ABEL-JACOBI MAPS ASSOCIATED TO ALGEBRAIC CYCLES

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Program with Matt Kerr

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§0 Notation

\[ \mathbb{P}^N := \frac{\mathbb{C}^{N+1} \setminus \{0\}}{\mathbb{C}^\times} \ni [z] := [z_0, \ldots, z_N] \]

\[ \mathbb{P}^N = \mathbb{C}^N \bigsqcup \mathbb{P}^{N-1}, \]

where \( \mathbb{C}^N = \{ [z] \in \mathbb{P}^N \mid z_0 \neq 0 \} \), with coordinates \( x := (x_1, \ldots, x_N) = (z_1/z_0, \ldots, z_N/z_0) \), and \( \mathbb{P}^{N-1} = \{ [z] \mid z_0 = 0 \} \) (hyperplane at infinity).

Let \( \{ f_1, \ldots, f_m \} \subset \mathbb{C}[z_0, \ldots, z_N] \) be homogeneous polynomials. Then

\[ V := \{ [z] \in \mathbb{P}^N \mid f_1(z) = \cdots = f_m(z) = 0 \}, \]

is called a projective variety. Any (reduced, irreducible) projective variety \( V \) stratifies in the form:

\[ V = V_{\text{smooth}} \bigsqcup V_{\text{sing}}, \]

where in the strong topology, \( V_{\text{smooth}} \neq \emptyset \) is open and locally a polydisk \( \Delta^d \subset \mathbb{C}^d \), and \( V_{\text{sing}} \) is a proper projective subvariety of \( V \), called the singular set. We define \( \dim V = d \). \( V \) is a projective algebraic manifold if \( V_{\text{sing}} = \emptyset \).
Examples. (i) \( V_1 = \{ z_0 z_2^2 = z_1 (z_1 + z_0) (z_1 - z_0) \} \subset \mathbb{P}^2 \). \( V_1 \cap \mathbb{C}^2 = \{ y^2 = x(x + 1)(x - 1) \} \subset \mathbb{C}^2 \ni (x, y) = (z_1/z_0, z_2/z_0) \). \( V_1 \) is a projective algebraic manifold of dimension 1 (\( V_1 \) is an example of an elliptic curve).

(ii) \( V_2 = \{ z_0 z_2^2 = z_1^3 \} \subset \mathbb{P}^2 \). \( V_2 \cap \mathbb{C}^2 = \{ y^2 = x^3 \} \subset \mathbb{C}^2 \). \( V_2 \) is a projective variety of dimension 1 (\( V_2 \) is a singular rational elliptic curve), and \((0, 0) = V_{\text{sing}}\) is its singular set (cusp).
\section{Classical story}

(I) **Definitions.** \(X/\mathbb{C}\) projective algebraic of dimension \(d\). Let \(0 \leq r \leq d\) be given. \(z^r(X) = z_{d-r}(X)\) = free abelian group generated by subvarieties of codimension \(r = \dim (d - r)\) in \(X\).

**Examples.** (i) \(z^d(X) = z_0(X) = \{\sum_{j=1}^{M} n_j p_j \mid n_j \in \mathbb{Z}, \ p_j \in X\}\).

(ii) \(z^0(X) = z_d(X) = \mathbb{Z}\{\mathbb{Z}\}\).

(iii) \(3V_1 - 5V_2 \in z^1(\mathbb{P}^2) = z_1(\mathbb{P}^2)\).

(iv) Let \(V \subset X\) be a subvariety of codimension \(r - 1\) in \(X\) and \(f \in \mathbb{C}(V)^\times\). Then \(\text{div}(f) := (f) := (f)_0 - (f)_\infty \in z^r(X)\) (principal divisor).

The principal divisors form a subgroup \(z^r_{\text{rat}}(X) \subset z^r(X)\). (Here \(\text{rat}\) stands for rational equivalence.) The Chow group is given by

\[
\text{CH}^r(X) = \text{CH}_{d-r}(X) = z^r(X)/z^r_{\text{rat}}(X)
\]

Evidently \(\text{CH}^\bullet(X) := \bigoplus_r \text{CH}^r(X)\) is a ring under \(\cap\). [Rational equivalence is the weakest equivalence relation to guarantee this.]
(II) Cycle class maps. Let \( i : V \hookrightarrow X \) be a subvariety of codimension \( r \), and \( \sigma : \tilde{V} \to V \) any desingularization. We have \( \{V\} \in H_{2d-2r}(\tilde{V}, \mathbb{Z}) \) the fundamental class generator and \([V] \in H^{2r}(X, \mathbb{Z})\) given by the composite

\[
H_{2d-2r}(\tilde{V}, \mathbb{Z}) \xrightarrow{(i \circ \sigma)^*} H_{2d-2r}(X, \mathbb{Z}) \xrightarrow{PD} H^{2r}(X, \mathbb{Z})
\]

\[
\{\tilde{V}\} \quad \mapsto \quad [V]
\]

[Note: \([V]\) does not depend on the choice of desingularization of \( V \), as one can dominate any two desingularizations by a third desingularization.] Extending by linearity gives \( \text{cl}_r : z^r(X) \to H^{2r}(X, \mathbb{Z}) \). Evidently \( \text{cl}_r(z^r_{rat}(X)) = 0 \), inducing \( \text{cl}_r : \text{CH}^r(X) \to H^{2r}(X, \mathbb{Z}) \). As we will see shortly this map is far from being an isomorphism. Even its image is rather complicated.
To clarify this latter point, we first recall de Rham cohomology and the Hodge decomposition theorem:

- A complex-valued $C^\infty$ $k$-form on $\mathbb{R}^n$ is given by
  \[
  \omega = \sum_{|I|=k} f_I(x_1, \ldots, x_n) dx^I, \quad f_I \in C^\infty(\mathbb{R}^n, \mathbb{C}),
  \]
  \[I = \{i_1 < \cdots < i_k\}, \quad dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}.\]
  One has the differential
  \[d\omega = \sum_{|I|=k} df_I \wedge dx^I,\]
  a $(k+1)$-form, with $d^2 = 0$. Note: $|I| > n \Rightarrow dx^I = 0$.

- A complex-valued $C^\infty$ $k$-form on $\mathbb{C}^d = \mathbb{R}^{2d}$ is given by
  \[\sum_{|I|+|J|=k} f_{IJ}(z_1, \ldots, z_d) dz^I \wedge d\overline{z}^J, \quad f_{IJ} \in C^\infty(\mathbb{C}^d, \mathbb{C}).\]
  Note $|I| > d$ or $|J| > d \Rightarrow dz^I \wedge d\overline{z}^J = 0$. Also $d = \partial + \overline{\partial}$.

This globalizes to a projective algebraic manifold $X$, viz.:

\[E^k_X = \bigoplus_{p+q=k} E^{p,q}_X, \quad d : E^k_X \to E^{k+1}_X, \quad d^2 = 0, \quad E^{p,q}_X = E^{q,p}_X,\]

where $E^k_X$ (resp. $E^{p,q}_X$) is the $\mathbb{C}$-vector space of $C^\infty$ $k$-forms (resp. $(p,q)$-forms) on $X$. 
Note: $\partial E^{p,q}_X \subset E^{p+1,q}_X, \overline{\partial} E^{p,q}_X \subset E^{p,q+1}_X$. $0 = d^2 = (\partial + \overline{\partial})^2 \Rightarrow 0 = \partial^2 = \overline{\partial}^2 = \partial \overline{\partial} + \overline{\partial} \partial$.

**Definition.** The $k$-th de Rham cohomology of $X$ is given by

$$H^k_{\text{DR}}(X) = \frac{\ker d : E^k_X \to E^{k+1}_X}{dE^{k-1}_X}.$$

**Hodge Decomposition Theorem.**

$$H^k(X, \mathbb{Z}) \otimes \mathbb{C} \simeq H^k_{\text{DR}}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $\overline{H^{p,q}(X)} = H^{q,p}(X)$. [Here we say that $H^k(X, \mathbb{Z})$ defines a (pure) Hodge structure (HS) of weight $k$. If we put $F^r H^k(X, \mathbb{C}) = \bigoplus_{p+q=k, p \geq r} H^{p,q}(X)$, then $H^{p,q}(X) = F^p \cap F^q$. Note that $H^k(X, \mathbb{C}) = F^r \oplus F^{k-r+1}$.]  

**Poincaré-Serre duality.** The pairings

$$H^k_{\text{DR}}(X) \times H^{2d-k}(X) \to \mathbb{C}$$

$$H^{p,q}(X) \times H^{d-p,d-q}(X) \to \mathbb{C}$$

induced by

$$(\omega, \eta) \mapsto \int_X \omega \wedge \eta \in \mathbb{C},$$

are nondegenerate.
Hodge decomposition theorem revisited. The complex \((E^\bullet_X, d)\) is filtered by subcomplexes \((F^p E^\bullet_X, d)\), \(p \geq 0\), where

\[
F^p E^k_X = \bigoplus_{i+j=k, i \geq p} E^i_X.
\]

Explicitly,

\[
F^p E^\bullet_X : 0 \to \cdots \to 0 \to F^p E^p_X \xrightarrow{d} \cdots \xrightarrow{d} F^p E^{2d}_X
\]

\[
\bigcap
E^\bullet_X : E^0_X \xrightarrow{d} \cdots \xrightarrow{d} E^{p-1}_X \xrightarrow{d} E^p_X \xrightarrow{d} \cdots \xrightarrow{d} E^{2d}_X
\]

The Hodge to de Rham spectral sequence is given by

\[
E_1^{p,q} := H^{p+q}(Gr^p_F E^\bullet_X) \Rightarrow H^{p+q}_{DR}(X).
\]

But

\[
H^{p+q}(Gr^p_F E^\bullet_X) = H^q_\partial(E^p_X) =: H^{p,q}(X).
\]
Let $\epsilon : H^{2r}(X, \mathbb{Z}) \to H^{2r}(X, \mathbb{C})$ be the natural map. Then
\[
\text{cl}_r(CH^r(X)) \subset \epsilon^{-1}(H^{r,r}(X)).
\]

One way to see this, if via Poincaré-Serre duality. If $V \subset X$ is a
subvariety of codimension $r$ ($\Rightarrow \dim V = d - r$, $\Rightarrow \dim_{\mathbb{R}} V = 2d - 2r$),
then for $\omega \in E^{2d-2r}_X$, define
\[
\delta_V(\omega) = \int_{V_{\text{smooth}}} \omega \in \mathbb{C}.
\]

Note: $\omega \in E^{d-p,d-q}_X$ with either $p > r$ or $q > r \Rightarrow \delta_V(\omega) = 0$, using
$\dim V = d - r$. Thus:

\[
V \mapsto \delta_V \in H^{d-r,d-r}(X)^\vee \left[ \sim H^{r,r}(X) \right] \subset H^{2d-2r}_{\text{DR}}(X, \mathbb{C})^\vee \left[ \sim H^{2r}_{\text{DR}}(X, \mathbb{C}) \right].
\]

[To show that $\delta_V$ makes sense as a closed current, hence descends on
the level of cohomology, we pass to a desingularization of $V$.]

If we declare $\mathbb{Z}$ a HS of pure type $(r, r)$ [weight $2r$], then
\[
\epsilon^{-1}(H^{r,r}(X)) = \text{hom}_{(M)HS}(\mathbb{Z}, H^{2r}(X, \mathbb{Z})).
\]
Let $\mathbb{A} \subset \mathbb{R}$ be a subring, finite type $\mathbb{A}$-modules $V, W$ HS of weight $m$. $\psi : V \to W$ is a morphism of HS if $\psi_\mathbb{C}(V^{p,q}_\mathbb{C}) \subset W^{p,q}_\mathbb{C}$, for all $p, q$ where $p + q = m$. [Here $V_\mathbb{C} := V \otimes_\mathbb{A} \mathbb{C}$, etc.]. Equivalently $\psi_\mathbb{C}(F^p V_\mathbb{C}) \subset F^p W_\mathbb{C}$ for all $p$. Recall that $\mathbb{Z}$ a HS of weight $2r$. Thus $\mathbb{Z} = \mathbb{Z} \Rightarrow \mathbb{Z} = \mathbb{Z}^{r,r}$. Morphisms of MHS (introduced below) must preserve the weight and Hodge filtrations.
Put $\text{CH}^r_{\text{hom}}(X) := \ker \text{cl}_r$. Our second cycle class map is of the form

$$\Phi : \text{CH}^r_{\text{hom}}(X) \to \text{Ext}^1_{\text{MHS}}(\mathbb{Z}, H^{2r-1}(X, \mathbb{Z})).$$

We need:

**Definition.** An $\mathbb{A}$-MHS is given by a finite type $\mathbb{A}$-module $H_{\mathbb{A}}$, together with an increasing “weight” filtration of the $\mathbb{A} \otimes_{\mathbb{Z}} \mathbb{Q}$-module $H_{\mathbb{A} \otimes \mathbb{Q}} := H_{\mathbb{A}} \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$0 \subset \cdots \subset W_i \subset W_{i+1} \subset \cdots \subset W_N = H_{\mathbb{A} \otimes \mathbb{Q}},$$

and a decreasing “Hodge filtration”

$$H_{\mathbb{C}} \supset \cdots \supset F^r \supset F^{r+1} \subset \cdots \subset F^M \supset \{0\},$$

such that the induced filtration on $Gr^i_W := W_i/W_{i-1}$ is a HS of weight $i$, (and where $N, M \in \mathbb{Z}$).

**Theorem [Deligne].** Let $X$ be a complex variety. Then $H^i(X, \mathbb{Z})$ carries a canonical and functorial MHS.
Example. Let $\overline{X}$ be a compact Riemann surface, $\Sigma \in \overline{X}$ a finite set of points and $X := \overline{X} \setminus \Sigma$. There is an exact sequence:

$$0 \to H^1(\overline{X}, \mathbb{Z}) \to H^1(X, \mathbb{Z}) \to [H^0(\Sigma, \mathbb{Z}) \cong H^2_\Sigma(X, \mathbb{Z})]$$

$$\to H^2(\overline{X}, \mathbb{Z}) \to 0$$

We put $W_2 := H^1(X)$, $W_1 := H^1(\overline{X})$, $W_0 = 0$. Thus $W_2/W_1 = H^0_{\text{deg}, 0}(\Sigma) \cong \mathbb{Z}^{|\Sigma| - 1}$ has a declared HS of weight 2, (and of pure Hodge type $(1, 1)$). Further $W_1/W_0 = H^1(\overline{X}, \mathbb{Z})$. 
Now let us return to the setting where $X$ is a smooth complex projective variety of dimension $d$. Let $\xi \in \text{CH}_{\text{hom}}^r(X)$, with support $|\xi| \subset X$. There is an exact sequence:

$$H^{2r-1}_{|\xi|}(X, \mathbb{Z}) \to H^{2r-1}(X, \mathbb{Z}) \to H^{2r-1}(X \setminus |\xi|, \mathbb{Z})$$

$$\to H^0_{|\xi|}(X, \mathbb{Z}) \to H^{2r}(X, \mathbb{Z})$$

By a weak purity argument, $H^{2r-1}_{|\xi|}(X, \mathbb{Z}) = 0$. [Alternatively, duality (Poincaré) gives us $H^{2r-1}_{|\xi|}(X, \mathbb{Z}) \simeq H_{2d-2r+1}(|\xi|, \mathbb{Z}) = 0$ since $\dim_{\mathbb{R}} |\xi| = 2d - 2r$.] Next, put

$$H^{2r}_{|\xi|}(X, \mathbb{Z})^\circ := \ker \left( H^{2r}_{|\xi|}(X, \mathbb{Z}) \to H^{2r}(X, \mathbb{Z}) \right).$$

There is a natural cycle class $[\xi] \in H^{2r}_{|\xi|}(X, \mathbb{Z})$ (here we can also use $H^{2r}_{|\xi|}(X, \mathbb{Z}) \simeq H_{2d-2r}(|\xi|, \mathbb{Z})$ via Poincaré duality, and the obvious class $[\xi] \in H^{2d-2r}(|\xi|, \mathbb{Z})$) and since $\xi$ is homologous to zero on $X$, it follows that $[\xi] \in H^{2r}_{|\xi|}(X, \mathbb{Z})^\circ$. Thus we have a diagram:

$$
\begin{align*}
H^{2r-1}(X, \mathbb{Z}) & \leftrightarrow H^{2r-1}(X \setminus |\xi|, \mathbb{Z}) \rightarrow H^{2r}_{|\xi|}(X, \mathbb{Z})^\circ \\
\uparrow & \quad \quad \uparrow \\
H^{2r-1}(X, \mathbb{Z}) & \leftrightarrow E \rightarrow \mathbb{Z}[\xi]
\end{align*}
$$

Thus we arrive at a class $\Phi_r(\xi) := \{E\} \in \text{Ext}_{\text{MHS}}^1(\mathbb{Z}, H^{2r-1}(X, \mathbb{Z})).$
Alternatively, we give a classical description of this map in terms of currents: By definition, the Griffiths jacobian is given by:

\[
J^r(X) := \frac{H^{2r-1}(X, \mathbb{C})}{F^r H^{2r-1}(X, \mathbb{C}) + H^{2r-1}(X, \mathbb{Z})} \approx \frac{F^{d-r+1} H^{2d-2r+1}(X, \mathbb{C})}{H_{2d+2r+1}(X, \mathbb{Z})}
\]

*Griffiths’ prescription.* \( \xi \in \text{CH}_r^\text{hom}(X), \xi = \partial \zeta. \) If we let \( \omega \in F^{d-r+1} H^{2d-2r+1}(X, \mathbb{C}), \) then

\[
\Phi_r(\xi)(\omega) := \int_{\zeta} \omega \bigg/ \{\text{Periods}\}.
\]

By Carlson, \( J^r(X) \approx \text{Ext}_{\text{MHS}}^1(\mathbb{Z}, H^{2r-1}(X, \mathbb{Z})), \) and the two definitions of the Abel-Jacobi map coincide. Incidentally, the isomorphism

\[
\text{Ext}_{\text{MHS}}^1(\mathbb{Z}, H^{2r-1}(X, \mathbb{Z})) \sim J^r(X),
\]

is given as follows. Consider a s.e.s. of MHS:

\[
0 \rightarrow H^{2r-1}(X, \mathbb{Z}) \rightarrow E \rightarrow \mathbb{Z} \rightarrow 0.
\]

By exactness of the Hodge filtration, there exists \( \mu \in F^r E_{\mathbb{C}} \mapsto 1 \in \mathbb{Z}. \) In the category of abelian groups, there is a retraction \( r : E \rightarrow H^{2r-1}(X, \mathbb{Z}). \) Then the isomorphism is given by \( \{E\} \mapsto \{r(\mu)\} \in J^r(X). \)
In summary, we can combine both maps $cl_r$ and $\Phi_r$ in a single diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\text{CH}^r_{\text{hom}}(X) & \xrightarrow{\Phi_r} & \text{Ext}^1_{\text{MHS}}(\mathbb{Z}, H^{2r-1}(X, \mathbb{Z})) \\
\downarrow & & \downarrow \\
\text{CH}^r(X) & \xrightarrow{\Psi_r} & H_{D^r}(X, \mathbb{Z}(r))) \\
\downarrow & & \downarrow \\
\frac{\text{CH}^r(X)}{\text{CH}^r_{\text{hom}}(X)} & \xrightarrow{cl_r} & \text{hom}(\mathbb{Z}, H^{2r}(X, \mathbb{Z})) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

In the case $r = 1$, $\Psi_1$ is an isomorphism, and hence all three vertical maps are isomorphisms. If however $r > 1$, then it is no longer the case that $\Psi_r$ is injective (even after tensoring over $\mathbb{Q}$). [Note: If $X$ is defined over a number field $L$ and if we restrict to those Chow cycles defined over $L$, then it is conjectured by Bloch/Belinson that $\Psi_r$ is injective on those cycles, after tensoring with $\mathbb{Q}$.]
§2 Higher Chow groups and Deligne cohomology

(I) Bloch’s higher Chow groups. Let $W/\mathbb{C}$ a quasiprojective variety. Put $z^r(W) = \text{free abelian group generated by subvarieties of codimension } r (= \text{dim } W - r) \text{ in } W$. The higher Chow groups are an algebraic version of ordinary simplicial homology. Consider the $m$-simplex:

$$\Delta^m = \text{Spec} \left\{ \frac{\mathbb{C}[t_0, \ldots, t_m]}{(1 - \sum_{j=0}^m t_j)} \right\} \cong \mathbb{C}^m.$$ 

We set

$$z^r(W, m) = \left\{ \xi \in z^r(W \times \Delta^m) \mid \xi \text{ meets all faces } \{t_{i_1} = \cdots = t_{i_\ell} = 0, \ \ell \geq 1\} \text{ properly} \right\}.$$ 

Note that $z^r(W, 0) = z^r(W)$. Now set $\partial_j : z^r(W, m) \to z^r(W, m - 1)$, the restriction to $j$-th face given by $t_j = 0$. The boundary map $\delta = \sum_{j=0}^m (-1)^j \partial_j : z^r(W, m) \to z^r(W, m - 1)$, satisfies $\delta^2 = 0$.

**Definition** [Bloch]. $\text{CH}^\bullet(W, \bullet) = \text{homology of } \{z^\bullet(W, \bullet), \delta\}$. We put $\text{CH}^r(W) := \text{CH}^r(W, 0)$. 
(II) **Alternate take**: Cubical version. Let $\square^m := (\mathbb{P}^1 \setminus \{1\})^m$ with coordinates $z_i$ and $2^m$ codimension one faces obtained by setting $z_i = 0, \infty$, and boundary maps

$$\partial = \sum (-1)^{i-1}(\partial_i^0 - \partial_i^\infty),$$

where $\partial_i^0$, $\partial_i^\infty$ denote the restriction maps to the faces $z_i = 0, z_i = \infty$ respectively. The rest of the definition is completely analogous except that one has to divide out degenerate cycles. It is known that both complexes are quasiisomorphic.

**Example.** ("Totaro cycle") $X = \text{Pt}$, $a \in \mathbb{C}^\times \setminus \{1\}$,

$$V_2(a) := \{(t, 1 - t, 1 - at^{-1}) \mid t \in \mathbb{P}^1\} \cap \square^3.$$

One computes

$$\partial V_2(a) = \left\{ \begin{array}{c}
[(1, \infty) - (\infty, 1)] \\
-[(1, 1 - a) - (\infty, 1)] \\
+[(a, 1 - a) - (0, 1)]
\end{array} \right\} \cap \square^2 = (a, 1 - a)$$
(iii) Gersten-Milnor version. For a field $\mathbb{F}$, one has the Milnor $K$-groups $K_M(\mathbb{F})$, where $K_0(\mathbb{F}) = \mathbb{Z}$, $K_1(\mathbb{F}) = \mathbb{F}^\times$ and $K_2(\mathbb{F}) = \bigg\{	ext{Symbols } \{a, b\} \mid a, b \in \mathbb{F}^\times \bigg\} / \bigg\{ \text{Steinberg relations} \bigg\}
\{a_1 a_2, b\} = \{a_1, b\}\{a_2, b\} \\
\{a, b\} = \{b, a\}^{-1} \\
\{a, 1 - a\} = \{a, -a\} = 1 \bigg\} .

One has a Gersten-Milnor resolution of a sheaf of Milnor $K$-groups on $X$, which leads to a complex whose last three terms and corresponding homologies for $0 \leq m \leq 2$ are:

$$\bigoplus_{c\text{d}_X Z=r-2} K^M_2(\mathbb{C}(Z)) \xrightarrow{T} \bigoplus_{c\text{d}_X Z=r-1} \mathbb{C}(Z)^\times \xrightarrow{\text{div}} \bigoplus_{c\text{d}_X Z=r} \mathbb{Z}$$

| \text{CH}^r(X, 2) | \text{CH}^r(X, 1) | \text{CH}^r(X, 0) |

where $\text{div}$ is the divisor map of zeros minus poles of a rational function, and $T$ is the Tame symbol map.
The Tame symbol map

\[ T : \bigoplus_{cd_X Z=r-2} K_2^M(\mathbb{C}(Z)) \rightarrow \bigoplus_{cd_X D=r-1} K_1^M(\mathbb{C}(D)), \]

is defined as follows. First \( K_2^M(\mathbb{C}(Z)) \) is generated by symbols \( \{f, g\}, \)
\( f, g \in \mathbb{C}(Z)^{\times}. \)

For \( f, g \in \mathbb{C}(Z)^{\times}, \)

\[ T(\{f, g\}) = \sum_D (-1)^{\nu_D(f)\nu_D(g)} \left( \frac{f^{\nu_D(g)}}{g^{\nu_D(f)}} \right)_D, \]

where \( \left( \cdots \right)_D \) means restriction to the generic point of \( D. \)
Examples

(i) $\text{CH}^r(X) := \text{CH}^r(X, 0) =$ free abelian group generated by subvarieties of codimension $k$ in $X$, modulo divisors of rational functions on subvarieties of codimension $k - 1$ in $X$.

(ii) $\text{CH}^r(X, 1)$ is represented by classes of the form $\xi = \sum_j (f_j, D_j)$, where $\text{codim}_X D_j = r - 1$, $f_j \in \mathbb{C}(D_j)^\times$, and $\sum \text{div}(f_j) = 0$ (and modulo the image of the Tame symbol).

(iii) $\text{CH}^r(X, 2)$ is represented by classes in the kernel of the Tame symbol, modulo the image of a higher Tame symbol.
Examples of cycles

(i) $X = \mathbb{P}^2$, with homogeneous coordinates $[z_0, z_1, z_2]$. \( \mathbb{P}^1 = \ell_j := V(z_j), j = 0, 1, 2. \) Let $P = [0, 0, 1] = \ell_0 \cap \ell_1$, $Q = [1, 0, 0] = \ell_1 \cap \ell_2$, $R = [0, 1, 0] = \ell_0 \cap \ell_2$. Introduce $f_j \in \mathbb{C}(\ell_j)^\times$, where \((f_0) = P - R, (f_1) = Q - P, (f_2) = R - Q. \) Then

\[
\xi := \sum_{j=0}^{2} (f_j, \ell_j) \in \text{CH}^2(\mathbb{P}^2, 1)
\]

represents a higher Chow cycle.

Exercise. Show that $\xi \neq 0$. [Hint: Use the fact that $\text{CH}^2(\mathbb{P}^1, 1) \simeq \mathbb{C}^\times$. Choose a suitable line $\mathbb{P}^1 \subset \mathbb{P}^2$ and show that $\xi|_{\mathbb{P}^1} \neq 1 \in \mathbb{C}^\times$.]
(ii) Again $X = \mathbb{P}^2$. Let $C \subset X$ be the nodal rational curve given by $z_2^2 z_0 = z_1^3 + z_0 z_1^2$ (In affine coordinates $(x, y) = (z_1/z_0, z_2/z_0) \in \mathbb{C}^2$, $C$ is given by $y^2 = x^3 + x^2$). Let $\tilde{C} \simeq \mathbb{P}^1$ be the normalization of $C$, with morphism $\pi : \tilde{C} \to C$. Put $P = (0, 0) \in C$ (node) and let $R + Q = \pi^{-1}(P)$. Choose $f \in \mathbb{C}(\tilde{C})^\times = \mathbb{C}(C)^\times$, such that $(f)_{\tilde{C}} = R - Q$. Then $(f)_{C} = 0$ and hence $(f, C) \in \text{CH}^2(\mathbb{P}^2, 1)$ defines a higher Chow cycle.
Hypercohomology. Let \((S^\bullet_{\geq 0}, d)\) be a (bounded) complex of sheaves on \(X\). One has a Cech double complex

\[
(C^\bullet(\mathcal{U}, S^\bullet), \ d, \ \delta),
\]

where \(\mathcal{U}\) is an open cover of \(X\). The \(k\)-th hypercohomology is given by the \(k\)-th total cohomology of the associated single complex

\[
(M^\bullet := \bigoplus_{i+j=k} C^i(\mathcal{U}, S^j), \ D = d \pm \delta),
\]

viz.,

\[
\mathbb{H}^k(S^\bullet) := \lim_{\rightarrow \mathcal{U}} H^k(M^\bullet).
\]

Associated to the double complex are two filtered subcomplexes of the associated single complex, with two associated Grothendieck spectral sequences abutting to \(\mathbb{H}^k(S^\bullet)\) (where \(p + q = k\)):

\[
\begin{align*}
\ '& E_2^{p,q} & := H^p_\delta(X, \mathcal{H}^q_d(S^\bullet)) \\
\ '" E_2^{p,q} & := H^p_d(H^q_\delta(X, S^\bullet))
\end{align*}
\]

The first spectral sequence shows that quasiisomorphic complexes yield the same hypercohomology.
Alternate take. Two complexes of sheaves $\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet$ are said to be quasiisomorphic if there is a morphism $h : \mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ inducing an isomorphism on cohomology $h_* : \mathcal{H}^\bullet(\mathcal{K}_1^\bullet) \sim\to \mathcal{H}^\bullet(\mathcal{K}_2^\bullet)$. Take a complex of acyclic sheaves $(\mathcal{K}^\bullet, d)$ (viz., $H^{i>0}(X, \mathcal{K}^j) = 0$ for all $j$) quasiisomorphic to $\mathcal{S}^\bullet$. Then

$$\mathfrak{H}^k(\mathcal{S}^\bullet) := H^i\left(\Gamma(\mathcal{K}^\bullet)\right).$$

For example if $\mathcal{L}^{\bullet, \bullet}$ is an [double complex] acyclic resolution of $\mathcal{S}^\bullet$, then the associated single complex $\mathcal{K}^\bullet = \oplus_{i+j=\bullet} \mathcal{L}^{i,j}$ is acyclic and quasiisomorphic to $\mathcal{S}^\bullet$. 
Examples. Let \((\Omega_X^\bullet, d), (\mathcal{E}_X^\bullet, d)\) be complexes of sheaves of holomorphic and \(\mathbb{C}\)-valued \(C^\infty\) forms respectively. By the holomorphic and \(C^\infty\) Poincaré lemmas, one has quasiisomorphisms:

\[
(\mathbb{C} \to 0 \to \cdots) \xrightarrow{\sim} (\Omega_X^\bullet, d) \xrightarrow{\sim} (\mathcal{E}_X^\bullet, d),
\]

where the latter two are Hodge filtered. The first spectral sequence of hypercohomology shows that

\[
H^k(X, \mathbb{C}) \simeq \mathbb{H}^k(\mathbb{C} \to 0 \to \cdots) \simeq \mathbb{H}^k((F^p)\Omega_X^\bullet) \simeq \mathbb{H}^k((F^p)\mathcal{E}_X^\bullet).
\]

The second spectral sequence of hypercohomology applied to the latter term, using the known acyclicity of \(\mathcal{E}_X^\bullet\), yields

\[
\mathbb{H}^k(F^p\mathcal{E}_X^\bullet) \simeq \frac{\ker d : F^p E_X^k \to F^p E_X^{k-1}}{d F^p E_X^{k-1}} \simeq F^p H^k_{\text{DR}}(X),
\]

where the latter isomorphism is due to the Hodge to de Rham spectral sequence.
**Deligne cohomology.** Let $\mathbb{A} \subset \mathbb{R}$ be a subring and $r \geq 0$ an integer. We put $\mathbb{A}(r) = (2\pi i)^r \cdot \mathbb{A}$, and declare $\mathbb{A}(r)$ a pure $\mathbb{A}$-Hodge structure of weight $-2r$ and of (pure) Hodge type $(-r, -r)$. So for example, when we declared $\mathbb{Z}$ a pure HS of weight $2r$ and of Hodge type $(r, r)$, we were really referring to $\mathbb{Z}(-r)$. We introduce the Deligne complex $\mathbb{A}_D(r)$:

$$\mathbb{A}(r) \rightarrow \mathcal{O}_X \rightarrow \Omega_X \rightarrow \cdots \rightarrow \Omega_X^{r-1} \quad =: \Omega_X^{<r}$$

**Definition.** Deligne cohomology is given by the hypercohomology:

$$H^i_D(X, \mathbb{A}(r)) = H^i(\mathbb{A}_D(r)),$$

namely one has a double Cech complex whose cohomology of the associated single complex is hypercohomology.
Alternate take. Let \( h : (A^\bullet, d) \to (B^\bullet, d) \) be a morphism of complexes. We define
\[
\text{Cone}(A^\bullet \xrightarrow{h} B^\bullet)
\]
by the formula
\[
[\text{Cone}(A^\bullet \xrightarrow{h} B^\bullet)]^q := A^{q+1} \oplus B^q, \quad \delta(a, b) = (-da, h(a) + db).
\]

Example. \( \text{Cone}(\mathbb{A}(r) \oplus F^r \Omega_X^\bullet \xrightarrow{e^{-l}} \Omega^\bullet)[−1] \) is given by:
\[
\mathbb{A}(r) \to \mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{X}^{r-2} \xrightarrow{(0, d)} (\Omega_{X}^r \oplus \Omega_{X}^{r-1})
\]
\[
\delta \left( \Omega_{X}^{r+1} \oplus \Omega_{X}^r \right) \delta \cdots \delta \left( \Omega_{X}^d \oplus \Omega_{X}^{d-1} \right) \to \Omega_{X}^d
\]

Using the holomorphic Poincaré lemma, one can show that the natural map
\[
\mathbb{A}_D(r) \to \text{Cone}(\mathbb{A}(r) \oplus F^r \Omega_X^\bullet \xrightarrow{e^{-l}} \Omega^\bullet)[−1],
\]
is a quasiisomorphism. Thus
\[
H^k_D(X, \mathbb{A}(r)) \simeq H^r \left( \text{Cone}(\mathbb{A}(r) \oplus F^r \Omega_X^\bullet \xrightarrow{e^{-l}} \Omega^\bullet)[−1] \right).
\]
Let $\mathcal{D}_X^\bullet$ be the sheaf of currents acting on $C^\infty(2d-\bullet)$-forms. Likewise, let $C_X^\bullet = C_{2d-\bullet,X}(\mathbb{A}(r))$ be the sheaf of (Borel-Moore) chains of real codimension $\bullet$. Identifying the constant sheaf $\mathbb{A}(r)$ with the complex $\mathbb{A}(r) \to 0 \to \cdots \to 0$, we have quasiisomorphisms

$$
\mathbb{A}(r) \xrightarrow{\sim} C_X^\bullet(\mathbb{A}(r)) \\
\Omega_X^\bullet \xrightarrow{\sim} \mathcal{E}_X^\bullet \\
\mathcal{E}_X^\bullet \xrightarrow{\sim} \mathcal{D}_X^\bullet
$$

where the latter two quasiisomorphisms are (Hodge) filtered. As the sheaves on the RHS are all known to be acyclic, we deduce:

$$H^k_{\mathcal{D}}(X, \mathbb{A}(r)) \simeq H^k(\text{Cone}(C_X^\bullet(X, \mathbb{A}(r)) \oplus F^r \mathcal{D}_X^\bullet(X) \xrightarrow{\varepsilon-l} \mathcal{D}_X^\bullet(X))[−1]).$$

Note that

$$\mathbb{H}^k(F^p \Omega_X^\bullet) \simeq \mathbb{H}^k(F^p \mathcal{E}_X^\bullet) \simeq F^p H^k_{\text{DR}}(X).$$

In particular

$$\mathbb{H}^k(\Omega_X^{\bullet<^p}) \simeq \frac{H^k_{\text{DR}}(X)}{F^p H^k_{\text{DR}}(X)}.$$
From the short exact sequence:

\[ 0 \to \Omega_X^{\bullet < r}[-1] \to \mathbb{A}_D(r) \to \mathbb{A}(r) \to 0, \]

together with Hodge theory, we arrive at the short exact sequence:†

\[ 0 \to \frac{H^{i-1}(X, \mathbb{C})}{H^{i-1}(X, \mathbb{A}(r)) + F^k H^{i-1}(X, \mathbb{C})} \to H^i_D(X, \mathbb{A}(r)) \]

\[ \to H^i(X, \mathbb{A}(r)) \cap F^r H^i(X, \mathbb{C}) \to 0. \]

In particular, if \((\mathbb{A}, i, r) = (\mathbb{Z}, 2r - m, r)\), we arrive at the s.e.s.:

\[ 0 \to \frac{H^{2r-m-1}(X, \mathbb{C})}{F^r H^{2k-m-1}(X, \mathbb{C}) + H^{2r-m-1}(X, \mathbb{Z}(r))} \to H^{2r-m}_D(X, \mathbb{Z}(r)) \]

\[ \to F^r H^{2r-m}(X, \mathbb{C}) \bigcap H^{2r-m}(X, \mathbb{Z}(r)) \to 0. \]

---

† Alternate take. \(D_X^\bullet(X))[-1]\) is a subcomplex of \(\text{Cone}(C_X^\bullet(X, \mathbb{A}(r)) \oplus F^r D_X^\bullet(X) \xrightarrow{\varepsilon \otimes l} D_X^\bullet(X))[-1]\). Hence the cone complex description of \(H^*_D(X, \mathbb{A}(r)) \simeq H^i(\text{Cone}(C_X^\bullet(X, \mathbb{A}(r)) \oplus F^r D_X^\bullet(X) \xrightarrow{\varepsilon \otimes l} D_X^\bullet(X))[-1]){\text{,}}\)

yields the exact sequence:

\[ \cdots \to H^{i-1}(X, \mathbb{A}(r)) \oplus F^r H^{i-1}(X, \mathbb{C}) \to H^{i-1}(X, \mathbb{C}) \]

\[ \to H^i_D(X, \mathbb{A}(r)) \to H^{i-1}(X, \mathbb{A}(r)) \oplus F^r H^{i-1}(X, \mathbb{C}) \to \cdots \]
Example. If \( \mathbb{A} = \mathbb{R} \) and \( i = 2r - m \), where \( m \geq 1 \), then \( H^i_{\text{tor}}(X, \mathbb{R}(r)) = 0 \); moreover if we set

\[
\pi_{r-1} : \mathbb{C} = \mathbb{R}(r) \oplus \mathbb{R}(r - 1) \to \mathbb{R}(r - 1)
\]
to be the projection, then we have the isomorphisms:

\[
H^{2r-m}_D(X, \mathbb{R}(r)) \cong \frac{H^{2r-m-1}(X, \mathbb{C})}{FrH^{2r-m-1}(X, \mathbb{C}) + H^{2r-m-1}(X, \mathbb{R}(r))}
\]

\[
\pi_{r-1} \mapsto \frac{H^{2r-m-1}(X, \mathbb{R}(r - 1))}{\pi_{r-1}(F^kH^{2r-m-1}(X, \mathbb{C}))}
\]

For example:

\[
H^{2r-1}_D(X, \mathbb{R}(r)) \cong \frac{H^{2r-2}(X, \mathbb{C})}{FrH^{2r-2}(X, \mathbb{C}) + H^{2r-2}(X, \mathbb{R}(r))}
\]

\[
\pi_{r-1} \mapsto H^{r-1,r-1}(X, \mathbb{R}) \otimes \mathbb{R}(r - 1)
\]

\[=: H^{r-1,r-1}(X, \mathbb{R}(r - 1)) \cong \left\{ H^{d-r+1,d-r+1}(X, \mathbb{R}(d - r + 1)) \right\}^\vee.
\]
§3 The higher Abel-Jacobi map – KLM formula

Bloch (and Beilinson via $K$-theory) constructed cycle class maps

$$cl_{r,m} : CH^r(X, m) \to H_{D}^{2r-m}(X, \mathbb{Z}(r)).$$

Recall that Deligne cohomology sits in an exact sequence:

$$0 \to \frac{H^{2r-m-1}(X, \mathbb{C})}{Fr H^{2r-m-1}(X, \mathbb{C}) + H^{2r-m-1}(X, \mathbb{Z}(r))} \to H_{D}^{2r-m}(X, \mathbb{Z}(r))$$

$$\to Fr H^{2r-m}(X, \mathbb{C}) \bigcap H^{2r-m}(X, \mathbb{Z}(r)) \to 0.$$

**Definition.** (i) $CH^r_{hom}(X, m)$ is the kernel of the composite map

$$CH^r(X, m) \to Fr H^{2r-m}(X, \mathbb{C}) \bigcap H^{2r-m}(X, \mathbb{Z}(r)).$$

(ii) From (i), we have an induced map

$$\Phi_{r,m} : CH^r_{hom}(X, m) \to \frac{H^{2r-m-1}(X, \mathbb{C})}{Fr H^{2r-m-1}(X, \mathbb{C}) + H^{2r-m-1}(X, \mathbb{Z}(r))},$$

called the Abel-Jacobi map.
We wish to give an explicit formula for $\Phi_{r,m}$. A first step is to apply Poincaré-Serre duality, viz.:

\[
\frac{H^{2r-m-1}(X, \mathbb{C})}{F^r H^{2r-m-1}(X, \mathbb{C}) + H^{2r-m-1}(X, \mathbb{Z}(r))} \cong \frac{F^{d-r+1} H^{2d+m-2r+1}(X, \mathbb{C})^\vee}{H_{2d+m-2r+1}(X, \mathbb{Z}(d-r))}.
\]

Thus in particular, the Abel-Jacobi map is:

\[
\Phi_{r,m} : \text{CH}^r_{\text{hom}}(X, m) \to \frac{F^{d-r+1} H^{2d+m-2r+1}(X, \mathbb{C})^\vee}{H_{2d+m-2r+1}(X, \mathbb{Z}(d-r))}.
\]
A meromorphic current

We wish to evaluate, for say smooth projective $Z$, and morphisms $f_1, \ldots, f_m \in \mathbb{C}(Z)^\times$ in general position, and form $\omega \in E^2_Z \text{dim } Z - m$, the “meromorphic current”:

$$\Omega(\omega) := \int_Z \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m} \wedge \omega.$$

We will assume the principal branch of the log function for our calculations below.

$$\int_Z \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m} \wedge \omega = \int_Z d\log f_1 \wedge \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_m}{f_m} \wedge \omega$$

$$= \int_Z d\left( \log f_1 \wedge \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_m}{f_m} \wedge \omega \right)$$

$$+ (-1)^m \int_{Z \setminus f_1^{-1}[-\infty, 0]} \log f_1 \wedge \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_m}{f_m} \wedge d\omega$$

$$= 2\pi i \int_{f_1^{-1}[-\infty, 0]} \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_m}{f_m} \wedge \omega$$

$$- 2\pi i \sum_{j=2}^{m} (-1)^j \int_{(f_j)} \log f_1 \wedge \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_j}{f_j} \wedge \cdots \wedge \frac{df_m}{f_m} \wedge \omega$$

$$+ (-1)^m \int_{Z \setminus f_1^{-1}[-\infty, 0]} \log f_1 \wedge \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_m}{f_m} \wedge d\omega.$$
In general, we put
\[
\mathbf{F} = (f_1, \ldots, f_m),
\]
\[
\Omega_{\mathbf{F}} = \bigwedge_1^m d \log f_j,
\]
\[
T_{\mathbf{F}} = (2\pi i)^m \int_{(f_1 \times \cdots \times f_m)^{-1}[-\infty,0]^m} (?,)
\]
\[
R_{\mathbf{F}} = \left[ \int_{Z \setminus f_1^{-1}[-\infty,0]} \log f_1 \wedge \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_m}{f_m} \wedge (?) \right.
\]
\[
-2\pi i \int_{f_1^{-1}[-\infty,0] \setminus (f_1 \times f_2)^{-1}[-\infty,0]^2} \log f_2 \wedge \frac{df_3}{f_3} \wedge \cdots \wedge \frac{df_m}{f_m} \wedge (?)
\]
\[
+ \cdots +
\]
\[
(-2\pi i)^{m-1} \int_{(f_1 \times \cdots \times f_{m-1})^{-1}[-\infty,0]^{m-1} \setminus (f_1 \times \cdots \times f_m)^{-1}[-\infty,0]^m} \log f_m \wedge (?) \right],
\]
and
\[
R_{\partial \mathbf{F}} = \sum_{j=1}^m (-1)^{j-1} R_{\{f_1, \ldots, \widehat{f_j}, \ldots, f_m\}} (f_j).
\]

Thus we have as currents:
\[
\Omega_{\mathbf{F}} = T_{\mathbf{F}} - d R_{\mathbf{F}} \pm 2\pi i R_{\partial \mathbf{F}}.
\]
We are now ready to cook up a formula on the level of cycles class groups and complexes: For $Z \subset X$ a subvariety of codimension $r - m$, the graph of $(f_1, \ldots, f_m)$ in $Z \times \square^m$ determines an algebraic cycle of codimension $r$ in $Z \times \square^m$, i.e. in $z^r(X, m)$ for $\{f_1, \ldots, f_m\}$ in general position. Quite generally, we consider a cycle $W \in z^r(X \times \square^m)$ in general position. One considers the projections $\pi_1 : W \to X$, $\pi_2 : W \to \square^m$. One has corresponding

$$\Omega_{\square} = \bigwedge_{1}^{m} d \log z_j$$

$$R_{\square} = \left[ \int_{Z \setminus z^{-1}_{1}[-\infty,0]} \log z_1 \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_m}{z_m} \wedge (?) \right]$$

$$-2\pi i \int_{z^{-1}_{1}[-\infty,0] \setminus (z_1 \times z_2)^{-1}[-\infty,0]^2} \log z_2 \wedge \frac{dz_3}{z_3} \wedge \cdots \wedge \frac{dz_m}{z_m} \wedge (?)$$

$$+ \cdots +$$

$$(-2\pi i)^{m-1} \int_{(z_1 \times \cdots \times z_{m-1})^{-1}[-\infty,0]^{m-1} \setminus (z_1 \times \cdots \times z_{m})^{-1}[-\infty,0]^{m}} \log z_m \wedge (?)$$

$$T_{\square} = (2\pi i)^{m} \int_{[-\infty,0]^{m}} (?) .$$
We put

\[ R_W = \pi_{1,*} \circ \pi_2^* R_\square, \quad \Omega_W = \pi_{1,*} \circ \pi_2^* \Omega_\square, \quad T_W = \pi_{1,*} \circ \pi_2^* T_\square. \]

Note that in the Deligne homology complex, the differential $\delta$ is given by:

\[
\delta \left( (2\pi i)^{-\dim W} (T_W, \Omega_W, R_W) \right)
= (2\pi i)^{-\dim W} (dT_W, d\Omega_W, T_W - \Omega_W - dR_W).
\]

Note that $\dim W = m + d - r$ where $d = \dim X$. Put:

\[
\mathcal{M}_D^\bullet = \text{Cone}\{ C^\bullet(X, \mathbb{Z}(r)) \oplus F^r \mathcal{D}_{X,\infty}^\bullet(X) \to \mathcal{D}_{X,\infty}^\bullet(X) \}[-1].
\]

The homology of this complex, at $\bullet = 2r - m$ is precisely the Deligne cohomology

\[ H_D^{2r-m}(X, \mathbb{Z}(r)). \]

There is a morphism of complexes

\[ z^r(X, \bullet) \to \mathcal{M}_D^\bullet \]

is induced by $W \mapsto (2\pi i)^{-\dim W} (T_W, \Omega_W, R_W)$. 
On the nullhomologous cycle level, with \( \{ W \} \in \text{CH}_{\text{hom}}^r(X, m) \) now a cycle with \( R_{\partial W} = 0 \), one has \( \gamma := \{ \pi_2^{-1}[-\infty, 0]^m \} \cap W = \partial \zeta \), and the AJ map is given by \( R_W + (-2\pi i)^m \int_{\zeta} \). More explicitly, using the cubical complex, the formula for the AJ map

\[
\Phi_{r,m} : \text{CH}_{\text{hom}}^k(X, m) \to J^{r,m}(X) := \frac{\{ F^{d-r+1} H^{2d-2r+m+1}(X, \mathbb{C}) \} \vee \{ F^{d-r+1} H^{2d-2r+m+1}(X, \mathbb{Z}(d-r)) \}}{H_{2d-2r+m+1}(X, \mathbb{Z}(d-r))},
\]

we have

\[
\frac{1}{(2\pi i)^{d-r+m}} \left[ \int_{\pi_2^*((\log z_1)d \log z_2 \wedge \cdots \wedge d \log z_m) \wedge \pi_1^*(\omega)}_{W \cap \pi_2^{-1}([-\infty, 0] \times \mathbb{C}^{m-1})} \right.
\]

\[
- (2\pi i) \int_{\pi_2^*((\log z_2)d \log z_3 \wedge \cdots \wedge d \log z_m) \wedge \pi_1^*(\omega)}_{\{ W \cap \pi_2^{-1}([-\infty, 0] \times \mathbb{C}^{m-1}) \} \setminus \{ W \cap \pi_2^{-1}([-\infty, 0]^2 \times \mathbb{C}^{m-2}) \}} \]

\[
+ \cdots + (-2\pi i)^{m-1} \int_{\pi_2^*(\log z_m) \wedge \pi_1^*(\omega)}_{\{ W \cap \pi_2^{-1}([-\infty, 0]^m \times \mathbb{C}^1) \} \setminus \{ W \cap \pi_2^{-1}([-\infty, 0]^m) \}} \left. \right]
\]

\[+ \left\{ (-2\pi i)^m \int_{\zeta} \pi_1^*(\omega) \right\}, \]

where the latter term is a membrane integral.
Examples

Recall
\[ \Phi_{r,m} : \text{CH}^r_{\text{hom}}(X,m) \to J^{r,m}(X) := \left\{ \frac{F^{n-r+1}H^{2d-2r+m+1}(X,\mathbb{C})}{\text{H}_{2d-2r+m+1}(X,\mathbb{Z}(d-r))} \right\}^\vee, \]

Example. \((m = 0)\). \(X = \{ y^2 = h(x) \} \subset \mathbb{P}^2\), where \(h(x)\) is a cubic polynomial with distinct roots. \(X\) is an elliptic curve. \(\text{CH}^1_{\text{hom}}(X,0) = \text{CH}_0(X)_{\text{deg}0}\) are the degree zero, 0-cycles. Then \((d,r,m) = (1,1,0)\) and
\[ J^{1,0}(X) = \frac{H^{1,0}(X)^\vee}{H_1(X,\mathbb{Z})} \cong \frac{\mathbb{C}}{\mathbb{Z}^2}. \]

Let \(\xi \in \text{CH}^1_{\text{hom}}(X,0)\) which we can write in the form \(\xi = \sum_j (p_j - q_j)\). Consider any real 1-chain \(\zeta\) on \(X\) such that \(\partial \zeta = \xi\). Note that \(H^{1,0}(X) = \mathbb{C}\omega\), where \(\omega = dx/y = dx/\sqrt{h(x)}\). Then
\[ \Phi_{1,0}(\xi)(\omega) = \int_\zeta \frac{dx}{\sqrt{h(x)}} = \sum_j \int_{q_j}^{p_j} \frac{dx}{\sqrt{h(x)}}, \]

is the classical elliptic integral.
Example. \((m = 1)\). Let \(X\) be a surface. Classes in \(\text{CH}^2(X, 1)\) are represented by

\[
\left\{ \xi = \sum_j (f_j, D_j) \mid \sum_j \text{div}(f_j) = 0 \right\},
\]

where the \(D_j\)'s \(\subset X\) are curves.

The map

\[
\Phi_{2,1} : \text{CH}^2_{\text{hom}}(X, 1) \to H^3_D(X, \mathbb{Z}(2)),
\]

is (here \((n, k, m) = (2, 2, 1)\))

\[
\Phi_{2,1} : \text{CH}^2_{\text{hom}}(X, 1) \to \frac{H^{2,0}(X) \oplus H^{1,1}(X)^\vee}{H_2(X, \mathbb{Z})},
\]

and is defined as follows. Assume given a higher Chow cycle \(\xi = \sum_j (f_j, D_j)\) representing a class in \(\text{CH}^2_{\text{hom}}(X, 1)\). Then via a proper modification, we can view \(f_j : D_i \to \mathbb{P}^1\), and consider the 1-chain \(\gamma_i = f_i^{-1}([0, \infty])\). Then \(\sum_j \text{div}(f_j) = 0\) implies that \(\gamma := \sum_j \gamma_j\) defines a 1-cycle. Since \(\xi\) is null-homologous, it is easy to show that \(\gamma\) bounds some real dimensional 2-chain \(\zeta\) in \(X\), viz., \(\partial \zeta = \gamma\).

For \(\omega \in H^{2,0}(X) \oplus H^{1,1}(X)\), the current defining \(\Phi_{k,1}(\xi)\) is given by ([Levine]):

\[
\Phi_{2,1}(\xi)(\omega) = \frac{1}{(2\pi i)} \left[ \sum_j \int_{D_j \setminus \gamma_j} \omega \log f_j + 2\pi i \int_{\zeta} \omega \right],
\]

where we choose a branch of the log function defined on \(\mathbb{C} \setminus [0, \infty)\).
**Real Regulator.** For $\mathbb{A} = \mathbb{R}$, we relabel $\Phi_{r,m} = r_{r,m}$. Using the description of real Deligne cohomology and the regulator formula above, we arrive at the formula for the real regulator for our surface $X$.

$$r_{2,1} : \text{CH}^2(X, 1) \rightarrow H^3_D(X, \mathbb{R}(2)) \simeq H^{1,1}(X, \mathbb{R}(1))^\vee.$$  

Namely:

$$r_{2,1}(\xi)(\omega) = \frac{1}{2\pi i} \sum_j \int_{D_j} \omega \log |f_j|.$$  

**Example.** ($m = 2$). In this case we give an explicit construction of the real regulator

$$r_{2,2} : \text{CH}^2_{\text{hom}}(X, 2) \rightarrow H^2_D(X, \mathbb{R}(k)),$$

in the case where $X$ is a compact Riemann surface. Now assume given a class $\{\xi\} \in \text{CH}^2_{\text{hom}}(X, 2)$:

$$\xi := \prod_{\alpha} \{f_\alpha, g_\alpha\}, X \xrightarrow{T} 0.$$  

[Here:

$$T(\xi) = \sum_{\alpha, p \in X} \left( (-1)^{\nu_p(f_\alpha)\nu_p(g_\alpha)} \frac{f_\alpha^{\nu_p(f_\alpha)}}{g_\alpha^{\nu_p(f_\alpha)}} \cdot p \right).$$]
In our case this amounts to a map:

\[ r : \text{CH}^2(X, 2) \rightarrow H^1(X, \mathbb{R}(1)) \simeq H^1(X, \mathbb{R})^\vee, \]

induced by

\[ \omega \in E_{X, \mathbb{R}}^1 \mapsto \frac{1}{2\pi} \int_X \left[ \log |f| \, d \text{arg} \, g - \log |g| \, d \text{arg} \, f \right] \wedge \omega. \]

“Mama’s Formula”

This formula has the following homological version. Fix \( p \in X \), and consider any loop \( \gamma \) in \( X \setminus \{(f) \cup (g)\} \) based at \( p \). Then via Poincaré duality \( H_1(X, \mathbb{R}) \simeq H^1(X, \mathbb{R}) \),

\[ \gamma \mapsto \frac{1}{2\pi} \, \text{Im} \left( \int_{\gamma} \log f \, \frac{dg}{g} - \log |g(p)| \int_{\gamma} \frac{df}{f} \right). \]
Example. Let $X$ be a point $\text{Pt}$. Then in this case

$$\text{CH}^r(\text{Pt}, 2r - 1) = \text{CH}_{\text{hom}}^r(\text{Pt}, 2r - 1)$$

and one has

$$\Phi_{r,2r-1} : \text{CH}^r(\text{Pt}, 2r - 1) \to H^1_D(\text{Pt}, \mathbb{Z}(r)),$$

moreover

$$H^1_D(\text{Pt}, \mathbb{Z}(r)) \simeq H^0(\text{Pt}, \mathbb{C}/\mathbb{Z}(r)) \simeq \mathbb{C}/\mathbb{Z}(r).$$

The map $\Phi_{r,2r-1}$ is given by

$$W \mapsto \frac{1}{(2\pi i)^{r-1}} \int_W R_W \in \mathbb{C}/\mathbb{Z}(r).$$
We illustrate a couple scenarios of the previous example in the case \( r = 2 \).

(I) [Lewis] Let \( a, b \in \mathbb{C}^\times \setminus \{1\} \), and put:

\[
V(a) = \left\{ \left( \frac{1 - a}{t}, 1 - t, t \right) \mid t \in \mathbb{P}^1 \right\} \bigcap \square^3,
\]

\[
W(a) = \left\{ \left( \frac{1 - b}{t}, t, 1 - t \right) \mid t \in \mathbb{P}^1 \right\} \bigcap \square^3,
\]

and note that

\[
\partial V(a) = (1 - a, a), \quad \partial W(b) = (b, 1 - b).
\]

Therefore

\[
\xi_a := V(a) - W(1 - a) \in \text{CH}^2(\text{Pt}, 3).
\]

The value \( \Phi_{2,3}(\xi_a) \in \mathbb{C}/\mathbb{Z}(2) \) is easy to compute:

\[
\Phi_{2,3}(\xi_a) = \text{Li}_2(a) + \text{Li}_2(1 - a) + \log a \log(1 - a),
\]

where \( \text{Li}_2 \) is the dilogarithm, and \( \log \) has the principal branch. By Beilinson’s rigidity

\[
\Phi_{2,3}(\xi_a) = \lim_{a \to 0} \Phi_{2,3}(\xi_a) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \in \mathbb{C}/\mathbb{Z}(2),
\]

which is a torsion class.
(II) [Kerr] Let $B_2(C)$ be the Bloch group, and let $st : B_2(C) \to C^\times \wedge_\Z C^\times$ be the standard map $\{a\}_2 \mapsto (1 - a) \wedge a$. Set $\rho(a) = alt_3(V(a))$. [Here $alt = 1/m! \sum \text{sgn}(\sigma)\sigma : z^k(Pt, m) \otimes \Q \to z^k(Pt, m) \otimes \Q$.] Given any element $\sum_j m_j \{a_j\}_2 \in \ker st$, one can complete $\sum_j m_j \rho(a_j)$ to a class $\xi \in CH^2(Pt, 3)$ by adding “decomposable” elements. One computes

$$r_{2,3}(\xi) = \sum_j m_j D_2(a_j) \in \R,$$

where $D_2$ is the Bloch-Wigner function. There are many examples where this is nonzero, hence $\Phi_{2,3}(\xi) \in C/\Z(2)$ is nontorsion in these cases.
STATEMENT OF THE PROBLEM

Let $\mathbb{A} \subset \mathbb{R}$ be a subring such that $\mathbb{A} \otimes \mathbb{Q}$ is a field. $X$ smooth projective, $\dim X = d$, and $Y \subset X$ a NCD. The KLM (= Kerr - Lewis - Müller-Stach) formula gives us an explicit description of a map constructed earlier by Bloch:

$$\text{CH}^{r}(X \setminus Y, m) \to H^{2r-m}_{\mathcal{D}}(X \setminus Y, \mathbb{A}(r)),$$

hence a corresponding map

$$\text{CH}^{r}(X, m) \to H^{2r-m}_{\mathcal{H}}(X, \mathbb{A}(r)) = H^{2r-m}_{\mathcal{D}}(X, \mathbb{A}(r)).$$

We provide an explicit description of the vertical maps in the diagram below, by first defining these maps on the level of complexes.

$$\text{CH}^{r}_{Y}(X, m)$$

$$\|$$

$$\text{CH}^{r-1}(Y, m) \to \text{CH}^{r}(X, m) \to \text{CH}^{r}(X \setminus Y, m)$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$H^{2r-m}_{\mathcal{H},Y}(X, \mathbb{A}(r)) \to H^{2r-m}_{\mathcal{H}}(X, \mathbb{A}(r)) \to H^{2r-m}_{\mathcal{H}}(X \setminus Y, \mathbb{A}(r))),$$

where $H^\bullet_{\mathcal{H}}$ is absolute Hodge cohomology.

One of our applications of this is a conjectural description of a Bloch-Beilinson filtration via kernels of variational regulators, whose formulas can be described explicitly in terms of membrane integrals.
Higher Chow groups revisited

We recall the following:

Let $T/\mathbb{C}$ be a quasiprojective variety. Put $Z^r(T) = \text{free abelian group generated by subvarieties of codimension } r \text{ in } T$. Consider the $m$-simplex:

$$\Delta^m = \text{Spec} \left\{ \frac{\mathbb{C}[t_0, \ldots, t_m]}{(1 - \sum_{j=0}^m t_j)} \right\} \simeq \mathbb{C}^m.$$

We set

$$Z^r(T, m) := \left\{ \xi \in Z^r(T \times \Delta^m) \mid \text{every component of } \xi \text{ meets all faces} \right\}$$

$$\{ t_{i_1} = \ldots = t_{i_\ell} = 0, \; \ell \geq 1 \}$$

properly

Note that $Z^r(T, 0) = Z^r(T)$. Now set $\partial_j : Z^r(T, m) \to Z^r(T, m - 1)$ to be the restriction to the $j$-th face (given by $t_j = 0$. The boundary map

$$\partial = \sum_{j=0}^m (-1)^j \partial_j : Z^r(T, m) \to Z^r(T, m - 1),$$

satisfies $\partial^2 = 0$.

**Definition.** [Bloch] $\text{CH}^k(T, \bullet) = \text{homology of } \{ Z^r(T, \bullet) \}.$

We put $\text{CH}^r(T) := \text{CH}^r(T, 0)$. 
Recall that $Y = Y_1 \cup \cdots \cup Y_N$ is a NCD in a smooth projective variety $X$, and where we assume that each $Y_i$ is smooth. For an integer $t \geq 0$, put $Y^{[t]} = \text{disjoint union of } t\text{-fold intersections of the various components of } Y$, with corresponding coskeleton $Y^{[\bullet]}$. We also put $Y^{(t)}$ to be the union of $t\text{-fold intersections of the various components of } Y$, where we observe that $Y^{[t]}$ is the ‘canonical’ desingularization of $Y^{(t)}$. Here we put $Y^{[0]} = X$, $Y^{[1]} = \bigsqcup_1^N Y_j$, and so on. (Note that $Y^{(0)} = X$, $Y^{(1)} = Y$, and so on.) One has a proper hypercovering $Y^{[\bullet]} \to Y$, viz.,

\[(*) \quad \to \quad \cdots \to Y^{[2]} \to Y^{[1]} \to Y,\]

where descent arguments enables on to compute homology of $Y$ in terms of $Y^{[\bullet]}$. One can think of the arrows as defining (alternating) Gysin maps on homology.
Consider the setting

\[ Y^\bullet \to Y \hookrightarrow X, \]

and a class \( \xi \in \text{CH}^r(X \setminus Y, m) \). Take the closure \( \overline{\xi} \), which we can assume (by a moving lemma of Bloch), that \( \overline{\xi} \in z^r(X, m) \). There is the possibility that

\[ \xi \in W_{-m} \text{CH}^r(X \setminus Y, m) := \text{Image}(\text{CH}^r(X, m) \to \text{CH}^r(X \setminus Y, m)). \]

In general, \( \partial \overline{\xi} \in z_Y^r(X, m - 1) := z^{r-1}(Y, m - 1) \), and we get a residue \( \tilde{\partial}_R^1(\xi) \in \text{CH}^{r-1}(Y^{(1)} \setminus Y^{(2)}, m - 1) \), which is an obstruction to \( \{\xi\} \) lying in \( W_{-m} \text{CH}^r(X \setminus Y, m) \). It is possible that

\[ \tilde{\partial}_R^1(\xi) \in \text{Image}(\text{CH}^{r-1}(Y^{[1]}, m - 1) \to \text{CH}^{r-1}(Y^{(1)} \setminus Y^{(2)}, m - 1)), \]

in which case we say \( \xi \in W_{-m+1} \text{CH}^r(X \setminus Y, m) \). One has a lifting \( \tilde{\partial} \overline{\xi} \in z^{r-1}(Y^{[1]}, m - 1) \), and again \( \partial (\tilde{\partial} \overline{\xi}) \) leads to a residue

\[ \tilde{\partial}_R^2(\xi) \in \text{CH}^{r-2}(Y^{(2)} \setminus Y^{(3)}, m - 2), \]

which is an obstruction to \( \xi \in W_{-m+1} \text{CH}^r(X \setminus Y, m) \).
Again if
$$\tilde{\partial}^r_{R}(\xi) \in \text{Image}(\text{CH}^{r-2}(Y^{[2]}, m - 2) \to \text{CH}^{r-2}(Y^{(2)} \setminus Y^{(3)}, m - 2)),$$
then we say $\xi \in W_{-m+2} \text{CH}^{r}X \setminus Y, m)$. This process ends at
$$\tilde{\partial}^{m}_{R}(\xi) \in \text{Image}(\text{CH}^{r-m}(Y^{[m]}, 0) \to \text{CH}^{r-m}(Y^{(m)} \setminus Y^{(m+1)}, 0)),$$
since there are no numerator conditions defining ordinary algebraic cycles. This really says that $\xi \in W_{0} \text{CH}^{r}(X \setminus Y, m) := \text{CH}^{r}(X \setminus Y, m)$.

Corresponding to this will be a sequence of Abel-Jacobi maps $AJ_{\ell}$, $\ell = 0, \ldots, m$, with the nonvanishing of $AJ_{\ell-1}$ providing an obstruction to defining $AJ_{\ell}$. 
We now give a spectral sequence interpretation to all of this. Corresponding to $Y^{[\bullet]} \to Y^{[0]} := X$ is a third quadrant double complex

\[
\mathcal{Z}^{i,j}_0 (r) := Z^{r+i}(Y^{[-i]}, -j), \ i, j \leq 0; \quad \mathcal{Z}^{i,j}_0 \xrightarrow{\partial} \mathcal{Z}^{i,j+1}_0, \quad \mathcal{Z}^{i,j}_0 \xrightarrow{Gy} \mathcal{Z}^{i,j+1}_0
\]

whose differentials are $\partial$ vertically and $Gy$ ( = “alternating” Gysin) horizontally. Corresponding to this double complex are associated first and second Grothendieck spectral sequences of the corresponding single complex $s^{\bullet} \mathcal{Z}(r)$ with $D = \partial \pm Gy$, which have $E_2$-terms:

\[\begin{align*}
'E_2^{p,q} & := H^p_{Gy}(H^q_{\partial}(s^{\bullet} \mathcal{Z}(r))) \\
''E_2^{p,q} & := H^p_{\partial}(H^q_{Gy}(s^{\bullet} \mathcal{Z}(r)))
\end{align*}\]

The second spectral sequence, together with Bloch’s quasiisomorphism

\[
\frac{Z^{\bullet}(X, *)}{Z_Y^{\bullet}(X, *)} \xrightarrow{\text{Restriction}} Z^{\bullet}(X \setminus Y, *),
\]

shows that

\[H^{-m}(s^{\bullet} \mathcal{Z}(r)) = ''E_2^{0,-m} = CH^r(X \setminus Y, m).\]

In particular we use the fact that $H^{\bullet < 0}_{Gy}(s^{\bullet} \mathcal{Z}(r)) = 0$, hence $''E_\infty^{p,q} = ''E_2^{p,q} = 0$ if $q < 0$. 
Thus with regard to $\text{CH}^r(X \setminus Y, m)$ we are really only considering the situation of $i \leq 0$, $j \leq 0$ with $-m = i + j$. The first spectral sequence of hypercohomology corresponds to a filtration on $\text{CH}^r(X \setminus Y, m)$ which we describe as follows. For $p \leq 0$, let

$$W^p s^* Z(r) = (s^* Z(r))^{(\bullet \geq p, \bullet)}$$

Then

$$W^p \text{CH}^r(X \setminus Y, m) := W^p H^{-m}(s^* Z(r))$$

$$:= \text{Image}(H^{-m}(W^p s^* Z(r)) \to H^{-m}(s^* Z(r)))$$

Thus:

$$Gr^p_W \text{CH}^r(X \setminus Y, m) = 'E^p_{\infty,-m-p}.$$ 

Now for $-m \leq \ell \leq 0$, put

$$W^\ell \text{CH}^r(X \setminus Y, m) := W^{-\ell-m} \text{CH}^r(X \setminus Y, m).$$
Then we have an increasing “weight” filtration

\[
\text{Image}(\text{CH}^r(X, m) \to \text{CH}^r(X \setminus Y, m)) = \text{CH}^r(X \setminus Y, m) \cap \cdots \subset W_0 \text{CH}^r(X \setminus Y, m) = \text{CH}^r(X \setminus Y, m).
\]

**Geometric interpretation.** Let \(\{\xi\} \in W_\ell \text{CH}^r(X \setminus Y, m).\) Then

\[
\tilde{\partial}^{\ell+m}_R(\xi) \in \text{Image}(\text{CH}^{r-\ell-m}(Y^{[\ell+m]}, -\ell) \to \text{CH}^{r-\ell-m}(Y^{(\ell+m)} \setminus Y^{(\ell+m+1)}, -\ell)).
\]

In general

\[
Gr^\ell_W \text{CH}^r(X \setminus Y, m) \rightarrow \begin{cases}
\text{A subquotient of} \\
\text{CH}^r(Y^{[\ell+m]}, -\ell) .
\end{cases}
\]

Notice that if we work with \(Y^{[t]}\) with \(t \geq 1\), then the same construction above leads to a spectral sequence abutting to \(\text{CH}^r_Y(X, m) := \text{CH}^{r-1}(Y, m)\) (where we think of Bloch’s higher Chow groups as defining Chow homology).
We summarize:

**Proposition.** (i) There is a third quadrant spectral sequence converging to $\text{CH}^r(X \setminus Y, m)$ with $E_1^{p,q} = \text{CH}^{p+r}(Y^{-p}, -q)$ (where $p + q = -m$), and $E_\infty^{0,-m} = \text{CH}^r(X \setminus Y, m)$. For the remaining $E_\infty$-terms, there are maps

$$\epsilon_{p,q} : E_\infty^{p,q} \to \text{CH}^{r+p}(Y^{(-p)} \setminus Y^{(-p+1)}, -q)$$

which are in general neither injective nor surjective.

(ii) The “weight” filtration given can be described as follows:

$$W_{\ell-1} \text{CH}^r(X \setminus Y, m) =$$

$$\ker \left[ W_\ell \text{CH}^r(X \setminus Y, m) \to \left\{ \begin{array}{c} \text{A subquotient of} \\ \text{CH}^r(Y^{[\ell+m]}, -\ell) \end{array} \right\} \right]$$

(iii) A similar story holds for $\text{CH}^r_Y(X, m)$. 
§4 Absolute Hodge cohomology

Absolute Hodge cohomology fits in an exact sequence:

$$0 \rightarrow \text{Ext}^1_{\mathbb{A} - \text{MHS}} \rightarrow \mathcal{H}^{2r-m}_{\mathcal{H}}(X \setminus Y, \mathbb{A}(r)) \rightarrow \text{Ext}^0_{\mathbb{A} - \text{MHS}} \rightarrow 0,$$

where

$$\text{Ext}^0_{\mathbb{A} - \text{MHS}} = \text{hom}_{\mathbb{A} - \text{MHS}} (\mathbb{A}(r), \mathcal{H}^{2r-m}(X \setminus Y, \mathbb{A}(r))),$$

and \(\text{Ext}^1_{\mathbb{A} - \text{MHS}} = \text{Cokernel}:

$$\left\{ \begin{array}{ccc}
\{ \mathcal{H}^{2r-m-1}(X \setminus Y, \mathbb{A}(r)) \\
\bigoplus \end{array} \bigoplus \right. \left. \begin{array}{ccc}
\mathcal{W}_0 \mathcal{H}^{2r-m-1}(X \setminus Y, \mathbb{A}(r)) \otimes \mathbb{Q} \\
F^0 \cap \mathcal{W}_0 \mathcal{H}^{2r-m-1}(X \setminus Y, \mathbb{C}) \\
\end{array} \bigoplus \right. \left. \begin{array}{ccc}
\mathcal{H}^{2r-m-1}(X \setminus Y, \mathbb{A}(r)) \otimes \mathbb{Q} \\
\mathcal{W}_0 \mathcal{H}^{2r-m-1}(X \setminus Y, \mathbb{C}) \\
\end{array} \right\}$$
Definition. A mixed $\mathbb{A}$-Hodge complex consists of the following datum:

- A bounded below complex $K^\bullet_{\mathbb{A}}$ of $\mathbb{A}$-modules (in the derived category), such that $H^p(K^\bullet_{\mathbb{A}})$ is an $\mathbb{A}$-module of finite type for all $p$.

- A bounded below filtered complex $(K^\bullet_{\mathbb{A} \otimes \mathbb{Q}}, W)$ of $\mathbb{A} \otimes \mathbb{Q}$-vector spaces, and an isomorphism $K^\bullet_{\mathbb{A} \otimes \mathbb{Q}} \xrightarrow{\sim} K^\bullet_{\mathbb{A}} \otimes \mathbb{Q}$ in the derived category.

- A bifiltered complex $(K^\bullet_{\mathbb{C}}, W, F)$ of $\mathbb{C}$-vector spaces, and a filtered isomorphism $\alpha : (K^\bullet_{\mathbb{C}}, W) \xrightarrow{\sim} (K^\bullet_{\mathbb{A} \otimes \mathbb{Q}}, W) \otimes \mathbb{C}$ in the filtered derived category.

Further,

- For every $m \in \mathbb{Z}$,

$$Gr^m_W K^\bullet_{\mathbb{A} \otimes \mathbb{Q}} \rightarrow (Gr^m_W K^\bullet_{\mathbb{C}}, F)$$

is a (polarizable) $\mathbb{A} \otimes \mathbb{Q}$-Hodge complex of weight $m$, i.e. the differentials of $Gr^m_W K^\bullet_{\mathbb{C}}$ are strictly compatible with the induced filtration $F$, and $F$ induces a pure (polarizable) $\mathbb{A} \otimes \mathbb{Q}$-Hodge structure of weight $m + r$ on $H^r(Gr^m_W K^\bullet_{\mathbb{A} \otimes \mathbb{Q}})$ for $r \in \mathbb{Z}$. 
A mixed $\mathbb{A}$-Hodge complex gives rise to a diagram:

\[
\begin{array}{ccc}
\begin{array}{c}
'K_{\mathbb{A} \otimes \mathbb{Q}}^\bullet \\
K_{\mathbb{A}}^\bullet
\end{array}
& 
\begin{array}{c}
('K_{C}^\bullet, W) \\
(K_{A \otimes \mathbb{Q}}^\bullet, W)
\end{array}
\end{array}
\begin{array}{c}
\xrightarrow{\alpha_1} \kappa \xleftarrow{\alpha_2} \\
\beta_1 \xrightarrow{\kappa} \beta_2
\end{array}
\begin{array}{c}
(K_{C}^\bullet, W, F),
\end{array}
\]

where $\alpha_j$, $\beta_j$, $j = 1, 2$, are morphisms of complexes, $\alpha_2$ being a quasiisomorphism, $\beta_1$ a filtered morphism, and $\beta_2$ a filtered quasiisomorphism. By the work of Deligne and Beilinson, the construction of mixed $\mathbb{A}$-Hodge complexes is equivalent to the construction of mixed $\mathbb{A}$-Hodge structures.

The absolute Hodge cohomology $H^\ell_{\mathcal{H}}(K^\bullet)$ (of a mixed $\mathbb{A}$-Hodge complex $K^\bullet$) is given by

\[
H^\ell(\text{Cone}\{K_A^\bullet \oplus \widehat{W}_0K_{\mathbb{A} \otimes \mathbb{Q}}^\bullet \oplus \widehat{W}_0 \cap F^0K_C^\bullet \xrightarrow{(\alpha, \beta)} 'K_{\mathbb{A} \otimes \mathbb{Q}}^\bullet \oplus \widehat{W}_0'K_C^\bullet \}[-1]),
\]

where $\widehat{W}_\bullet := (\text{Dec} W)_\bullet$ is the filtration décalée, and

\[
(\alpha, \beta)(\xi_A, \xi_Q, \xi_C) = (\alpha_1 \xi_A - \alpha_2 \xi_Q, \beta_1 \xi_Q - \beta_2 \xi_C).
\]
Main Example. Recall $Y = Y_1 \cup \cdots \cup Y_N$ a NCD in a smooth projective variety $X$, $\dim X = d$, and where for an integer $t \geq 0$, we put $Y^{[t]}$ = disjoint union of $t$-fold intersections of the various components of $Y$, with corresponding coskeleton $Y^{[\bullet]}$. [Here we recall that $Y^{[0]} = X$.] Now put (for $i \leq 0$):

\[ D(r)^{i,j} := D^{2r+2i+j}(Y^{[-i]}) \]
\[ C(r)^{i,j} = C^{2r+2i+j}(Y^{[-i]}, \mathbb{A}(r+i)) \]

double complexes with differentials $G_y$ and $d$ (and $D = d \pm G_y$). Let

\[ s^\bullet D(r) := \bigoplus_i D(r)^{i,\bullet-i} = \bigoplus_i D^{2r+i+\bullet}(Y^{[-i]}), \]
\[ s^\bullet C(r) := \bigoplus_i C(r)^{i,\bullet-i} = \bigoplus_i C^{2r+i+\bullet}(Y^{[-i]}, \mathbb{A}(r+i)), \]

$(i + j = \bullet)$ represent the associated single complexes with differential $D$. 
Consider the *decreasing* “weight” filtration given by

\[ 'W^\ell (\mathbf{s} \cdot \mathcal{D}(r)) := \bigoplus_{i \geq \ell} \mathcal{D}(r)^{i, \bullet - i} \]

\[ 'W^\ell (\mathbf{s} \cdot \mathcal{C}(r)) := \bigoplus_{i \geq \ell} \mathcal{C}(r)^{i, \bullet - i} \]

We now consider \( \mathcal{D}(r)^{i,j} \), \( \mathcal{C}(r)^{i,j} \) as above with \( i \leq 0 \), with corresponding \( \mathbf{s} \cdot \mathcal{D}(r) := \bigoplus_i \mathcal{D}(r)^{i, \bullet - i} \), \( \mathbf{s} \cdot \mathcal{C}(r) := \bigoplus_i \mathcal{C}(r)^{i, \bullet - i} \).

\[
\mathbb{A}^\bullet_H (r) := \text{Cone}\{ \mathbf{s} \cdot \mathcal{C}(r) \bigoplus \widehat{W}_0 \mathbf{s} \cdot \mathcal{C}(r) \otimes \mathbb{Q} \bigoplus \widehat{W}_0 F^0 \mathbf{s} \cdot \mathcal{D}(r) \\
\rightarrow \mathbf{s} \cdot \mathcal{C}(r) \otimes \mathbb{Q} \bigoplus \widehat{W}_0 \mathbf{s} \cdot \mathcal{D}(r) \}\left[-1\right]
\]

where

\[
\widehat{W}_0 \mathbf{s}^s = \ker \left( D : 'W^s \mathbf{s}^s (r) \rightarrow \frac{s^{s+1}(r)}{W^{s+1} \mathbf{s}^s (r)} \right),
\]

and accordingly,

\[ H^\ell (\mathbb{A}^\bullet_H (r)) = H^{2r+\ell}_H (X \setminus Y, \mathbb{A}(r)). \]
We want to express $\mathbb{A}_H^\bullet(r)$ as the single complex $s^\bullet H(r)$ associated to a double complex. For simplicity, we do this in the case $\mathbb{A} = \mathbb{Q}$ (a similar story works for general $\mathbb{A}$).

Towards this goal, we correspondingly make use of the decreasing “weight” filtrations:

\[ 'W^\ell (s^\bullet D(r)) := \bigoplus_{i \geq \ell} D(r)^i \cdot_{i} \]
\[ 'W^\ell (s^\bullet C(r)) := \bigoplus_{i \geq \ell} C(r)^i \cdot_{i} \]

The dictionary between the decreasing “weight” filtration $'W^\bullet$ and the corresponding increasing filtration $W_\bullet$ on cohomology is given by:

\[ 'W^\ell H^{-p} (s^\bullet C(r)) := \text{Image}\{ H^{-p} (W^\ell C(r)) \to H^{-p} (s^\bullet C(r)) \} \]
\[ =: W_{-p-\ell} H^{2r-p} (X \setminus Y, \mathbb{Q}(r)) \]
The associated double complex is:

\[
\mathcal{H}^{i,j}_0(r) := \begin{cases} 
0 & \text{if } j > 1 \\
\ker d \subset D^{2r+i}(Y^{[-i]}) & \text{if } j = 1 \\
\ker d \subset C^{2r+i}(Y^{[-i]}, \mathbb{Q}(r+i)) \bigoplus \ker d \subset F^{0}D^{2r+i}(Y^{[-i]}) \bigoplus D^{2r+2i-1}(Y^{[-i]}) \quad & \text{if } j = 0 \\
C^{2r+2i+j}_D(Y^{[-i]}, \mathbb{Q}(r+i)) & \text{if } j < 0 
\end{cases}
\]

Recall that \( \mathcal{Z}^{i,j}_0(r) := Z^{r+i}(Y^{[-i]}, -j) \). The KLM map

\[
\mathcal{Z}^{i,j}_0(r) \to \mathcal{H}^{i,j}_0(r),
\]

induces a morphism of double complexes. At the infinity level, and via the quasiisomorphism \( C^{\bullet}_H(Y^{[-i]}, \mathbb{Q}(r)) \to C^{\bullet}_D(Y^{[-i]}, \mathbb{Q}(r)) \) (as \( Y^{[-i]} \) is smooth projective), we have

\[
\mathcal{H}^{i,j}(r) \simeq Gr^{-i}_W H^{i+j}(s^{\bullet}_H(r)) = Gr^{i}_W H^{2r+i+j}_H(X \setminus Y, \mathbb{Q}(r)).
\]

In particular, if we put \( i + j = -m \), then \( -m \leq i \leq 0 \), and the RHS of the above becomes (with \( \ell = -i - m \)):

\[
\mathcal{H}^{-\ell,-m,\ell}(r) \simeq Gr^\ell_W H^{2r-m}_H(X \setminus Y, \mathbb{Q}(r)), \quad \ell = -m, \ldots, 0.
\]
Conjecture. [Generalization of Bloch-Beilinson] Suppose that $X$, $Y$ are defined over $\overline{\mathbb{Q}}$ and that $\mathbb{A} = \mathbb{Q}$. Then the induced map

$$Gr^\ell_W CH^r(X \setminus Y, m) \otimes \mathbb{Q} \hookrightarrow Gr^\ell_W H^{2r-m}_H(X \setminus Y, \mathbb{Q}(r)),$$

is an injection.

Remark. The same construction enables one to arrive at a similar description of the map $CH^r_Y(X) \to H^{2r-m}_{H,Y}(X, \mathbb{A}(r))$. 
Before working on an example, we first compute target graded pieces of absolute Hodge cohomology. Consider the s.e.s.:

$$0 \to W_{\ell-1} \to W_0 \to Gr^0_W \to 0$$

applied to $H^{2r-m-1}(X \setminus Y, \mathbb{Q}(r))$

Put

$$\Xi_\ell := \text{Image} \left( \begin{array}{c}
\text{hom}_{\text{MHS}}(\mathbb{Q}(0), Gr^0_W H^{2r-m-1}(X \setminus Y, \mathbb{Q}(r))) \\
\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), Gr^\ell_W H^{2r-m-1}(X \setminus Y, \mathbb{Q}(r)))
\end{array} \right).$$

**Proposition.**

$$\mathcal{H}^{-m-\ell, \ell}(r) \cong \frac{\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), Gr^\ell_W H^{2r-m-1}(X \setminus Y, \mathbb{Q}(r)))}{\Xi_\ell},$$

for $-m \leq \ell < 0$, and that for $\ell = 0$, there is a s.e.s.:

$$0 \to \frac{\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), Gr^{-1}_W H^{2r-m-1}(X \setminus Y, \mathbb{Q}(r)))}{\Xi_0}$$

$$\to \mathcal{H}^{-m, 0}(r) \to \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^{2r-m}(X \setminus Y, \mathbb{Q}(r))) \to 0.$$
The Abel-Jacobi map

Let us define

\[ \text{CH}^r_{\text{hom}}(X \setminus Y, m) := \ker \{ \text{CH}^r(X \setminus Y, m) \rightarrow \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^{2r-m}(X \setminus Y, \mathbb{Q}(r))) \}, \]

with induced Abel-Jacobi map

\[ AJ_{r,m} : \text{CH}^r_{\text{hom}}(X \setminus Y, m) \rightarrow \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{2r-m-1}(X \setminus Y, \mathbb{Q}(r))) . \]

Thus we are really looking at \( Gr^\ell AJ : \]

\[ Gr^\ell_W \text{CH}^r_{\text{hom}}(X \setminus Y, m) \otimes \mathbb{Q} \]

\[ \downarrow \]

\[ \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), Gr^{\ell-1}_W H^{2r-m-1}(X \setminus Y, \mathbb{Q}(r))) \]

\[ \Xi_\ell \]

for \(-m \leq \ell \leq 0\). This is evaluated using the KLM formula for smooth projective varieties.
Example. We will work out the case of the map $\text{CH}^r(X \setminus Y, 1) \to H^{2r-1}_\mathcal{H}(X \setminus Y, \mathbb{Q}(r))$, where $Y = Y_1 \cup Y_2$. In this case

$$Y^{[1]} = Y_1 \coprod Y_2$$

$$Y^{[2]} = Y^{(2)} = Y_1 \cap Y_2$$

Regardless of the NCD $Y$, the weight filtration on $\text{CH}^r(X \setminus Y, 1)$ is 2-step, viz.,

$$\text{CH}^r(X \setminus Y, 1) = W_0 \text{CH}^r(X \setminus Y, 1) \supset W_{-1} \text{CH}^r(X, 1) = G_{r-1}^r(X \setminus Y, m) =: \text{CH}^r(X \setminus Y, m) = \mathcal{Z}_{\infty}^{0,-1}(r).$$

We calculate the highest weight situation first. In this case we have $E^{-1,0}_\infty = E^{-1,0}_2 =: \mathcal{Z}_{\infty}^{-1,0}(r) = G_{rW}^0 \text{CH}^r(X \setminus Y, m) = \frac{\ker G_Y : \text{CH}^{r-1}(Y^{[1]}) \to \text{CH}^r(X)}{G_Y(\text{CH}^{r-2}(Y^{[2]}))}. $
We have a diagram

\[
\begin{array}{c}
0 \\
\uparrow \\
\hom_{\text{MHS}}(\mathbb{Q}(0), H^{2r-1}(X \setminus Y, \mathbb{Q}(r))) \\
\uparrow \\
\mathcal{Z}_{\infty}^{-1,0}(r) \rightarrow \mathcal{H}_{\infty}^{-1,0}(r) \\
\uparrow \\
\text{AJ} \\
\uparrow \\
\text{Ext}_{\text{MHS}}^{1}(\mathbb{Q}(0), G_{W}^{-1,0} H^{2r-2}(X \setminus Y, \mathbb{Q}(r))) \\
\uparrow \\
0
\end{array}
\]

where for \( \xi \in \text{CH}^{r}(X \setminus Y, 1) \), the map \( \text{AJ}(\xi) \) is only defined if \( \xi \in \text{CH}_{\text{hom}}^{r}(X \setminus Y, 1) \), i.e. \( [\xi] = 0 \).
We ‘calculate’ the morphism $Z_{\infty}^{-1,0} \to H_{\infty}^{-1,0}(r)$. First of all, working with an exact sequence

$$0 \to W_{-1} \to W_0 \to Gr^0_W \to 0,$$

we have $\text{hom}_{\text{MHS}}(\mathbb{Q}(0), W_{-1}) = 0$, hence

$$\text{hom}_{\text{MHS}}(\mathbb{Q}(0), W_{-1}) \hookrightarrow \text{hom}_{\text{MHS}}(\mathbb{Q}(0), Gr^0_W) \hookrightarrow Gr^0_W.$$

An explicit computation gives

$$Gr^0_W H^{2r-1}(X \setminus Y, \mathbb{Q}(r)) \simeq$$

$$\frac{\ker \text{Gy} : H^{2r-2}(Y^{[1]}, \mathbb{Q}(r-1)) \to H^{2r}(X, \mathbb{Z}(r))}{\text{Gy}(H^{2r-4}(Y^{[2]}, \mathbb{Q}(r-2)))}$$

Let $\xi \in Z^r(X \setminus Y, 1)$ represent a class $\{\xi\} \in \text{CH}^r(X \setminus Y, 1)$, with closure $\overline{\xi} \in Z^r(X, 1)$. Then $\xi$ being a cycle on $X \setminus Y$ implies that $\partial \overline{\xi}$ is supported on $Y$. One has a lift $\widetilde{\partial \overline{\xi}}$ on $Y^{[1]}$, with fundamental class $[\widetilde{\partial \overline{\xi}}] \in (*)$, agreeing with $[\xi] \in \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^{2r-1}(X \setminus Y, \mathbb{Q}(r)))$. If $[\widetilde{\partial \overline{\xi}}] \neq 0$, then we have ‘detected’ the cycle $\xi$. 

Next, suppose that \( \widetilde{\partial \xi} = 0 \). We are now looking at the Abel-Jacobi value

\[
AJ(\xi) \in \frac{\Ext^1_{\text{MHS}}(\mathbb{Q}(0), Gr^{-1}_W H^{2r-2}(X \setminus Y, \mathbb{Q}(r)))}{\Xi_0}.
\]

Note that \( Gr^{-1}_W H^{2r-2}(X \setminus Y, \mathbb{Q}(r)) \) is identified with the cohomology of:

\[
H^{2r-5}(Y^{[2]}, \mathbb{Q}(r - 2)) \xrightarrow{G_y} H^{2r-3}(Y^{[1]}, \mathbb{Q}(r - 1)) \xrightarrow{G_y} H^{2r-1}(X, \mathbb{Q}(r))
\]

and hence the dual is given by the cohomology of

\[
H^{2d-2r+1}(X, \mathbb{Q}(d - r)) \xrightarrow{G_y^*} H^{2d-2r+1}(Y^{[1]}, \mathbb{Q}(d - r)) \xrightarrow{G_y^*} H^{2d-2r+1}(Y^{[2]}, \mathbb{Q}(d - r))
\]

We now apply the Hodge filter \( F^{d-r+1} \) after tensoring the latter sequence with \( \mathbb{C} \). So for

\[
\omega \in F^{d-r+1} H^{2d-2r+1}(Y^{[1]}, \mathbb{C}), \text{ } G_y^*(\omega) = d\eta \text{ on } Y^{[2]},
\]

(\( \eta \) having Hodge type \( F^{d-r+1} \)), we compute \( AJ(\xi)(\omega) \).
Thus we can assume given a class \([\gamma] \in H^{r-2,r-2}(Y[2], \mathbb{Q}(r-2))\), such that
\[
\left[ \partial \xi \right] = 2\pi i G_Y[\gamma] \in H^{2r-2}(Y[1], \mathbb{Q}(r-1)).
\]
In this case \(\partial \xi - 2\pi i G_Y(\gamma) \sim_{\text{hom}} 0\) on \(Y[1]\), and therefore
\[
\partial \xi - 2\pi i G_Y(\gamma) = \partial \zeta,
\]
bounds an integral (real) chain \(\zeta\). One shows that
\[
AJ(\xi)(\omega) = \int_{\zeta} \omega + 2\pi i \int_{\gamma} \eta.
\]
Note that if \(\gamma\) is algebraic, then by Hodge type of \(\eta\), we have
\[
\int_{\gamma} \eta = 0.
\]
This AJ map calculation is well defined modulo "periods". Note that
\[
\Xi_0 \simeq \text{Image}(H^{r-2,r-2}(Y[2], \mathbb{Q}(r-2)) \to \text{Ext}_{\mathbb{MHS}}^1(\cdots)).
\]
If \( \tilde{\partial} \xi \) determines a nontrivial value in
\[
\Ext^1_{\text{MHS}} \left( \mathbb{Z}(0), \frac{\ker \text{Gy} : H^{2r-3}(Y^{[1]}, \mathbb{Z}(r-1)) \to H^{2r-1}(X, \mathbb{Z}(r))}{\text{Gy}(H^{2r-5}(Y^{[2]}, \mathbb{Z}(r-2)))} \right),
\]
then again we have detected \( \xi \). But suppose this value is zero. Then there is still no guarantee that \( \xi \) belongs to the lower weight filtration. However, if we work with \( X, Y \) over \( \overline{\mathbb{Q}} \), and assume the aforementioned generalization of the Bloch-Beilinson conjecture, and work modulo torsion, then we can now calculate detect \( \xi \) cohomologically on the lowest weight \( W_{-1} \text{CH}^r(X \setminus Y, 1) \). Specifically we are in the situation where we can assume (after possibly modifying \( \xi \) by a cycle supported on \( Y \)) that \( \partial \xi = 0 \) on \( X \). Then \( \xi \in W_{-1} \text{CH}^r(X \setminus Y, 1) = \text{CH}^r(X \setminus Y, 1) \), and the formula, which again will involve additional “periods” \( \Xi_{-1} \) as above:
\[
\text{CH}^r(X \setminus Y, 1) \to \frac{\Ext^1_{\text{MHS}} \left( \mathbb{Z}(0), \frac{H^{2r-2}(X, \mathbb{Z}(r))}{H^{2r-2}_Y(X, \mathbb{Z}(r))} \right)}{\Xi_{-1}},
\]
and which now involves the pure HS
\[
\frac{H^{2r-2}(X, \mathbb{Q}(r))}{H^{2r-2}_Y(X, \mathbb{Q}(r))} \simeq \ker \left\{ H^{2d-2r+2}(X, \mathbb{Q}(d - r)) \to H^{2d-2r+2}(Y, \mathbb{Q}(d - r)) \right\}^\vee,
\]
is given by the same formulas as in the literature (e.g. KLM).
NORMAL FUNCTIONS

Recall that $X/\mathbb{C}$ is a projective algebraic manifold of dimension $d$. We can consider a $\overline{\mathbb{Q}}$-spread

$$\mathcal{X} \xrightarrow{\rho} S,$$

where $\mathcal{X}$ and $S$ are smooth quasiprojective varieties over $\overline{\mathbb{Q}}$ and $\rho$ is smooth and proper, and where if $\eta$ is the generic point of $S$, then $X/\mathbb{C} = \mathcal{X}_\eta \times \mathbb{C}$. Roughly speaking, the resulting cohomological data defines an arithmetic VHS. For the moment we will adopt the formalism of M. Saito’s mixed Hodge modules, and consider the same situation with $\mathcal{X}$ and $S$ defined over any subfield $K \subset \mathbb{C}$. Here $\text{MHM}(\mathcal{X}) = \text{category of mixed Hodge modules on } \mathcal{X}$. There is a cycle class map

$$\text{CH}^r(\mathcal{X}, m; \mathbb{Q}) \to \text{Ext}^{2r-m}_{\text{MHM}(\mathcal{X})}(\mathbb{Q}\mathcal{X}(0), \mathbb{Q}\mathcal{X}(r)),$$

where $\mathbb{Q}\mathcal{X}(r)$ is the Tate object in $\text{MHM}(\mathcal{X})$. There is the Leray spectral sequence $E_2^{p,q} :=$

$$\text{Ext}^p_{\text{MHM}(S)}(\mathbb{Q}_S(0), R^q \rho_* \mathbb{Q}\mathcal{X}(r)) \Rightarrow \text{Ext}_{\text{MHM}(\mathcal{X})}^{p+q}(\mathbb{Q}\mathcal{X}(0), \mathbb{Q}\mathcal{X}(r)),$$

which degenerates at $E_2$ (Deligne), and where $R^q \rho_* : \text{MHM}(\mathcal{X}) \to \text{MHM}(S)$ is the derived functor in the category of mixed Hodge modules.
Further, the Leray spectral sequence associated to the derived functor of bounded complexes

$$Rg_* : D^b(\text{MHM}(S)) \to D^b(\text{MHM}(\text{Spec}(K)))$$

and the natural map $\text{MHM}(\text{Spec}(K)) \to \text{MHS}$, with $\text{Ext}_{\text{MHS}}^{>2} = 0$ implies the s.e.s.:

$$0 \to E_{\infty}^{\nu,2r-\nu-m} \to E_{\infty}^{\nu,2r-\nu-m} \to E_{\infty}^{\nu,2r-\nu-m} \to 0,$$

where

\[
\begin{align*}
E_{\infty}^{\nu,2r-\nu-m} &= \text{Ext}_{\text{MHS}}^1(\mathbb{Q}(0), H^{\nu-1}(S, R^{2r-\nu-m} \rho_* \mathbb{Q}(r))) \\
E_{\infty}^{\nu,2r-\nu-m} &= \text{Ext}_{\text{MHM}(S)}^\nu(\mathbb{Q}_S(0), R^{2r-\nu-m} \rho_* \mathbb{Q}(r)) \\
E_{\infty}^{\nu,2r-\nu-m} &= \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^{\nu}(S, R^{2r-\nu-m} \rho_* \mathbb{Q}(r)))
\end{align*}
\]
Remark. One can introduce a filtration $F'\nu^r(\mathcal{X}/S, m; \mathbb{Q})$ on the variational piece by the condition

$$Gr_{F}^{\nu}CH^{r}(\mathcal{X}/S, m; \mathbb{Q}) \hookrightarrow E_{\infty}^{\nu,2r-\nu-m}. $$

If one further assumes that $K = \overline{\mathbb{Q}}$, then after taking direct limits over open subsets $U \subset S/\overline{\mathbb{Q}}$, followed by a limit over subfields $L = \overline{\mathbb{Q}}(S) \subset \mathbb{C}$ of finite transcendence degree over $\overline{\mathbb{Q}}$, one gets a filtration on $CH^{r}(X/\mathbb{C}, m; \mathbb{Q})$ which coincides with the filtration by Asakura/M. Saito,

Let us further assume that $S$ is affine, and let $V \subset S(\mathbb{C})$ be a smooth irreducible variety of dimension $\nu - 1$. One has this diagram

$$
\begin{array}{ccc}
\mathcal{X}_V & \hookrightarrow & \mathcal{X}(\mathbb{C}) \\
\rho_V & & \downarrow \rho \\
V & \hookrightarrow & S(\mathbb{C})
\end{array}
$$

This gives rise to a commutative diagram via restriction maps:

$$
\begin{array}{ccc}
E_{\infty}^{\nu,2r-\nu-m}(\rho) & \hookrightarrow & E_{\infty}^{\nu,2r-\nu-m}(\rho) \\
\downarrow & & \downarrow \text{“0”} \\
E_{\infty}^{\nu,2r-\nu-m}(\rho_V) & \hookrightarrow & E_{\infty}^{\nu,2r-\nu-m}(\rho_V)
\end{array}
$$
Of course, here we are using the characterization of the Leray filtration on cohomology as a motivic filtration, via Arapura’s work. Thus via the map $F^\nu CH^r(\mathcal{X}/S, m; \mathbb{Q}) \to E_{\infty}^{2r-\nu-m}(\rho)$, a class $\xi \in F^\nu CH^r(\mathcal{X}/S, m; \mathbb{Q})$ gives rise to an element of

$$E_{\infty}^{2r-\nu-m}(\rho_\mathcal{V}) = \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{\nu-1}(V, R^{2r-\nu-m} \rho_\star \mathbb{Q}(r))),$$

and hence by varying $V \subset S(\mathbb{C})$, a normal function. For example, if we have a smooth family $\{V_t\}_{t \in B}$ of such subvarieties in $S(\mathbb{C})$, then $\xi$ determines a holomorphic cross-section

$$\eta_\xi : B \to \coprod_{t \in B} \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{\nu-1}(V_t, R^{2r-\nu-m} \rho_\star \mathbb{Q}(r))).$$

One has the notion of infinitesimal invariants of normal functions, and so on.
We now assume $K = \overline{\mathbb{Q}}$. The corresponding group of normal functions obtained by varying $V$ in $S(\mathbb{C})$, will be called arithmetic normal functions. Let us assume $S = S/\overline{\mathbb{Q}}$ is affine. Note that $\mathcal{X}_V := \rho^{-1}(V)$ is no longer complete if $\dim V > 0$. Furthermore, since $V$ is affine,

$$H^{\nu - 1}(V, R^{2r - \nu - m} \rho_* \mathbb{Q}(r)) \rightarrow H^{2r - m - 1}(\mathcal{X}_V, \mathbb{Q}(r)),$$

is the lowest Leray filtered piece. Thus there is a natural map:

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{\nu - 1}(V, R^{2r - \nu - m} \rho_* \mathbb{Q}(r))) \rightarrow \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{2r - m - 1}(\mathcal{X}_V, \mathbb{Q}(r))).$$

If a normal function is zero, then it will be zero at all “points” $V \subset S(\mathbb{C})$, a fortiori at all “points” $V \subset S(\mathbb{Q})$. In short, to detect whether a normal is zero, amounts to checking that the corresponding Abel-Jacobi maps at all $V \subset S(\mathbb{Q})$ are zero. However if one accepts the aforementioned generalization of the Bloch-Beilinson conjecture, then this is reduced to a very explicit calculation.
A natural question is,

"Is it possible to describe the conjectured Bloch-Beilinson filtration in terms of kernels of variational regulators?"

We think so, viz.:

**Conjecture.** The arithmetical normal functions are by themselves, enough to give a complete characterization of the [conjectured] Bloch-Beilinson filtration \( \{F^\nu CH^r(X, m; \mathbb{Q})\} \). More specifically, a class \( \xi \in F^\nu CH^r(X, m; \mathbb{Q}) \) belongs to \( F^{\nu+1}CH^r(X, m; \mathbb{Q}) \) iff for some \( \overline{Q} \)-spread, \( \xi \) has a lifting to some \( \tilde{\xi} \in F^\nu CH^r(X/S, m; \mathbb{Q}) \) for which the corresponding normal function is zero.