CENTRAL UNITS OF INTEGRAL GROUP RINGS
OF NILPOTENT GROUPS

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ABSTRACT. In this paper a finite set of generators is given for a subgroup of
finite index in the group of central units of the integral group ring of a finitely
generated nilpotent group.

In this paper we construct explicitly a finite set of generators for a subgroup of
finite index in the centre $Z(U(ZG))$ of the unit group $U(ZG)$ of the integral group
ring $ZG$ of a finitely generated nilpotent group $G$. Ritter and Sehgal [4] did the
same for finite groups $G$, giving generators which are a little more complicated.
They also gave in [2] necessary and sufficient conditions for $Z(U(ZG))$ to be trivial;
recall that the units $G$ are called the trivial units. We first give a finite set of
generators for a subgroup of finite index in $Z(U(ZG))$ when $G$ is a finite nilpotent
group. Next we consider an arbitrary finitely generated nilpotent group and prove
that a central unit of $ZG$ is a product of a trivial unit and a unit of $ZT$, where $T$
is the torsion subgroup of $G$. As an application we obtain that the central units
of $ZG$ form a finitely generated group and we are able to give an explicit set of
generators for a subgroup of finite index.

1. FINITE NILPOTENT GROUPS

Throughout this section $G$ is a finite group. When $G$ is Abelian, it was shown in
[1] that the Bass cyclic units generate a subgroup of finite index in the unit group.
Using a stronger version of this result, also proved by Bass in [1], we will construct
a finite set of generators from the Bass cyclic units when $G$ is a finite nilpotent

Our notation will follow that in [6]. The following lemma is proved in [1].

Lemma 1. The images of the Bass cyclic units of $ZG$ under the natural homomorphism
$j : U(ZG) \to K_1(ZG)$ generate a subgroup of finite index.

Let $L$ denote the kernel of this map $j$, and $B$ the subgroup of $U(ZG)$ generated
by the Bass cyclic units. It follows that there exists an integer $m$ such that $z^m \in LB$
for all $z \in Z(U(ZG))$, and so we can write $z^m = lb_1b_2\ldots b_k$ for some $l \in L$ and
Bass cyclic units $b_i$.
Next, let $Z_i$ denote the $i$-th centre of $G$, and suppose from now on that $G$ is nilpotent of class $n$. For any $x \in G$ and Bass cyclic unit $b \in Z(x)$, we define

$$b^{(1)} = b$$

and for $2 \leq i \leq n$

$$b^{(i)} = \prod_{\alpha \in Z_i} b^{(i-1)}_{\alpha},$$

where $\alpha^g = g^{-1}\alpha g$ for $\alpha \in ZG$. Note that by induction $b^{(i)}$ is central in $Z(Z_i, x)$ and independent of the order of the conjugates in the product expression. In particular, $b^{(n)} \in Z(U(ZG))$.

Recall again that if $z \in Z(U(ZG))$, then $z^m = lb_1b_2...b_k$ for some $l \in L$ and Bass cyclic units $b_i$. Since $K_1(ZG)$ is Abelian, we can write

$$z^m[l_1|Z_2||Z_3|...|Z_n] = (lb_1b_2...b_k)[l_1|Z_2||Z_3|...|Z_n]$$

$$= l_1\prod_{1 \leq i \leq k} b^{(i)}_{l(i)}$$

for some $l_1 \in L$

$$= l_2\prod_{1 \leq i \leq k} b^{(i)(2)}_{l(i)}$$

for some $l_2 \in L$

$$= l'^{1}\prod_{1 \leq i \leq k} b^{(i)(n)}_{l(i)}$$

for some $l' \in L$.

Since each $b^{(i)}$ is in $Z(U(ZG))$, we conclude that $l' \in L \cap Z(U(ZG))$. But we shall show next that $L \cap Z(U(ZG))$ is trivial, so $l' \in \pm Z(G)$. The argument uses the same idea as in [3, Lemma 3.2].

For every primitive central idempotent $e$ in the rational group algebra $QG$, the simple ring $QGe$ has a reduced norm which we denote by $nr_e$. Further, denote

$$m_e = \sqrt{QGe : Z(QGe)}$$

and let

$$r = \prod_e m_e.$$ 

Now let $l' \in L \cap Z(U(ZG))$. By definition of $K_1(ZG)$ this means that a suitable matrix

$$\begin{bmatrix}
  l' \\
  1 \\
  \vdots \\
  1
\end{bmatrix}$$

is a product of commutators. Therefore $l'e$ has reduced norm one. Since $l'e$ is also central, we obtain

$$(l'e)^{m_e} = nr(l'e)e = e.$$ 

Hence

$$l'^{r} = 1.$$ 

So $l'$ is a torsion central unit, and therefore is trivial [7, Corollary 1.7, page 4].

Since $Z(U(ZG))$ is finitely generated (see, e.g., [2]), $(Z(U(ZG)))^{m|Z_2||Z_3|...|Z_n}$ is of finite index. But we have just seen that the latter subgroup is contained in the subgroup generated by $\pm Z(G)$ and $\{b^{(n)}_b | b \text{ a Bass cyclic unit}\}$. We have proved
Proposition 2. Let $G$ be a finite nilpotent group of class $n$. Then
\[
\langle b(n) \mid b \text{ a Bass cyclic} \rangle
\]
is of finite index in $\mathcal{U}(\mathbb{Z}G)$.

Remark. Note that our method for constructing generators for a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$ can be adapted for some other classes of finite groups $G$. For example if $G = D_{2n} = \langle a, b \mid x^n = 1, y^2 = 1, yx = x^{n-1}y \rangle$, the dihedral group of order $2n$, then the only nontrivial Bass cyclic units $b$ of $\mathbb{Z}D_{2n}$ belong to $\mathbb{Z}\langle x \rangle$. It follows that $bb^y = b^yb$ is central. Our proof now remains valid and yields that $\langle bb^y \mid b \text{ a Bass cyclic} \rangle$ is of finite index in $\mathcal{U}(\mathbb{Z}D_{2n})$.

2. FINITELY GENERATED NILPOTENT GROUPS

We will now consider central units of an integral group ring of an arbitrary finitely generated nilpotent group $G$. The torsion subgroup of $G$ is denoted $T$. First we show that central units of $\mathbb{Z}G$ have the following decomposition.

Proposition 3. Let $G$ be a finitely generated nilpotent group. Every $u \in \mathcal{U}(\mathbb{Z}G)$ can be written as $u = rg$, $r \in \mathbb{Z}T$, $g \in G$.

Proof. Let $F = G/T$. Since $T$ is finite and $F$ acts on the set of primitive central idempotents of $\mathbb{Q}T$ by conjugation, by adding the idempotents in an orbit of this action we obtain
\[
\mathbb{Q}T = \bigoplus (\mathbb{Q}T)e_i = \bigoplus R_i,
\]
where $e_i$ are primitive central idempotents of $\mathbb{Q}G$. Then $\mathbb{Q}G$ is the crossed product
\[
(*) \quad \mathbb{Q}G = \mathbb{Q}T \rtimes F = \left( \bigoplus R_i \right) \rtimes F = \bigoplus R_i \rtimes F.
\]
Decompose $u$ as a sum of elements in $(*)$:
\[
u = \bigoplus_i \left( \sum_{j=1}^n u_j f_j \right), \quad 0 \neq u_j \in R_i, \ f_j \in G, \ \text{for each } j.
\]
We assume that we have put together the $u_j$'s with the same $f_j T \in G/T$, namely for $k \neq j$, $f_k T \neq f_j T$.

We claim that $n = 1$. Let us denote by $-\bar{u}$ the projection of $\mathbb{Q}G$ onto $R_i \rtimes F$. Then since $u$ is central we have $\mathbb{Q}T \bar{u} = \bar{u} \mathbb{Q}T$, which implies $\mathbb{Q}T u_j f_j = u_j f_j \mathbb{Q}T$ for all $j$. It follows that $u_j$ is not a zero divisor provided $R_i$ has only one simple (artinian) component, and so $u_j$ is a unit. The only time $u_j$ can be a nonunit is when it has some zero components in the simple components of $R_i$. However, by the construction of $R_i$, these latter components can be moved to any other place by conjugating suitably. But they must stay put due to the facts that $F$ is ordered and $\bar{u}$ is central. It follows that $u_j$ is a unit for all $j$. Hence, working in $R_i \rtimes F$ and using again that $F$ is ordered, it follows by a classical argument that $\bar{u} = \sum_j u_j f_j$ is simply equal to $u_n f_n$ as claimed.

Changing notation, we write
\[
u = \bigoplus_i \alpha f, \quad \alpha \in R_i, \ f \in G.
\]
Let \( k = |\text{Aut}(T)| \), so \( f^k \) commutes with \( T \) for \( f \in G \). Hence

\[
  u^k = \bigoplus (\alpha f)^k = \bigoplus \beta f^k, \quad \beta \in R_i
\]

(note that the number of summands in \( u^k \) is the same as the number of summands in \( u \), because each \( \alpha \) is a unit in \( R_i \)), and thus

\[
  u^k = (u^k)^f = \bigoplus (\beta f^k)^t = \bigoplus \beta f^k, \quad t \in T.
\]

The last step follows from the fact that conjugation will preserve the order on the \( fT \)'s in the ordered group \( F \). Since \( (f^k)^f = tf^k \), we can choose \( k \) large enough so that all the \( f^k \) commute with each other and with \( T \). Thus we may assume that

\[
  u^k = \bigoplus \beta f^k.
\]

Again, we put together all \( \beta \) with the same \( f^kT \). In other words, we assume that \( u^k = \bigoplus \beta f^k \) with all \( f^kT \) different. Note that these new values of \( \beta \) all lie in \( ZT \).

Furthermore, we now obtain for each \( t \in T \),

\[
  u^k = (u^k)^t = \bigoplus (\beta)^t f^k,
\]

and thus \( \beta^t = \beta \). So the ring \( R \) generated by all the \( \beta \) is commutative. Again, if necessary, replacing \( k \) by a high enough power, we may assume that the group \( A \) generated by all the \( f^k \) in the summation of \( u^k \) is a torsion-free Abelian group, and thus a free Abelian group. Consequently

\[
  u^k \in RA,
\]

the commutative group ring of \( A \) over \( R \). Let \( N = \text{Rad}(R) \) be the set of nilpotent elements of \( R \). Now \( ZT \) has only trivial idempotents [6, Theorem 2.20, page 25]. Hence since \( R \subseteq ZT \) and since idempotents of \( R/N \) can be lifted to \( R \), it follows that \( R/N \), also has only trivial idempotents. Therefore [6, Lemma 3.3, page 55] together with an inductive argument tells us that \( (R/N)A \) has only trivial units. It follows that

\[
  u^k = \beta f^k + \text{nilpotent elements}.
\]

But as each \( \beta \) is a sum of units in various \( R_i \), it follows that the last term must be zero. Hence \( u^k = \beta f^k \), and thus all \( f \)'s in the original decomposition of \( u = \bigoplus \alpha f \) were in the same coset of \( T \). Thus \( u = rf \) as required. \( \Box \)

We give two important consequences of the last result. We say that \( Z(U(ZG)) \) is trivial if it contains only trivial units.

**Corollary 4.** Let \( G \) be a finitely generated nilpotent group. If \( Z(U(ZT)) \) is trivial, then \( Z(U(ZG)) \) is trivial.

**Proof.** Let \( u \in Z(U(ZG)) \) be nontrivial. Then the support of \( u \) contains two different elements, say \( x \) and \( y \). Since finitely generated nilpotent groups are residually finite, there exists a finite factor \( G/N = \overline{G} \) so that \( x \neq y \) in \( \overline{G} \) (see [5, page 149]). Hence \( \overline{u} \) has in its support at least two different elements, and thus \( \overline{u} \) is of infinite order ([7, Corollary 1.7, page 4]). By Proposition 3 we write \( u = rg, \ r \in ZT, \ g \in G \). Since \( u \) is central, \( r \) commutes with \( g \). It then follows easily that \( \overline{r} \), and hence also \( r \), is of infinite order. Moreover, there exists a positive integer \( n \) such that \( (g^n, T) = 1 \). Consequently it follows from \( u^n = r^ng^n \) that \( r^n \) commutes with \( T \). Thus \( r^n \) is a nontrivial unit of \( Z(U(ZT)) \). \( \Box \)
Corollary 5. Let $G$ be a finitely generated nilpotent group. Then $Z(U(ZG))$ is finitely generated. Furthermore, $(Z(U(ZG)) \cap Z(U(ZT))) Z(G)$ is of finite index in $Z(U(ZG))$.

Proof. Let $S = Z(U(ZG)) \cap Z(U(ZT))$. First we show that $Z(U(ZG))/SZ(G)$ is a torsion group of bounded exponent. Indeed, let $u \in Z(U(ZG))$. Because of Proposition 3 write $u = rg$, with $r \in U(ZT)$ and $g \in G$. Considering the natural epimorphism $ZT \to Z(G/T)$ and using the fact that $Z(U(Z(G/T)))$ is trivial because $G/T$ is ordered, it follows that $gT \in Z(G/T)$. Hence $(g^kT) = 1$ and $g^k \in Z(G)$ for $k = |Aut(T)|$ and $l = k|T|$. Now since $u$ is central, $r$ and $g$ commute. Therefore

$$u^l = r^l g^l$$

and $r^l \in S$.

Consequently $u^l \in SZ(G)$, and the claim follows.

As a subgroup of the finitely generated group $Z(U(ZT))$, the group $S$ is itself finitely generated. Hence so is $SZ(G)$. Since the torsion subgroup of $Z(U(ZG))$ is finite (see for example [6, page 46]), the above claim now easily yields that $Z(U(ZG))$ is indeed finitely generated.

We will now construct finitely many generators for the central units of any finitely generated nilpotent group.

Let $n$ be the nilpotency class of $T$ and $h$ the Hirsch number of $G/T$. Let $k = |Aut(T)|$. Further let $x_1, \ldots, x_h$ be elements of $G$ such that for each $1 \leq i \leq h$ the group $G_i = \langle T, x_1, \ldots, x_i \rangle$ is normal in $G$ and $G_i/G_{i-1} \cong \mathbb{Z}$, where $G_0 = T$. For any generator $b(n)$ described in Proposition 2 define

$$b^{(0)}(n) = b(n)$$

and for $1 \leq i \leq h$

$$b^{(i)}(n) = \prod_{0 \leq j < k} (b^{(i-1)}(n))^{x_j^i}.$$ 

Since each $b^{(n)}(n)$ is in $Z(U(ZT))$, the order of the conjugates in the product expression for $b^{(i)}(n)$ is unimportant. It follows by induction that $b^{(i)}(n)$ is in $Z(U(ZG_i))$. In particular, $b^{(h)}(n) \in Z(U(ZG))$.

Theorem 6. Let $G$ be a finitely generated nilpotent group. Suppose $n$ is the nilpotency class of $T$ and $h$ is the Hirsch number of $G/T$. Then

$$\langle b^{(h)}(n) \mid b \text{ a Bass cyclic of } ZT \rangle Z(G)$$

is of finite index in $Z(U(ZG))$.

Proof. Because of Corollary 5 the group $SZ(G)$ with $S = Z(U(ZG)) \cap Z(U(ZT))$ is of finite index in $Z(U(ZG))$. Let $\alpha_1, \ldots, \alpha_p$ be a finite set of generators for $S$. By Proposition 2 there exists a positive integer $m$ such that all $\alpha_1^m, \ldots, \alpha_p^m$ are in $\langle b(n) \mid b \text{ a Bass cyclic of } ZT \rangle$. For simplicity, write $\alpha = \alpha_1^m$. Then

$$\alpha = \prod b^{(n)}(n),$$

where the product runs over a finite number of Bass cyclic units of $ZT$. Since $\alpha$ is in $Z(U(ZG))$, and using the notation introduced above, we obtain

$$\alpha^k = \alpha x_1^1 \cdots x_i^{k-1}.$$
As each $b(n)$ is central in $ZT$, this implies

$$\alpha^k = \prod b_{(n)}^{(i)}.$$ 

Continuing this process one obtains that

$$\alpha^{kh} = \prod b_{(n)}^{(h)}.$$ 

Since the group generated by $\alpha^{kh}, \ldots, \alpha^{kh}$ is of finite index in $S$, the result follows.

Note that Corollary 4 can now also be obtained as an easy consequence of Theorem 6.

We now give an example showing that the converse of Corollary 4 does not hold.

**Example.** Let $G = \langle a, x | a^x = a^3, a^8 = 1 \rangle$. Clearly $G$ is a nilpotent group with $T = \langle a \rangle$, a cyclic group of order 8. From Higman's result (see [6]) it follows that $Z(U(ZT))$, modulo the trivial units, is a free Abelian group of rank 1. We now show that $Z(U(ZG))$ contains only trivial units. For this suppose $u$ is a nontrivial central unit in $ZG$. By Proposition 3, we can write $u = rx^i$ for some integer $i$ and $r \in U(ZT))$. We know from the above that $r$ is of infinite order, and since $r$ commutes with $x$, it must be in $Z(U(ZG))$.

Because the only Bass cyclic unit, up to inverses, in $ZT$ is

$$b = (1 + a + a^2)^4 - 10a, \quad \hat{a} = 1 + a + \cdots + a^7,$$

Proposition 2 yields that

$$r^k = b^l,$$

for some nonzero integers $k, l$. Observe, however, that $b^x = b^{-1}$. Since $b^l = r^k$ is central in $ZG$, we obtain $b^l = b^{-l}$, contradicting the fact that $b$ is of infinite order.

**REFERENCES**


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