proof. Let C be the centralizer of A in G. Then $C \lhd G$ and the index $(G : C)$ is finite. Let $H = A \times G/C$ where $a^{\bar{g}} = a^g$, $\forall a \in A$, $\bar{g} \in G/C$. Then H is finite. Moreover, an element of $\mathbb{Z}A$ is in $\mathfrak{Z}$ if and only if it is central in $\mathbb{Z}H$. □

Lemma 11. *$\mathfrak{Z}$ trivial $\Rightarrow$ Every finite normal subgroup A of G satisfies (RS) in G.*

Proof. Let H be as in Lemma 10. Then $\mathfrak{Z}(\mathcal{U}\mathbb{Z}H) \cap \mathbb{Z}A$ is trivial. Then it follows by Theorem 9 that A satisfies (RS) in H, hence also in G. □

Lemma 12. *Let A be a finite normal subgroup of G. Then*

$$A \text{ satisfies (RS) in } G \Rightarrow \mathfrak{Z} \cap \mathbb{Z}A \text{ is trivial.}$$

Proof. Let H be as in Lemma 10. Then A satisfies (RS) in G and hence in H which isfinite. Also $\mathfrak{Z} \cap \mathbb{Z}A = \mathfrak{Z}(\mathcal{U}\mathbb{Z}H) \cap \mathbb{Z}A$. By Theorem 8, we conclude that $\mathfrak{Z} \cap \mathbb{Z}A$ is trivial. □

Lemma 13. *If every finite normal subgroup of G satisfies (RS) in G then $\mathfrak{Z}$ is trivial.*

Proof. Let $\mu \in \mathfrak{Z}$. Then $\mu = \omega g$ with $\omega \in \mathbb{Z}T$, $g \in G$ where T is a finite normal subgroup contained in $\phi(G)$ and $\omega g = g\omega$ by Lemma 4. Consider the element

$$\mu\mu^* = \omega g g^{-1} \omega^* = \omega\omega^* \in \mathbb{Z}T.$$

This element is central in $\mathbb{Z}G$. It follows by Lemma 12 that $\mu\mu^*$ is trivial. Thus μ is trivial as well, as desired. □

Proof of the Theorem. Lemmas 11 and 13. □

5. TRIVIAL UNITS

As mentioned in the introduction, the following result can be read off from Sehgal (1978). The condition of nilpotence of G/T in the following can be relaxed.

Theorem 14. *Let G be an arbitrary group and T the set of its torsion elements. Suppose $\mathcal{U}(\mathbb{Z}G) = \mathcal{U}$ is trivial. Then*

(1) *Every subgroup $\langle t \rangle$, $t \in T$, is normal in G and hence T is a subgroup.*
(2) *Either T is abelian of exponent 2, 3, 4 or 6 or $T = K_8 \times E$ where K_8 is thequaternion group of order 8 and $E^2 = 1$.*

Conversely, if G satisfies (1) and (2) and G/T is nilpotent then $\mathcal{U}$ is trivial.

Proof. Assume $\mathcal{U}$ is trivial. As seen on page 51 of Sehgal (1978), $\mathbb{Z}G$ has no nilpotent elements and so every idempotent of $\mathbb{Q}G$ is central and for every $t \in T$, we have $\langle t \rangle \lhd G$. Then T is a subgroup and (2) is Higmans theorem. Conversely, it follows by Lemma 3.22, p. 194 of Sehgal (1978) that $\mathcal{U} = (\mathcal{U}\mathbb{Z}T)G$. The result now follows from the one for T. □

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