THREE LECTURES ON THE HODGE CONJECTURE

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Abstract. The statement of the Hodge conjecture for projective algebraic manifolds is presented in its classical form, as well as the general (Grothendieck amended) version. The intent of these lectures is to focus on some specific examples, rather than present a general survey overview, as can be found in [Lew2] and [Shi]. A number of exercises for the reader are sprinkled throughout the lectures. For background material, the reader is assumed to have some familiarity with the geometry of complex manifolds, such as can be found in chapter 0 of [G-H1].

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Lecture 1: The statement and some standard examples

Some preliminary material. Let \( \mathbb{P}^N = \{ \mathbb{C}^{N+1} \setminus \{0\} \}/\mathbb{C}^\times \), be complex projective \( N \)-space. A projective algebraic manifold \( X \) is a closed embedded submanifold of \( \mathbb{P}^N \). By a theorem of Chow, \( X \) is cut out by the zeros of a finite number of homogenous polynomials, satisfying a certain jacobian criterion (so that \( X \subset \mathbb{P}^N \) is indeed smooth). The fact that \( X \) is projective algebraic implies that \( X \) contains “plenty” of subvarieties. Let \( z^k(X) \) be the free abelian group generated by [irreducible] subvarieties of codimension \( k \) in \( X \). If \( \dim X = n \), then \( z^k(X) = z_{n-k}(X) \), the group generated by dimension \( n-k \) subvarieties of \( X \). For example, \( z_0(X) = \{ \sum_{i=1}^r n_i p_i \mid p_1, \ldots, p_r \in X, r \geq 1, n_i \in \mathbb{Z} \} = \) group of 0-cycles. Now let \( E^k_X = \mathbb{C} \)-valued \( C^\infty \) \( k \)-forms on \( X \). We have the decomposition:

\[
E^k_X = \bigoplus_{p+q=k} E^{p,q}_X, \quad \overline{E^{p,q}_X} = E^{q,p}_X,
\]

where \( E^{p,q}_X \) are the \( C^\infty \) \( (p,q) \)-forms which in local holomorphic coordinates \( z = (z_1, \ldots, z_n) \in X \), are of the form:

\[
\sum_{|I| = p, |J| = q} f_{I} J d\bar{z}_I \wedge d\bar{z}_J, \quad I = 1 \leq i_1 < \cdots < i_p \leq n,
J = 1 \leq j_1 < \cdots < j_q \leq n
\]

\[
dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p},
\]
\[
d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.
\]

The differential \( d : E^k_X \to E^{k+1}_X \) splits into \( d = \partial + \overline{\partial} \), where \( \overline{\partial} E^{p,q}_X \subset E^{q,p+1}_X, \partial E^{p,q}_X \subset E^{p+1,q}_X \). Since \( d^2 = 0 \), by Hodge type considerations, \( 0 = \overline{\partial}^2 = \partial^2 = \partial \overline{\partial} + \overline{\partial} \partial \). The decomposition in (1.0) descends to the cohomological level, viz.,

**Theorem 1.1. [Hodge decomposition]**

\[
H^k_{\text{sing}}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\text{de Rham isomorphism}} H^k_{\text{DR}}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),
\]

where \( H^{p,q}(X) = d\text{-closed (p,q)-forms (modulo coboundaries)} \), and \( \overline{H^{p,q}(X)} = H^{q,p}(X) \). Moreover, all such cohomology groups are finite
dimensional. Furthermore, \(^1\)
\[
H^{p,q}(X) \simeq \frac{E^{p,q}_{X,d}-\text{closed}}{\partial \bar{\partial} E^{p-1,q-1}_X}.
\]

Now recall \(\dim X = n\).

**Theorem 1.3.** [Poincaré and Serre Duality] The following pairings induced by
\[
(w_1, w_2) \mapsto \int_X w_1 \wedge w_2,
\]
are non-degenerate:
\[
H^k_{\text{DR}}(X, \mathbb{C}) \times H^{2n-k}_{\text{DR}}(X, \mathbb{C}) \to \mathbb{C},
\]
\[
H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{C}.
\]
Therefore \(H^k(X) \simeq H^{2n-k}(X)^{\vee}\), \(H^{p,q}(X) \simeq H^{n-p,n-q}(X)^{\vee}\).

**The cycle class map:** The fundamental class.
\[
\text{cl}_k : \mathcal{Z}^k(X) \to H^k_{\text{DR}}(X, \mathbb{C}) \simeq H^{2n-2k}_{\text{DR}}(X, \mathbb{C})^{\vee}.
\]
Let \(V \subset X\) be a subvariety of codimension \(k\) in \(X\), \(\{w\} \in H^{2n-k}_{\text{DR}}(X, \mathbb{C})\). Define \(\text{cl}_k(V)(w) = \int_{V \cdot \ast} w\), and extend to \(\mathcal{Z}^k(X)\) by linearity. [Note that \(\dim_{\mathbb{R}} V = 2n - 2k\), and where \(V^{\ast} = V \setminus \text{V}_{\text{sing}}\).] We must show that \(\int_{V^{\ast}} w\) has finite volume, and that the corresponding current is closed, viz., descends to the cohomology level. All this is essentially a result of \(\dim_{\mathbb{R}} \text{V}_{\text{sing}} \geq 2\); but we can argue differently as follows. One can construct a desingularization \(\sigma : \tilde{V} \xrightarrow{\approx} V\), where say \(\sigma^{-1}(V_{\text{sing}})\) is a divisor with normal crossings (locally a cut out of \(\tilde{V}\) by \(z_1 \cdots z_l = 0\)). It is obvious then that via \(V \to V \leftrightarrow X\), \(\sigma^{\ast} w\) is a \(C^\infty\) form on \(\tilde{V}\), and therefore
\[
\int_{V^{\ast}} w = \int_{\tilde{V}} \sigma^{\ast} w
\]

\(^1\)For the reader’s convenience, we prove the latter statement here, assuming some knowledge of the Hodge-Kähler identities. Up to a factor of 2, the Laplacians \(\Delta_{\partial}\), \(\Delta_{\bar{\partial}}\), \(\Delta_{\bar{\partial}}\) agree, hence define the same harmonic space. The Greens operator \(G\) commutes with any operator that commutes with the Laplacian. Now suppose \(\omega \in E^{p,q}_X\) is an exact form (coboundary), and let \((\cdot)^{\ast}\) stand for adjoint. Then by Hodge type, \(\partial \omega = \bar{\partial} \omega = 0\), and \(\omega = \bar{\partial}^{\ast} \bar{\partial} G_{\bar{\partial}}(\omega) + \partial^{\ast} \partial G_{\partial}(\omega)\). But \(\bar{\partial} G_{\bar{\partial}}(\omega) = G_{\bar{\partial}}(\partial \omega) = 0\). Thus if we set \(\eta = \bar{\partial}^{\ast} G_{\bar{\partial}}(\omega) \in E^{p,q-1}_X\), we have \(\omega = \bar{\partial} \eta\). Next, by the same reasoning, \(\omega = \partial \partial^{\ast} G_{\partial}(\omega) = \partial \partial^{\ast} G_{\partial}(\partial \eta) = \partial \partial^{\ast} \partial G_{\partial}(\partial \eta)\). Now one uses the fact that \([\partial^{\ast}, \partial] = 0\).
has finite volume, since \( \tilde{V} \) is compact. Furthermore, if \( w = d\eta \) on \( X \), then
\[
\int_{\tilde{V}} w = \int_{\tilde{V}} \sigma^* d\eta = \int_{\tilde{V}} d(\sigma^* \eta) = \int_{\partial \tilde{V} = 0} \sigma^* \eta = 0,
\]
by Stokes’ theorem. Suppose \( \{w\} \in H^{p,q}(X) \), where \( p + q = 2n - 2k \). Then \( (p, q) \neq (n - k, n - k) \) \( \Rightarrow \) either \( p > n - k \) or \( q > n - k \). In either case \( \int_{\tilde{V}} w = 0 \), since \( \dim V := \dim_{\mathbb{C}} V = n - k \).

Evidently, \( \text{cl}_k(z^k(X)) \subset H^{2k}(X, \mathbb{Z}) \), thus \( \text{cl}_k(z^k(X)) \subset H^{k,k}(X) \cap H^{2k}(X, \mathbb{Z}) \) from the above discussion. There are known examples where \( \text{cl}_k(z^k(X)) \neq H^{k,k}(X) \cap H^{2k}(X, \mathbb{Z}) \), even modulo torsion (see 1.5.2 below); however, the following is still open:

**Hodge Conjecture 1.4.** \( H^{k,k}(X, \mathbb{Q}) : \)
\[
\text{cl}_k(z^k(X) \otimes \mathbb{Z} \mathbb{Q}) = H^{k,k}(X, \mathbb{Q}) := H^{k,k}(X) \cap H^{2k}(X, \mathbb{Q}).
\]

**Some examples.**

(1) The Lefschetz (1, 1) theorem: Hodge\(^{1,1}(X, \mathbb{Z}) \). The (additive abelian) group \( z^1(X) \) is called the group of Weil divisors on \( X \). We introduce the (multiplicative abelian) group of Cartier divisors. Let \( \mathbb{C}(X) \) be the rational function field of \( X \), and \( \mathbb{C}(X)^\times \) the multiplicative group. Further, let \( \mathcal{D}_X = \mathbb{C}(X)^\times / \mathcal{O}_X^\times \), where \( \mathcal{O}_X \) is the sheaf of germs of regular functions on \( X \), and \( \mathcal{O}_X^\times \) is the corresponding sheaf of nowhere vanishing regular functions. A section \( D \in \Gamma(X, \mathcal{D}_X) \) is called a Cartier divisor. In terms of local data (and working in the Zariski topology),
\[
D \in \Gamma(X, \mathcal{D}_X) \Leftrightarrow
\]
\[
D = \{ f_\alpha \in \Gamma(U_\alpha, \mathbb{C}(X)^\times) = \mathcal{C}(X)^\times \ | \ f_\alpha / f_\beta \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^\times) \}\text{.}
\]
There is a natural divisor map \( \text{div} : \Gamma(X, \mathcal{D}_X) \to z^1(X) \), which in fact is an isomorphism between these groups. There are two short exact sequences:
\[
0 \to \mathcal{O}_X^\times \to \mathbb{C}(X)^\times \to \mathcal{D}_X \to 0,
\]
\[
0 \to \mathbb{Z} \to \mathcal{O}_{X,an} \xrightarrow{\exp(2\pi \sqrt{-1}z)} \mathcal{O}_{X,an}^\times \to 0,
\]

\(^2\)We assume that the reader is familiar with the cohomology of \( \mathbb{P}^n \), and more generally the fact that the cohomology of a Grassmannian manifold \( \{ \mathbb{C}^k's \subset \mathbb{C}^n \} \) is generated by algebraic cocycles.
where $\mathcal{O}_{X,\text{an}}$ is the corresponding analytic sheaf\textsuperscript{3}. The first sequence yields the short exact sequence:

$$
\frac{\Gamma(X, \mathcal{D}_X)}{\Gamma(X, \mathcal{C}(X)\times)} \simeq H^1(X, \mathcal{O}_X)^{\times} =: \text{Pic}(X) = \text{Picard group of } X
$$

[This uses the fact that $H^1(X, \mathcal{C}(X)\times) = 0$ (use Čech cohomology).] The second sequence yields:

$$
0 \rightarrow \frac{H^1(X, \mathcal{O}_{X,\text{an}})}{H^1(X, \mathcal{C}(X)\times)} \rightarrow H^1(X, \mathcal{O}_{X,\text{an}}^\times) \overset{c}{\rightarrow} H^2(X, \mathbb{Z})
$$

$$
\rightarrow H^2(X, \mathcal{O}_{X,\text{an}}) \rightarrow \cdots
$$

where $c$ is the first Chern class map, and where $H^1(X, \mathcal{O}_{X,\text{an}}^\times)$ is interpreted as the group of isomorphism classes of holomorphic line bundles over $X$. The Dolbeault isomorphism theorem gives $H^q(X, \mathcal{O}_{X,\text{an}}) \simeq H^{0,q}(\mathbb{C})$, and by a comparison theorem between analytic and algebraic data (use Chow’s theorem or Serre’s GAGA), $H^1(X, \mathcal{O}_{X,\text{an}}^\times) = H^1(X, \mathcal{O}_X^\times)$. Putting it all together, there is a diagram:

$$
\text{Pic}(X) \overset{c}{\rightarrow} H^2(X, \mathbb{Z}) \rightarrow H^{0,2}(X)
$$

$$
z^1(X) \quad H^2(X, \mathbb{C})
$$

This shows that $\text{cl}_1(z^1(X)) = \{\eta \in H^2(X, \mathbb{Z}) \mid \eta \in H^1,1(X)\}$, i.e. the famous Lefschetz (1,1) theorem holds.

**Exercise.** Recall the following theorems:

**Lefschetz Theorems 1.5.** (i) (Weak) Let $j : Y \hookrightarrow X$ be a smooth hyperplane section of a smooth $X$. Then $j_* : H_i(Y, \mathbb{Z}) \rightarrow H_i(X, \mathbb{Z})$ is an isomorphism for $i < \dim Y$, and surjective for $i = \dim Y$.

(ii) (Hard) Let $L_X : H^i(X, \mathbb{Q}) \rightarrow H^{i+2}(X, \mathbb{Q})$ be the operator defined by cupping with the first Chern class of the hyperplane bundle over $X$, i.e. the cohomology class of a hyperplane section of $X$. Then for $i \leq n := \dim X$,

$$
L_X^{n-i} : H^i(X, \mathbb{Q}) \overset{\sim}{\rightarrow} H^{2n-i}(X, \mathbb{Q})
$$

is an isomorphism.

\textsuperscript{3}It is customary to write this as $0 \rightarrow \mathbb{Z}(1) \rightarrow \mathcal{O}_{X,\text{an}} \overset{\exp}{\rightarrow} \mathcal{O}_{X,\text{an}}^\times \rightarrow 0$, where $\mathbb{Z}(1) = 2\pi\sqrt{-1}\mathbb{Z}$. This is because there is no canonical choice of $\sqrt{-1}$. 

Deduce from this that for $2k \leq n$, $H^{k,k}(X, \mathbb{Q}) \Rightarrow H^{n-k,n-k}(X, \mathbb{Q})$, where $n = \dim X$. Further deduce that the Hodge conjecture is true for all $X$ of dimension $\leq 3$.

Griffiths and Harris introduced a series of conjectures in [G-H2], the weakest of which is the following:

**Conjecture 1.5.1** Let $X \subset \mathbb{P}^4$ be a sufficiently general\(^4\) threefold of degree $d \geq 6$. Then for any curve $C \subset X$, we have $d|\deg C$.

**Claim 1.5.2.** Assume that Conjecture 1.5.1 holds\(^5\), and let $X$ be given as in 1.5.1. If $\beta$ is a generator for $H^{2,2}(X, \mathbb{Z}) \simeq \mathbb{Z}$, then $\beta$ is non-algebraic.

**Proof.** For our purposes the degree of $C$ can be defined as follows. The image of the fundamental class $\{C\}$ in the map $H_2(C, \mathbb{Z}) \to H_2(\mathbb{P}^4, \mathbb{Z}) = \mathbb{Z} \times \mathbb{P}^1$, induced by inclusion, determines a cycle $k \cdot \mathbb{P}^1$ for some integer $k \geq 1$. We define $\deg(C) = k$. The weak Lefschetz theorem in our situation implies that the inclusion map $j : X \hookrightarrow \mathbb{P}^4$ induces an isomorphism $j_* : H_2(X, \mathbb{Z}) \simeq H_2(\mathbb{P}^4, \mathbb{Z})$. From Poincaré duality, we deduce that $H_2(X, \mathbb{Z}) \simeq H^1(X, \mathbb{Z}) = H^{2,2}(X, \mathbb{Z})$, i.e. every class in $H_2(X, \mathbb{Z})$ is non-torsion and has Poincaré dual of type $(2,2)$. Now let us assume the above conjecture is answered in the affirmative. Then $j_*(H_2, \text{alg}(X, \mathbb{Z})) = d \cdot H_2(\mathbb{P}^4, \mathbb{Z})$ ($d \geq 6$), and therefore if $\beta$ is a generator for $H_2(X, \mathbb{Z})$, then $[\beta] \in H^{2,2}(X, \mathbb{Z})$ and yet $[\beta] \notin H^4_{\text{alg}}(X, \mathbb{Z})$.

**II** Projective bundles and blow-ups. Recall that $\mathbb{P}^N$ describes the family of 1-dimensional subspaces in $\mathbb{C}^{N+1}$, and thus there is an associated tautological (universal) line bundle over $\mathbb{P}^N$, that associates with each point $p \in \mathbb{P}^N$, the corresponding line in $\mathbb{C}^{N+1}$. Let $W \hookrightarrow X$ be a vector bundle of rank $r$ on $X$, and $\pi : \mathbb{P}[W] \to X$ the corresponding projective bundle. The cohomology of $\mathbb{P}[W]$ is well-known, in terms of some Chern class invariants associated to $W$, and the corresponding tautological line bundle over $\mathbb{P}[W]$. In particular, if $\mu$ is the (first) Chern class of the tautological line bundle over $\mathbb{P}[W]$, then identifying

\(^4\)General in this case meaning in a transcendental sense. More specifically, where $X$ corresponds to a point in the complement of a countable union of proper subvarieties, of the projective space of threefolds in $\mathbb{P}^4$ of a given degree.

\(^5\)According to the work of J. Kollár, et al, in Trento Examples (Springer Lecture Notes 1515 (1992), pp. 134-135), the partial results obtained there in the direction of the above conjecture, lead to a counterexample to the statement of the Hodge conjecture with $\mathbb{Z}$-coefficients for certain hypersurface threefolds.
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$H^\bullet(X)$ with $\pi^* H^\bullet(X)$, we have:

$$H^\bullet(\mathbb{P}[W], \mathbb{Q}) \simeq \frac{H^\bullet(X, \mathbb{Q})[\omega]}{(\mu^{r-c_1(W)}\mu^{r-1} + \cdots + (-1)^r c_r(W))}$$

$$\simeq H^\bullet(X, \mathbb{Q}) \oplus H^{\bullet-2}(X, \mathbb{Q}) \wedge \mu \oplus \cdots \oplus H^{\bullet-2(r-1)}(X, \mathbb{Q}) \wedge \mu^{r-1}.$$ 

Now let $D \subset X$ be a smooth subvariety of codimension $r \geq 2$ in $X$, and $N(D/X)$ the corresponding normal bundle (of rank $r$). Further, let $E = \mathbb{P}[N(D/X)]$, and $\mu \in H^{1,1}(E, \mathbb{Z})$ be the first Chern class of the tautological line bundle over $E$. Then for example $H^{j-2}(E, \mathbb{Q}) \simeq H^{j-2}(D, \mathbb{Q}) \oplus H^{j-4}(D, \mathbb{Q}) \wedge \mu \oplus \cdots \oplus H^{j-2r}(D, \mathbb{Q}) \wedge \mu^{r-1}$, where $H^*(D, \mathbb{Q})$ is identified with its image $\pi^*: H^*(D, \mathbb{Q}) \to H^*(E, \mathbb{Q})$. Now set $V = H^{j-2}(D, \mathbb{Q}) \oplus H^{j-4}(D, \mathbb{Q}) \wedge \mu \oplus \cdots \oplus H^{j-2r+2}(D, \mathbb{Q}) \wedge \mu^{r-2}$. Using the fact that the fibers of $\pi: E \to D$ are positive dimensional ($\simeq \mathbb{P}^{r-1}$, recall $r \geq 2$) and working on the cycle level via Poincaré duality, it follows that cycles represented by cocycles in $V$ map to cycles of strictly lower dimension in $D$. Therefore $\pi_* (V) = 0$ where $\pi_* : H^{j-2}(E, \mathbb{Q}) \to H^{j-2r}(D, \mathbb{Q})$; moreover since $H^{j-2}(E, \mathbb{Q}) = V \oplus H^{j-2r}(D, \mathbb{Q}) \wedge \mu^{r-1}$, it follows for dimension reasons that $\pi_* : H^{j-2r}(D, \mathbb{Q}) \wedge \mu^{r-1} \cong H^{j-2r}(D, \mathbb{Q})$ is an isomorphism; moreover one has the Gysin map $H^{j-2r}(D, \mathbb{Q}) \to H^j(X, \mathbb{Q})$, induced by inclusion. It is easy to verify that

$$H^j(B_D(X), \mathbb{Q}) \cong V \oplus H^j(X, \mathbb{Q}).$$

**Exercise.** Let $h : X_1 \to X_2$ be a dominating rational map between fourfolds. Show that Hodge$^{2,2}(X_1, \mathbb{Q})$ holds $\Rightarrow$ Hodge$^{2,2}(X_2, \mathbb{Q})$ holds.

**[Hint: First, if $h$ is a morphism, then use the fact that $h_* \circ h^* = \times d$, where $d$ is the degree of $h$. To reduce to the case of a morphism, use the above results on blow-ups after constructing a “Hironaka house”:**

```
X_1, N
|   ↓ j_N
X_1, N-1
|  ↓ j_{N-1}
...  
|  ↓
X_1,1
|  ↓ j_2
X_1,0
```

$h: X_1 \to X_2$
where (i) \( \tilde{f}_j : X_{1,j} \to X_{1,j-1} \) is a blow-up along a nonsingular center, i.e. \( X_{1,j} = B_{D_j}(X_{1,j-1}) \) where \( D_j \) is a smooth variety of codimension \( \geq 2 \) in \( X_{1,j-1} \) and \( \tilde{f}_j \) is the corresponding blow-down morphism.

(ii) The above diagram is commutative and \( \tilde{f} : X_{1,N} \to X_2 \) is a morphism (i.e. regular).

(III) Uniruled fourfolds ([C-M1]). A variety \( X \) is uniruled if it is covered by a family \( \{C_b\}_{b\in \Omega} \) of rational curves. One can assume that \( \dim \Omega = \dim X - 1 \), and where \( \Omega \) is irreducible. If \( \eta \) is the generic point of \( \Omega \), then there is a finite field extension \( L \) over \( \mathbb{C}(\eta) \) such that \( C_\eta \times L \cong \mathbb{P}^1(L) \). This translates to a generically finite to one dominating rational map \( h : \Omega' \times \mathbb{P}^1 \to X \) where \( \Omega' \) is smooth.\(^6\)

Exercise. Show that if \( \dim \Omega' = 3 \), then Hodge\(^{2,2}(\Omega' \times \mathbb{P}^1, \mathbb{Q}) \) holds.

Note that the Hodge conjecture is true through dimension \( \leq 3 \); moreover by a sequence of blow-ups, we can assume that \( h \) is a morphism. Thus Hodge\(^{2,2}(\Omega' \times \mathbb{P}^1, \mathbb{Q}) \Rightarrow \text{Hodge}^{2,2}(X, \mathbb{Q}) \). This result includes unirational fourfolds (also see [Mu1]). For example, Conte and Murre show that a smooth complete intersection fourfold with \( H^{4,0}(X) = 0 \) is uniruled.

Exercise. Show that a smooth projective uniruled variety \( X \) of dimension \( n \) satisfies \( H^{n,0}(X) = 0 \).

The general Hodge conjecture. Grothendieck ([Gro]) was the first to introduce the notion of the coniveau filtration.

Proposition-Definition 1.6. The (descending) filtration by coniveau on singular cohomology is given by any of the 3 equivalent definitions below:

\[
N^p H^i(X, \mathbb{Q}) := \ker \left( H^i(X, \mathbb{Q}) \to \lim_{\cd Y \geq p} H^i(Y \setminus Y, \mathbb{Q}) \right)
\]

\[
:= \text{Image} \left( \sum_{\cd Y \geq p} H^i(Y, \mathbb{Q}) \to H^i(X, \mathbb{Q}) \right)
\]

\[
:= \text{Gysin Images} \left( \sum_{\cd Y = r \geq p} H^{i-2r}(\tilde{Y}, \mathbb{Q}) \to H^i(X, \mathbb{Q}) \right),
\]

where \( \tilde{Y} \to Y \) is a desingularization. [The third characterization follows from the work of Deligne ([De] (Prop. 8.2.7, Cor. 8.2.8)).]}

\(^6\)A geometric proof of this fact that gives a fairly explicit description of such an \( h \), is given in [Lew2] (pp. 216–218).
Recall that a Hodge structure of weight \( m \in \mathbb{Z} \) is a finite dimensional vector space \( H = H_\mathbb{Q} \) with a decomposition \( H \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{p,q} \), where \( H^{p,q} = H^{q,p} \). It is easy to verify that \( N^p H^i(X, \mathbb{Q}) \subset F^p H^i(X, \mathbb{C}) \) (exercise!); however, as Grothendieck pointed out (op. cit.), the inclusion \( N^p H^i(X, \mathbb{Q}) \subset F^p H^i(X, \mathbb{C}) \cap H^i(X, \mathbb{Q}) \) is not in general an equality, since the LHS is a Hodge substructure of \( H^i(X, \mathbb{Q}) \) by the third characterization of \( N^p H^i(X, \mathbb{Q}) \) above, whereas the RHS need not be (see Appendix IV). Let \( F_H^p H^i(X, \mathbb{Q}) \) be the maximum Hodge substructure of \( F^p H^i(X, \mathbb{C}) \cap H^i(X, \mathbb{Q}) \). The [Grothendieck amended] General Hodge Conjecture (GHC) asserts that:

\[
\text{GHC}(p, i, X): \text{ The inclusion } \quad N^p H^i(X, \mathbb{Q}) \subset F_H^p H^i(X, \mathbb{Q})
\]
is an equality.

**Exercise.** Let \( X \) be a fourfold. Show that

\[
\text{GHC}(1, 4, X) \Rightarrow \text{Hodge}^{2,2}(X, \mathbb{Q}) \text{ holds},
\]

and that

\[
\text{GHC}(2, 4, X) \Leftrightarrow \text{Hodge}^{2,2}(X, \mathbb{Q}) \text{ holds}.
\]

We now recall the Chow group \( \text{CH}_{\text{alg}}^k(X) \) of algebraic cycles, algebraically equivalent to zero (see Appendix I to lecture 2). We put \( J_a^k(X) = \Phi_k(\text{CH}_{\text{alg}}^k(X)) \), where \( \Phi_k : \text{CH}_{\text{hom}}^k(X) \to J^k(X) \) is the Abel-Jacobi map (discussed in Appendix I below).

**Proposition 1.7.** \( J_a^k(X) \) is an Abelian variety (even though \( J^k(X) \) is in general only a complex torus).

**Proof (Outline).** First of all, \( \xi \in \text{CH}_{\text{alg}}^k(X) \Rightarrow \xi \in w_* (\text{CH}_{\text{alg}}^1(\Gamma)) \) for some smooth curve \( \Gamma \). But \( \text{CH}_{\text{alg}}^1(\Gamma) \simeq J^1(\Gamma) \). One can likewise argue that \( \exists \tilde{w} \in z^k(J^1(\Gamma) \times X) \) such that \( \xi \in \tilde{w}_*(J^1(\Gamma)) \). Further, \( \Phi_k \circ \tilde{w}_* : J^1(\Gamma) \to J_a^k(X) \) is a homomorphism. Next, assume given an Abelian variety \( A \) and a homomorphism \( w_* : A \to J_a^k(X) \) induced by a cycle \( w \in z^k(A \times X) \), such that \( \dim \Phi_k \circ w_*(A) \) is maximal. I claim that \( \Phi_k \circ w_*(A) = J_a^k(X) \); otherwise \( \exists [\xi] \in J_a^k(X) \setminus \{ \Phi_k \circ w_*(A) \} \). But \( [\xi] \in \tilde{w}_*(J^1(\Gamma)) \) for some smooth curve \( \Gamma \) and cycle \( \tilde{w} \in z^k(J^1(\Gamma) \times X) \), hence we can replace \( A \) by \( A \times J^1(\Gamma) \), \( w \) by \( \text{Pr}^*_{\Gamma} w + \text{Pr}^*_{\Gamma} \tilde{w} \) in \( z^k(A \times J^1(\Gamma) \times X) \). By maximality of dimension, it therefore follows that \( \Phi_{k,*} \circ w_*(A) = J_a^k(X) \). Next, by Poincare’s complete reducibility theorem, \( \exists B, C \) Abelian varieties in \( A \), where \( B = \text{connected component of} \ \ker( A \to J_a^k(X)) \), such that \( B + C = A \) and \( B \cap C \) is finite. Thus
the corresponding map \( C \to J_a^k(X) \) is finite, hence the lattices defining \( C \) and \( J_a^k(X) \) coincide when tensored with \( \mathbb{Q} \). Therefore \( J_a^k(X) \) is an Abelian variety.

\[ \square \]

**Exercise.** Show that the Lie algebra, \( \text{Lie}(J_a^k(X)) = N_k H^{2k-1} (X, \mathbb{Q}) \otimes \mathbb{R} \), where \( N_k H^{2k-1} (X, \mathbb{Q}) \otimes \mathbb{R} \) has a suitable complex structure. Deduce that:

\[
\text{GHC}(k-1, 2k-1, X) \text{ holds } \Leftrightarrow \text{Lie}(J_a^k(X)) = F_{H}^{k-1} H^{2k-1} (X, \mathbb{Q}) \otimes \mathbb{R}.
\]

**Exercise.** Let \( X \) be a threefold. Show that:

\[
\text{GHC}(1, 3, X) \text{ holds } \Leftrightarrow \text{Hodge}^{2,2}(\Gamma \times X, \mathbb{Q}) \text{ holds } \forall \text{ smooth curves } \Gamma.
\]

**Exercise.** Let \( n = \text{dim } X \). Show that the statement:

\[ \text{“GHC}(1, n, X) \text{ holds for } X” \]

is a birational invariant statement about \( X \).

**Exercise.** Assume \( \text{dim } X = n \), and that the GHC holds. Further, assume that \( \text{Level}(H^*(X)) \leq \ell \), where \( \text{Level}(H^*(X)) = \max\{ p - q \mid H^{p,q}(X) \neq 0 \} \). Then if \( [\Delta(p, q)] \), \( p + q = 2n \), is the corresponding \( \text{K"unneth component of the diagonal class} \):

\[
H^{2n}(X \times X, \mathbb{Q}) \ni [\Delta] = \bigoplus_{p+q=2n} [\Delta(p, q)] \in \bigoplus_{p+q=2n} H^p(X, \mathbb{Q}) \otimes H^q(X, \mathbb{Q}),
\]

show that we can arrange for \( [\Delta(p, q)] \subset Y_p \times W_q \), where \( \text{codim}_X Y_p \geq \frac{p-\ell}{2} \) and \( \text{codim}_X W_q \geq \frac{q-\ell}{2} \).

**Exercise.** Assume that the Hodge conjecture (1.4) holds for projective algebraic manifolds. Show that \( \xi \in N_j H^p(X, \mathbb{Q}) \Leftrightarrow \exists \text{ a smooth projective } S \text{ of dimension } \ell = p - 2j, \text{ and a cycle } w \in CH^{p-j}(S \times X) \text{ such that } \xi \in [w]_* H^{\ell}(S, \mathbb{Q}). \) Deduce that there exists \( S \) and \( w \) as above, such that \( N_j H^p(X, \mathbb{Q}) = [w]_* H^{\ell}(S, \mathbb{Q}). \)

**Exercise.** Assume given \( X \) of dimension \( n \). Further, let

\[
\mu : H^1(X, \mathbb{Q}) \to H^{2n-1}(X, \mathbb{Q}),
\]

\[
\nu : H^{2n-1}(X, \mathbb{Q}) \to H^1(X, \mathbb{Q}),
\]

be morphisms of Hodge structures. Show that \( \mu \) and \( \nu \) are algebraic cycle induced. Under the assumption of the Hodge conjecture (1.4), give a generalization of this result.
APPENDIX I: THE ABEL-JACOBI MAP

Set $z^k_{\text{hom}}(X) = \ker(\text{cl}_k : z^k(X) \to H^{2k}(X, \mathbb{Z}))$. We construct a second cycle class map:

$$\Phi_k : z^k_{\text{hom}}(X) \to J^k(X),$$

where $J^k(X)$ is a certain compact complex torus. Let

$$F'' H^i(X, \mathbb{C}) = \bigoplus_{p+q=i, \sigma \geq \tau} H^{p,q}(X),$$

and note that:

$$H^{2k-1}(X, \mathbb{C}) = F^k H^{2k-1}(X, \mathbb{C}) \bigoplus F^k H^{2k-1}(X, \mathbb{C}).$$

Thus

$$J^k(X) := \frac{H^{2k-1}(X, \mathbb{C})}{F^k H^{2k-1}(X, \mathbb{C}) + H^{2k-1}(X, \mathbb{Z})}$$

is a compact complex torus, called the Griffiths jacobian. Serre duality induces the perfect pairing

$$\frac{H^{2k-1}(X, \mathbb{C})}{F^k H^{2k-1}(X, \mathbb{C})} \times F^{\sigma-k+1} H^{2n-2k+1}(X, \mathbb{C}) \to \mathbb{C}.$$ 

Thus by compatibility of Poincaré and Serre duality, we arrive at:

**Corollary 1.8.**

$$J^k(X) \simeq \frac{F^{\sigma-k+1} H^{2n-2k+1}(X, \mathbb{C})^\vee}{H^{2n-2k+1}(X, \mathbb{Z})},$$

where the denominator [group of periods] is identified with its image:

$$H^{2n-2k+1}(X, \mathbb{Z}) \to F^{\sigma-k+1} H^{2n-2k+1}(X, \mathbb{C})^\vee$$

by the formula:

$$\{\gamma\} \mapsto \left(\{w\} \in F^{\sigma-k+1} H^{2n-2k+1}(X, \mathbb{C}) \mapsto \int_{\gamma} w\right),$$

which is a well-defined map by Stokes’ theorem.

**Prescription for $\Phi_k$:** Let $\xi \in z^k_{\text{hom}}(X)$. Then $\xi = \partial \zeta$ bounds a $2n - 2k + 1$ real dimensional chain $\zeta$ in $X$. Let $\{w\} \in F^{\sigma-k+1} H^{2n-2k+1}(X, \mathbb{C})$. Define:

$$\Phi_k(\xi)(\{w\}) = \int_{\zeta} w \pmod{\text{periods}}.$$

**Proposition 1.9.** $\Phi_k(\xi)$ is well-defined.

**Outline of Proof.** First, if $\partial \zeta' = \xi$, then set $\varphi = \zeta - \zeta'$, hence $\partial \varphi = \partial \zeta - \partial \zeta' = \xi - \xi = 0$, i.e. $\{\varphi\} \in H_{2n-2k+1}(X, \mathbb{Z})$. Thus $\int_{\zeta} w$ —
\[ \int_{w} \alpha = \int_{w'} \alpha \in \text{group of periods.} \] Next, we must show that \( \Phi_k \) depends only on the complex structure of \( X \), i.e., independent of cohomological representative of \( \{w\} \), in terms of the Hodge structure. For this, we use a result of Dolbeault:

**Lemma 1.10.**

\[ F^k H^\ell(X, \mathbb{C}) = \frac{\ker d : F^k E^\ell_X \to E^{\ell+1}_X}{d(F^k E^{\ell-1}_X)}. \]

Therefore, by (1.10), \( \{w\} = \{w'\} \Rightarrow w - w' = d\eta, \eta \in F^{n-k+1} E^{2n-2k}_X. \) Therefore \( \int_{w} \alpha = \int_{w'} \alpha \in \frac{\ker d : F^k E^\ell_X \to E^{\ell+1}_X}{d(F^k E^{\ell-1}_X)}. \) But \( \xi \) involves subvarieties of dimension \( n - k \) and \( \eta \in F^{n-k+1} E^{2n-2k}_X. \) Therefore \( \int_{\xi} \alpha = 0, \) hence \( \int_{\xi} w = \int_{\xi} w'. \)

**APPENDIX II: NORMAL FUNCTIONS AND THE LEFSCHETZ (1,1) THEOREM**

Lefschetz’s original approach to his (1,1) involved Poincaré normal functions. This appears in [Lef]. He proved his theorem for surfaces. We thus consider the case where \( X \) is a smooth surface. Consider a

general (Lefschetz) pencil \( \{X_t\}_{t \in \mathbb{P}^1} \) of curves in \( X \). Roughly speaking, a holomorphic cross-section:

\[ \nu : \mathbb{P}^1 \to \bigsqcup_{t \in \mathbb{P}^1} J(X_t), \]

is called a normal function. Fix a point \( p_0 \in \cap_{t \in \mathbb{P}^1} X_t \) (base locus). For smooth \( X_t \), one has a bimeromorphic [= birational] morphism (“Jacobi inversion”):

\[ S^{(g)}(X_t) \to \mathbb{P}^1_{\text{hom}}(X_t) \to J^1(X_t), \quad g = \text{genus of } X_t, \]

given by:

\[ p_1 + \cdots + p_g \mapsto \Phi_1(p_1 + \cdots + p_g - g \cdot p_0), \]

where \( S^{(g)}(X_t) \) is the \( g \)-th symmetric product. Thus as \( t \in \mathbb{P}^1 \) varies, the \( \nu(t) \) traces out a codimensional one cycle on \( X \). One way to construct a family of such normal functions is via the restriction:

\[ H^{0,1}(X) \to H^{0,1}(X_t) \to J^1(X_t) \]

\[ \xi \mapsto \nu_{\xi}(t). \]

From this construction, Poincaré was able to construct a family of linearly inequivalent divisors on \( X \). This was the original motivation for introducing normal functions. Now consider the holomorphic vector
bundle \( \mathcal{F} := \coprod_{t \in \mathbb{P}^1} H^{0,1}(X_t) \) with general fiber \( H^{0,1}(X_t) \), and the “sheaf of lattices” \( \mathbb{L}_Z \) with fiber \( H^1(X_t, \mathbb{Z}) \). Further let \( \mathcal{J} = \coprod_{t \in \mathbb{P}^1} J(X_t) \).

There is a short exact sequence:

\[ 0 \to \mathbb{L}_Z \to \mathcal{O}_{\mathbb{P}^1}(\mathcal{F}) \to \mathcal{O}_{\mathbb{P}^1}(\mathcal{J}) \to 0. \]

This gives rise to a diagram:

\[
\begin{array}{ccc}
\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathcal{J})) & \xrightarrow{\delta} & H^1(\mathbb{P}^1, \mathbb{L}_Z) \\
\| & & \uparrow \\
\text{Group of} & \text{Can be identified} & \text{This turns out} \\
\text{normal fns} & \text{with the key part} & \text{to be roughly} \\
& \text{of } H^2(X, \mathbb{Z}) & H^{0,2}(X, \mathbb{C}) \\
& \text{by} & \\
& \text{the Leray spectral} & \\
& \text{sequence associated} & \\
& \text{to } \coprod_{t \in \mathbb{P}^1} X_t \to \mathbb{P}^1 & \\
\end{array}
\]

For \( \nu \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathcal{J})) \), \( \delta(\nu) \) is the topological invariant associated to \( \nu \), or also called the cohomology class of \( \nu \). Thus roughly, \( H^{1,1}(X, \mathbb{Z}) = \delta(\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathcal{J}))) \), i.e. every integral cohomology class of type \( (1, 1) \), is the cohomology class of a normal function. Thus the main result in Lefschetz’s proof was the identification of the Hodge type of \( H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathcal{F})) \). This was complicated by the fact that although the family of curves \( \{X_t\} \) on \( X \) are topologically the same, their corresponding analytic structures are not the same. Finally, by using Jacobi inversion, every normal function traces out an algebraic cycle on \( X \) which agrees with the original cohomology class in \( H^{1,1}(X, \mathbb{Z}) \). Thus we again arrive at the Lefschetz \((1, 1)\) theorem for \( X \).

The main supporting points of Lefschetz’s approach to his theorem, is that the role of normal functions generalizes, in the sense that every (primitive) cohomology class in \( H^{k,k}(X, \mathbb{Z}) \) is the cohomology class of a normal function. This was the Griffiths’ program, which is described in [Gr2], as well as in [Z1], together with a modern treatment in [Z2]. The hope was to arrive at an inductive proof of the Hodge conjecture. Unfortunately, Jacobi inversion fails in general. Specifically, Griffiths was the first to exhibit examples (e.g. a general\(^\text{7}\) quintic threefold) where \( \Phi_2 : z^k_{\text{hom}}(X) \to J^k(X) \) has at most a countable image.

\(^7\)In a transcendental sense. Namely, the parameter space of smooth quintic threefolds is not compact, and it is the complement of a certain countable union of proper analytic subvarieties that defines the transcendental quintics. See [Gr1], §13 for details.
APPENDIX III: A REAL REGULATOR ON $K_1(X)$ AND THE HODGE-\(D\)-CONJECTURE

Let $X/\mathbb{C}$ be a projective algebraic manifold of dimension $n$, let $z^k(X)$ be the free abelian group generated by irreducible subvarieties of codimension $k$ in $X$, and let

$$z^k(X, 1) = \ker \left[ \Gamma \left( \bigoplus_{\text{codim} Z = k-1} \mathbb{C}(Z)^\times \right) \xrightarrow{\text{div}} z^k(X) \right],$$

where $\mathbb{C}(Z)^\times$ is the multiplicative group of the rational function field $\mathbb{C}(Z)$, and “div” is the divisor map. Further, let $T^k(X, 1)$ be the subgroup generated by the Tame symbols:

$$T(\{f, g\}) := \sum_{\text{codim} Z = k-1} (-1)^{\nu_Z(f)\nu_Z(g)} \left( \frac{f^{\nu_Z(f)}}{g^{\nu_Z(g)}} \right)_Z$$

where $f, g \in \mathbb{C}(Z)^\times$ as $Z$ runs over all irreducible subvarieties of codimension $k - 2$ on $X$, and $\nu_Z(h)$ is the order of vanishing of a rational function $h$ along a codimension one $Z$. The higher Chow group $\text{CH}^k(X, 1)$, first introduced by Bloch as the homology of a certain simplicial complex, can alternately be defined as the quotient group:

$$\text{CH}^k(X, 1) = \frac{z^k(X, 1)}{T^k(X, 1)}.$$  

[Note that the “usual” $k$-th Chow group is given by

$$\text{CH}^k(X) := \text{coker} \left[ \Gamma \left( \bigoplus_{\text{codim} Z = k-1} \mathbb{C}(Z)^\times \right) \xrightarrow{\text{div}} z^k(X) \right],$$

and that there are induced cycle class maps:

$$\text{cl}_k : \text{CH}^k(X) \to H^{k,k}(X, \mathbb{Z}) \quad \text{(fundamental class)},$$

$$\Phi_k : \text{CH}^k_{\text{hom}}(X) \to J^k(X) \quad \text{(Abel-Jacobi map).}$$]

The Riemann-Roch theorem furnishes isomorphisms from $K$-theory, (via a Chern character map $\text{ch}$):

$$\text{ch} : K_0(X)_\mathbb{Q} \simeq \text{CH}^*(X)_\mathbb{Q} \quad \text{[Grothendieck]}$$

$$\text{ch} : K_1(X)_\mathbb{Q} \simeq \text{CH}^*(X, 1)_\mathbb{Q} \quad \text{[Bloch]}$$

**Proposition 1.10.** There is a well-defined regulator map

$$r : \text{CH}^k(X, 1) \to H^{k-1,-k-1}(X, \mathbb{R}) \simeq H^{n-k+1,n-k+1}(X, \mathbb{R})^\vee,$$
given by:

$$\xi = \sum_i f_i \otimes Z_i \mapsto r(\xi)(\omega) = \sum_i \int_{Z_i} \omega \log |f_i|,$$

where \( \{\omega\} \in H^{n-k+1,n-k+1}(X, \mathbb{R}) \).

**Idea of Proof.** One first shows that the current \( r(\xi) \) in 1.10 acting on \( E_X^{n-k+1,n-k+1} \) is \( \partial \bar{\partial} \)-closed. This uses the fact that \( \sum_i \text{div}(f_i) = 0 \), together with a residue argument. Here are some details: Let \( \omega \in \partial \bar{\partial} E_X^{n-k,n-k} \). Then we can write \( \omega = d \bar{\partial} \eta \) for some \( \eta \in E_X^{n-k,n-k} \). Also let

$$\xi = \sum_i f_i \otimes Z_i \in z^k(X,1)$$

be given as above, and consider the corresponding integral

$$r(\xi)(\omega) = \sum_i \int_{Z_i} \omega \log |f_i|.$$

By Stokes’ theorem and a standard calculation (below):

$$\int_{Z_i} (d \bar{\partial} \eta) \log |f_i| = \int_{Z_i} \bar{\partial} \eta \wedge d \log |f_i|$$

$$= \int_{Z_i} \bar{\partial} \eta \wedge \partial \log |f_i| = \int_{Z_i} d \eta \wedge \partial \log |f_i|,$$

where the latter two equalities follow by Hodge type, and the former uses

$$d((\bar{\partial} \eta) \log |f|) = (d \bar{\partial} \eta) \log |f| - \partial \eta \wedge d \log |f|.$$ 

More specifically, by taking \( \epsilon \)-tubes about the components \( D \subset \text{div}(f) \), and using that \( \dim D = n-k \) and \( \partial \bar{\partial} \eta \in E_X^{n-k,n-k+1} \), and that \( \log |f| \) is locally \( L^1 \), we find

$$\lim_{\epsilon \to 0} \int_{\text{Tube}_\epsilon((f_i))} (\bar{\partial} \eta) \log |f| = 0.$$

Note that \( \bar{\partial} \eta \wedge \partial \log |f| \in E_X^{n-k,n-k+2+1} \) (locally \( L^1 \) forms) and that \( \dim Z_i = n-k + 1 \). Thus we are left with an integral of the form

$$\int_Z d \eta \wedge \partial \log |f|$$

as indicated above. Next,

$$d(\eta \wedge \partial \log |f|) = d \eta \wedge \partial \log |f|,$$

since \( \partial \bar{\partial} \log |f| = 0 \). Thus

$$\int_{Z_i} (d \bar{\partial} \eta) \log |f_i| = \int_{Z_i} d(\eta \wedge \partial \log |f_i|) = \lim_{\epsilon \to 0} \int_{\text{Tube}_\epsilon((f_i))} \eta \wedge \partial \log |f_i|.$$
If we put \( Z = Z_i \), for a given \( i \), with \( z \) a local coordinate on \( Z_{\text{reg}} \), then using the dictionary \( |f_i| \leftrightarrow |z| \) we have the residue integral:

\[
\lim_{\epsilon \to 0} \int_{|z| = \epsilon} \eta \wedge \partial \log |z|^2 = \lim_{\epsilon \to 0} \int_{|z| = \epsilon} \eta \wedge \frac{dz}{z} = 2\pi \sqrt{-1} \int_{\{z = 0\} \cap Z} \eta_{\{z = 0\} \cap Z},
\]

where

\[
\eta_{\{z = 0\} \cap Z} = \text{Residue}_{\{z = 0\} \cap Z} \left( \eta \wedge \frac{dz}{z} \right)
\]

(i.e., taking “tubes” is dual to taking “residues”). Then by computing the residue, and linearity, we find that

\[
r(\xi)(\omega) = 2\pi \sqrt{-1} \sum_D \left[ \left( \sum_i \nu_D(f_i) \right) \int_D \eta \right].
\]

(We note that there remains the possibility that \( Z = Z_i \) is singular along \( D \). To remedy this, one may pass to a normalization of \( Z \) with the same calculations above.) Since \( \sum_i \text{div}(f_i) = 0 \), we have \( r(\xi)(\omega) = 0 \). Thus we obtain a well-defined map \( r : z^k(X, 1) \to H^{k-1,k-1}(X, \mathbb{R}) \) by theorem (1.1). To show that \( r(T(\{f, g\})) = 0 \), for \( f, g \in \mathbb{C}(Z)^k \), \( \text{codim}_X Z = k - 2 \), we consider the following. Set \( F := (f, g) : Z \to \mathbb{P}^1 \times \mathbb{P}^1 \), and \( t, s \in \mathbb{C}(\mathbb{P}^1)^k \) corresponding to the affine coordinates on \( \mathbb{P}^1 \). By a proper modification, we can assume that \( F \) is a morphism. Then \( F^*(\{t, s\}) = \{f, g\} \), and by functoriality and a standard calculation, \( r(T(\{f, g\})) = 0 \).

Beilinson’s Hodge-\( D \)-conjecture ([Be]) asserts that

\[(1.11) \quad r : \text{CH}^k(X, 1)_{\mathbb{R}} \to H^{k-1,k-1}(X, \mathbb{R}),\]

is surjective. This was recently proven to be false (see [MS]), by establishing a Noether-Lefschetz theorem analog in this setting. It uses a connectedness result of Nori ([No]). For example, the Hodge-\( D \)-conjecture fails when \( X \subset \mathbb{P}^3 \) is a general surface of degree \( \geq 5 \), with \( k = 2 \) ([MS], op. cit.). There remains the possibility, however, that the Hodge-\( D \)-conjecture is true for smooth and proper \( X \) defined over a number field. It is reasonable to expect the following:

\textbf{Conjecture 1.12.} Let \( X \) be a \( K3 \) surface with general moduli. Then the Hodge-\( D \)-conjecture holds for \( X \). Specifically,

\[r : \text{CH}^2(X, 1)_{\mathbb{R}} \to H^{1,1}(X, \mathbb{R}),\]

is onto.

\textbf{Question 1.13.} (Variant of the Hodge conjecture.) Let \( X/\mathbb{C} \) be a projective algebraic manifold. Is it the case that the Hodge classes \( H^{k-1,k-1}(X, \mathbb{Q}) \) are in the image of \( r \) in (1.11)?

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8B. Gordon and J. D. Lewis ([G-L]) introduce a twisted version \( \text{CH}^k(X, 1) \) of \( \text{CH}^k(X, 1) \) involving “flat line bundles”, and construct a corresponding regulator
APPENDIX IV: GROTHENDIECK’S COUNTEREXAMPLE

Let $X = T_1 \times T_2 \times T_3$ where $T_j = \mathbb{C}/\Lambda_j$ and where $\Lambda_j = \mathbb{Z} \oplus \mathbb{Z} \tau_j$ and say $\text{Im}(\tau_j) > 0$ for $j = 1, 2, 3$. Now by the Künneth formula, we have

$$\dim_{\mathbb{Q}} \left[ \left\{ F^1 H^3(X, \mathbb{C}) \right\} \cap H^3(X, \mathbb{Q}) \right] \equiv \dim_{\mathbb{Q}} \left[ \left\{ F^1 H^3(X, \mathbb{C}) \right\} \cap H^1(T_1, \mathbb{Q}) \otimes H^1(T_2, \mathbb{Q}) \otimes H^1(T_3, \mathbb{Q}) \right] \mod 2.$$

Now set $\omega = dz_1 \wedge dz_2 \wedge dz_3$ and $V = \{ \xi \in H_1(T_1, \mathbb{Q}) \otimes H_1(T_2, \mathbb{Q}) \otimes H_1(T_3, \mathbb{Q}) \mid \int_\xi \omega = 0 \}$. By Poincaré and Serre duality $\dim_{\mathbb{Q}} V \equiv \dim_{\mathbb{Q}} \left[ \left\{ F^1 H^3(X, \mathbb{C}) \right\} \cap H^3(X, \mathbb{Q}) \right] \mod 2$. For $j = 1, 2, 3$, let $\{ \alpha_j, \beta_j \}$ be the generators of $H_1(T_j, \mathbb{Z}) \simeq \mathbb{Z}^2$ corresponding to $\{ 1, \tau_j \}$ respectively. Consider the basis of $H_1(T_1, \mathbb{Z}) \otimes H_1(T_2, \mathbb{Z}) \otimes H_1(T_3, \mathbb{Z}) \simeq \mathbb{Z}^8$ given by:

$$\xi_1 = \alpha_1 \times \alpha_2 \times \alpha_3 \quad \xi_5 = \beta_1 \times \beta_2 \times \alpha_3$$
$$\xi_2 = \beta_1 \times \alpha_2 \times \alpha_3 \quad \xi_6 = \beta_1 \times \alpha_2 \times \beta_3$$
$$\xi_3 = \alpha_1 \times \beta_2 \times \alpha_3 \quad \xi_7 = \alpha_1 \times \beta_2 \times \beta_3$$
$$\xi_4 = \alpha_1 \times \alpha_2 \times \beta_3 \quad \xi_8 = \beta_1 \times \beta_2 \times \beta_3$$

We now compute.

$$\int_{\xi_1} \omega = 1 \quad \int_{\xi_2} \omega = \tau_1 \quad \int_{\xi_3} \omega = \tau_2 \quad \int_{\xi_4} \omega = \tau_3$$
$$\int_{\xi_5} \omega = \tau_1 \tau_2 \quad \int_{\xi_6} \omega = \tau_1 \tau_3 \quad \int_{\xi_7} \omega = \tau_2 \tau_3 \quad \int_{\xi_8} \omega = \tau_1 \tau_2 \tau_3$$

Let $\xi = \sum_j r_j \xi_j$ where $r_j \in \mathbb{Q}$ (for $j = 1, \ldots, 8$). Then:

$$** \quad 0 = \int_{\xi} \omega = r_1 + r_2 \tau_1 + r_3 \tau_2 + r_4 \tau_3 + r_5 \tau_1 \tau_2 + r_6 \tau_1 \tau_3 + r_7 \tau_2 \tau_3 + r_8 \tau_1 \tau_2 \tau_3.$$

Now for example set $\tau_1 = \tau_2 = \tau_3 = (\sqrt{2}) e^{2\pi \sqrt{-1}/3} = \tau$ say. Then (***) becomes:

$$*** \quad 0 = r_1 + (r_2 + r_3 + r_4) \tau + (r_5 + r_6 + r_7) \tau^2 + r_8 \tau^3,$$

and hence

$$r_1 + 2r_8 = 0 \quad r_2 + r_3 + r_4 = 0 \quad r_5 + r_6 + r_7 = 0,$$

a rank 3 linear system of equations, and therefore in this case $\dim_{\mathbb{Q}} V = 8 - 3 = 5 \equiv 1 \mod 2$, a fortiori $\dim_{\mathbb{Q}} \left[ \left\{ F^1 H^3(X, \mathbb{C}) \right\} \cap H^3(X, \mathbb{Q}) \right] \equiv 1 (2)$. But any Hodge substructure of $H^3(X, \mathbb{Q})$ must be even dimensional (why?). Hence $N^1 H^3(X, \mathbb{Q}) \neq F^1 H^3(X, \mathbb{C}) \cap H^3(X, \mathbb{Q})$. 

---

$\varphi : \text{CH}^k(X, 1) \rightarrow H^{k-1,k-1}(X, \mathbb{R})$. In this setting, the corresponding analog of question (1.13) seems much more tractable.
Lecture 2: A Geometric Approach

The cylinder map for the quintic fourfold. The verification of the GHC in various geometrically interesting cases, using a cylinder map construction, abound in the literature. We work out a prototypical example situation. We construct the cylinder map associated to the family of lines on a general\(^9\) quintic fourfold \(X\), and deduce Hodge\(^2,2\)(\(X, \mathbb{Q}\)) from (2.0) below. This construction is taken from [Lew], which goes back to some ideas of Clemens (see [T] (p.42), and also [B-M]). For a number of applications of cylinder maps, the reader can also consult for example [C-G], [Co], [Co-M], [Mu2], [Te], [Lew2] (Ch. 13), [Lew3]-[Lew4], [Sh1]-[Sh4], to name a few. Let \(G := \text{Grass}(2, 6) = \{\mathbb{C}^2 : s \in \mathbb{C}^n\}\) be the Grassmanian of 2-dimensional subspaces of \(\mathbb{C}^6\). Then \(G\) can be identified with \(\{\mathbb{P}^1 : s \in \mathbb{P}^5\}\); moreover \(\dim G = 8\). Any hypersurface of degree \(d\) in \(\mathbb{P}^n\), determines a corresponding point in \(\mathbb{P}^N\), where \(N = (n+d) - 1\), viz.:

\[
\sum_{\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}^{n+1}} a_{\alpha} z_0^{\alpha_0} \cdots z_n^{\alpha_n} = 0 \quad \mapsto \quad [\ldots, a_{\alpha}, \ldots] \in \mathbb{P}^N.
\]

Let \(\mathbb{P}^N (N = 251)\) be the projective space of hypersurfaces of degree 5 in \(\mathbb{P}^5\), and put \(\mathcal{H} = \{(c, t) \in G \times \mathbb{P}^N \mid \mathbb{P}^1 \subset X_t\}\), with projection diagram:

\[
\begin{array}{c}
\mathcal{H} \\
\pi_1 \leftarrow \qquad \pi_2 \rightarrow \\
G \quad \mathbb{P}^N
\end{array}
\]

An elementary topological argument shows that any such quintic fourfold contains a line \(\mathbb{P}^1\), and hence as we will see below, at the very least, a two dimensional family of lines. [Otherwise, the techniques below will imply that the cylinder map \(\Phi_x\) introduced below would be zero. Thus \(H^4(X, \mathbb{Q}) \simeq \mathbb{Q}\), which is far from being the case! (See [Lew3] (p.194)).] Thus \(\pi_2\) is surjective. On the other hand, \(\pi_1\) defines a \(\mathbb{P}^{N-6}\)-bundle over \(G\). [Hint: Up to a PGL action, we can assume \(\mathbb{P}^4\) is cut out of \(\mathbb{P}^5\) by \(\{z_2 = z_3 = z_4 = z_5 = 0\} \subset \mathbb{P}^5\). Then any homogeneous polynomial of degree 5 in the coordinates \([z_0, z_1, z_2, z_3, z_4, z_5]\) cannot involve \(\sum_{j=0}^{5} z_j z_i^{5-j}\).] Thus \(\dim \mathcal{H} = N + 2\), and hence if \(X \subset \mathbb{P}^5\) is a general hypersurface of degree 5, and if we

---

\(^9\)“General” is interpreted here to mean the following. Given a family of varieties \(\{Y_t\}_{t \in S}\), over some base variety \(S\), a general \(Y_t\) means that \(t\) belongs to some given non-empty Zariski open subset of \(S\), satisfying a number of desirable features (e.g. \(Y_t\) smooth, etc.).
set \( \Omega_X := \{ \mathbb{P}^1 \cap X \} \subset G \), then \( \Omega_X \) is a smooth surface. It turns out to be irreducible as well. If \( Z \subset \mathbb{P}^6 \) is a given general quintic hypersurface, then likewise \( \Omega_Z := \{ \mathbb{P}^1 \cap Z \} \) is smooth (and irreducible) and of dimension 4; moreover in this case through a generic point \( p \in Z \) passes 5! lines. \( \text{[Hint: We can assume by a PGL action that } p = [1,0,0,0,0,0,0] \in X \subset \mathbb{P}^6 \text{, and therefore } X \text{ is the hypersurface defined by } F(z_0, \ldots, z_6) = \sum_{j=1}^{5} z_0^{5-j} F_j(z_1, \ldots, z_6), \text{ where } F_j(z_1, \ldots, z_6) \text{ is homogeneous of degree } j. \text{ Thus in the affine coordinates } (x_1, \ldots, x_6) = (z_1/z_0, \ldots, z_6/z_0), p = (0,0,0,0,0,0), \text{ and } Z \cap \mathbb{C}^6 \text{ is cut out by } z_0^{-5} F(z_0, \ldots, z_6) : = f(x_1, \ldots, x_6) = \sum_{j=1}^{5} f_j(x_1, \ldots, x_6) = 0, \text{ where } \]

\[ f_j(x_1, \ldots, x_6) = z_0^{-5} F_j(z_1, \ldots, z_6), \]

is homogeneous of degree \( j \). Then in affine coordinates, any line \( \ell \) through \( p \) in \( Z \), must be described in the form \( \ell = \{ t \cdot v \mid t \in \mathbb{C} \} \), for some non-zero \( v \in \mathbb{C}^6 \). Thus \( \ell \subset Z \iff [v] \in \Sigma_p := \{ f_1 = f_2 = f_3 = f_4 = f_5 = 0 \} \subset \mathbb{P}^5 \). Note that generically, \( \Sigma_p \) consists of 5! points by Bezout’s theorem.] Let:

\[
\begin{align*}
P(X) &= \{(c, x) \in \Omega_X \times X \mid x \in \mathbb{P}^1_c \} \\
P(Z) &= \{(c, z) \in \Omega_Z \times Z \mid z \in \mathbb{P}^1_c \}
\end{align*}
\]

Note that \( P(X) \) is the projectivization of the pullback to \( \Omega_X \) of the universal rank 2 vector bundle over \( G \), namely the bundle over \( G \) that associates to each point \( c \in G \), the corresponding 2-dimensional subspace \( \mathbb{C}^2_c \subset \mathbb{C}^6 \). Similarly for \( P(Z) \). Hence \( P(X) \) and \( P(Z) \) are smooth and irreducible. Note that the projection \( \pi_Z : P(Z) \to Z \) is onto, since through every point of \( Z \), there passes a line. In particular, \( \deg \pi_Z = 5! \). By Bertini’s theorem, \( \tilde{X} := \pi_Z^{-1}(X) \) is smooth, since \( X \) is general. One has an obvious diagram of projections (or restrictions of projections),
and inclusions:

\[
P(X) \xrightarrow{\pi_X} X
\]

\[
\downarrow i_1 \quad \quad \pi \quad \quad \downarrow j
\]

\[
\hat{X} \quad \quad \quad \quad Z
\]

\[
\downarrow i \quad \quad \quad \pi_Z \quad \quad \downarrow
\]

\[
\rho_X \downarrow \quad \quad \rho \downarrow \quad \quad P(Z)
\]

\[
\check{\rho}_Z
\]

\[
\Omega_X \hookrightarrow \Omega_Z
\]

Note that any line \( \mathbb{P}^1 \subset Z \subset \mathbb{P}^6 \) either meets \( X = \mathbb{P}^5 \cap Z \) at a single point, or lies entirely in \( X \). Thus it is reasonably obvious that \( \hat{X} = B_{\Omega_X}(\Omega_Z) = \) blow-up of \( \Omega_Z \) along \( \Omega_X \). We introduce the cylinder map \( \Phi_* := \pi_{X,*} \circ \rho_X^* \). For a cycle \( \gamma \subset \Omega_X \), \( \Phi_*(\gamma) = \{ U_{\xi} \subset \mathbb{P}^1 \} \) determines a corresponding (co-)cycle on \( X \). Now for any cycle \( \xi \subset X \), \( \pi_* \circ \pi^*(\xi) = (\deg \pi)\xi \). [Since \( \deg \pi = 5! \) is finite.] Now recall that \( H^4(\hat{X}, \mathbb{Q}) \simeq H^4(\Omega_Z, \mathbb{Q}) \oplus H^2(\Omega_X, \mathbb{Q}) \), the isomorphism is in fact given by \( \rho_* \oplus \rho_{X,*} \circ i_1^* \), with inverse \( \rho^* + i_{1,*} \circ \rho_X^* \). Thus under this decomposition of \( H^4(\hat{X}, \mathbb{Q}) \), the surjective morphism \( \pi_* \) is a sum of two morphisms:

\[
\pi_* : \rho^*(H^4(\Omega_Z, \mathbb{Q})) \to H^4(X, \mathbb{Q}),
\]

\[
\pi_* : i_{1,*} \circ \rho_X^*(H^2(\Omega_X, \mathbb{Q})) \to H^4(X, \mathbb{Q}).
\]

But \( \pi_* \circ \rho^* = \pi_* \circ \{ \rho_Z \circ i \}^* = \{ \pi_* \circ i^* \} \circ \rho_Z^* = j^* \circ \{ \pi_{Z,*} \circ \rho_{Z,*} \} \), hence \( \pi_* \circ \rho^*(H^4(\Omega_Z, \mathbb{Q})) \subset j^* H^4(Z, \mathbb{Q}) \). In fact, by the weak Lefschetz theorem for cohomology, \( H^4(Z, \mathbb{Z}) \simeq H^4(\mathbb{P}^6, \mathbb{Z}) \simeq \mathbb{Z} \). In particular, \( \pi_* \circ \rho^*(H^2(\Omega_X, \mathbb{Q})) = \mathbb{Q} \omega \wedge \omega \), where \( \omega \) is a Kähler class (first Chern class of the hyperplane bundle) of \( X \). Next \( \pi_* \circ i_{1,*} \circ \rho_X^* = \pi_{X,*} \circ \rho_X^* = \Phi_* \). Thus \( \Phi_* : H^2(\Omega_X, \mathbb{Q}) \to H^4(X, \mathbb{Q})/\mathbb{Q} \omega \wedge \omega \) is onto. This, together with the Noether-Lefschetz theorem implies that:

\[
(2.0) \quad \Phi_* : H^2(\Omega_X, \mathbb{Q}) \to H^4(X, \mathbb{Q}) \text{ is onto.}
\]

In particular, \( \text{Hodge}^{2,2}(X, \mathbb{Q}) \) holds for \( X \).

## Appendix I

In this part, we need to introduce the following terminology. Let \( \xi_1, \xi_2 \in z^k(X) \). We say that \( \xi_1 \) and \( \xi_2 \) are rationally equivalent to each other, and write \( \xi_1 \sim_{\text{rat}} \xi_2 \), if there is a cycle \( w \in z^k(\mathbb{P}^1 \times X) \) in
general position, such that $\xi_1 - \xi_2 = w(0) - w(\infty)$. We say that $\xi_1$ and $\xi_2$ are algebraically equivalent to each other, and write $\xi_1 \sim_{\text{alg}} \xi_2$, if there is a smooth connected curve $\Gamma$, points $P, Q \in \Gamma$, and a cycle $w \in z^k(\Gamma \times X)$ in general position, such that $\xi_1 - \xi_2 = w(P) - w(Q)$. The subgroups of cycles rationally, algebraically and homologically equivalent to zero are denoted by $z^k_{\text{rat}}(X)$, $z^k_{\text{alg}}(X)$, $z^k_{\text{hom}}(X)$. We put $CH^k(X) := z^k(X)/z^k_{\text{rat}}(X)$, $CH^k_{\text{alg}}(X) := z^k_{\text{alg}}(X)/z^k_{\text{rat}}(X)$, and $\text{Griff}^k(X) := z^k_{\text{hom}}(X)/z^k_{\text{alg}}(X)$ (the Griffiths group). [Note : By a norm argument, this definition of $CH^k(X)$ agrees with the earlier definition given in Appendix III to lecture I.] Now let $X \subseteq \mathbb{P}^{n+1}$ be any smooth hypersurface of degree $d$ cut out by an irreducible homogeneous polynomial $F(z_0, \ldots, z_{n+1})$ (deg $F = d$). Set $G(z_0, \ldots, z_{n+1}) = F + z_{n+2}$, $Z = \{G = 0\} \subseteq \mathbb{P}^{n+2}$ and note that $X = \mathbb{P}^{n+1} \cap Z$, where $\mathbb{P}^{n+1} = \{z_{n+2} = 0\} \subseteq \mathbb{P}^{n+2}$. One can easily check that $Z$ is smooth. Let $j : X \hookrightarrow Z$ be the inclusion and $\nu : Z \to \mathbb{P}^{n+1}$ the projection from $[0, \ldots, 0, 1] \in \mathbb{P}^{n+2}$. (Explicitly: $\nu([z_0, \ldots, z_{n+2}]) = [z_0, \ldots, z_{n+1}]$.) We will also consider the inclusion $i : X \hookrightarrow \mathbb{P}^{n+1}$. The following result is a Chow analogue of the weak Lefschetz theorem:

**Proposition 2.1.** ([Lew4]) The following diagram is commutative:

$$
\begin{array}{ccc}
CH^\bullet(Z) & \xrightarrow{dj^*} & CH^\bullet(X) \\
\downarrow{\iota^*} & & \uparrow{i^*} \\
CH^\bullet(\mathbb{P}^{n+1})
\end{array}
$$

Using (2.1), the results of this lecture generalize as follows:

**Theorem 2.2.** ([Lew3]) Let $X$ be a general hypersurface of degree $d \geq 3$ in $\mathbb{P}^{n+1}$, and let $k = \lceil \frac{n+1}{d} \rceil$. Let us further assume that $k(n + 2 - k) + 1 - \binom{d+k}{k} \geq 0$. Then there is a smooth subvariety $\Omega_X(k) \subset \text{Grass}\{\mathbb{P}^k's \subset X\}$ of dimension $n - 2k$ such that the following holds for the cylinder map $\Phi_*$:

1. If $n$ is odd, then $\Phi_* : H^{n-2k}(\Omega_X(k), \mathbb{Z}) \to H^n(X, \mathbb{Z})$ is onto,

2. If $n$ is even, then $\Phi_* : H^{n-2k}(\Omega_X(k), \mathbb{Q}) \to H^n(X, \mathbb{Q})$ is onto,

3. $\Phi_* : CH^{n-k}(\Omega_X) \otimes \mathbb{Q} \to \{CH^\bullet(X)/\mathbb{Z}H^*_\mathbb{X}\} \otimes \mathbb{Q}$ is onto, where $H_X := \mathbb{P}^n \cap X$ is a hyperplane section of $X$.

4. $\Phi_* : CH^\bullet_{\text{alg}}(\Omega_X) \to CH^\bullet_{\text{alg}}(X)$ is onto,
(5) \( \Phi_* : CH^{*-k}_{hom}(\Omega_X) \otimes \mathbb{Q} \to CH^{*-k}_{hom}(X) \otimes \mathbb{Q} \) is onto,

(6) \( \Phi_* : Griff^{*-k}(\Omega_X) \otimes \mathbb{Q} \to Griff^{*}(X) \otimes \mathbb{Q} \) is onto.

**Corollary 2.3.** ([Lew3]) Let \( X \) be given as in (2.2). Then:

1. \( CH^r_{alg}(X) = 0 \) for \( r \leq k \) and for \( r \geq n - k + 1 \),

2. \( CH^r_{hom}(X) \otimes \mathbb{Q} = 0 \) for \( r \leq k \) and for \( r \geq n - k + 1 \),

3. \( CH^{k+1}_{alg}(X) \) is finite dimensional\(^{10}\),

4. \( Griff^{k+1}(X) \otimes \mathbb{Q} = Griff^{n-k}(X) \otimes \mathbb{Q} = 0 \).

**Exercise.** Let \( Q_n \subset \mathbb{P}^{n+1} \) be a smooth quadric hypersurface (i.e. \( \text{deg} \ Q_n = 2 \)). Show that \( H^*(Q_n, \mathbb{Q}) \) is generated by algebraic cocyles. [Hint: Let \( \nu_p : Q_n \to \mathbb{P}^n \) be the projection from a general point \( p \in Q_n \). Describe the cohomology of the blow-up \( B_p(Q_n) \) in terms of the cohomology of \( \mathbb{P}^n \) and the variety of lines \( \{ \mathbb{P}^1 \subset Q_n \mid p \in \mathbb{P}^1 \} \) through \( p \).]

**Problem** Note that theorem (2.2) establishes \( GHC(k, n, X) \) for \( X, k \) given there, satisfying the numerical condition in (2.2). In fact, a deformation argument allows one to deduce the same result, even if \( X \) is not general (provided, however, that \( X \) is smooth). Investigate \( GHC(k, n, X) \) for those smooth \( X \) which do not satisfy the numerical condition. For example, the first 5 cases of \( X \) where this condition fails are the following:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( d )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>

\(^{10}\) Finite dimensional means that there is a cycle induced surjective homomorphism \( A \to CH^{k+1}_{alg}(X) \), for some Abelian variety \( A \).
APPENDIX II

Another generalization of lecture two is the following:

**Theorem 2.4.** ([Lew1]) Let $X \subset Z$ be an inclusion of algebraic manifolds with respective dimensions 4, 5. Let $\{C_b\}_{b \in \Omega}$ be a family of curves covering $Z$ where $\dim \Omega = 4$, and set $\Omega_X = \{b \in \Omega \mid \dim C_b \cap X = 1\}$. Assume:

(i) $X$ meets the generic curve $C_b$ transversally in a single point.

(ii) $\text{Codim}_\Omega \Omega_X \geq 2$.

Then $\text{Hodge}^{2,2}(Z, \mathbb{Q}) \Rightarrow \text{Hodge}^{2,2}(X, \mathbb{Q})$.

**Remark.** In the case $\{C_b\}_{b \in \Omega}$ is a family of rational curves, condition (ii) can be dropped.

**Lecture 3: The Method Via the Diagonal Class**

**Motivation** Let $X$ be a projective algebraic manifold of dimension $n$, and consider the diagonal class $\Delta \in \text{CH}^n(X \times X)$. By collecting the coefficients of the polynomials cutting out $X$ in some $\mathbb{P}^N$ [a finite number by Hilbert’s basis theorem], we can assume that $X = X/k$ is defined over a field $k$, of finite transcendence degree over $\mathbb{Q}$. Let $\eta \in \text{Spec}(X/k)$ be the generic point. Then $\mathcal{O}_{X,\eta} = k(\eta) = k(X)$, the function field of $X$. Set $L = k(\eta)$. Then clearly we can view $\Delta$ as defining a family of 0-cycles in $X$, parameterized by $X$, with generic fiber the generic point $\eta = \Delta_\eta = \Delta_\ast(\eta) \in \text{CH}^n(X_L)$. The idea of studying the Chow group via the generic 0-cycle $\{\eta\} \in \text{CH}^n(X_L)$ was suggested by J.-L. Colliot-Thélène, and exploited by S. Bloch and V. Srinivas (see [Blo] and [B-S]). We explain the ideas below.

**Lemma 3.0.**

$$\text{CH}^\bullet(X_L) = \lim_{U \subset X/k} \text{CH}^\bullet(X/k \times_k U)_{\text{Zariski open}}$$

**Proof.** First of all,

$$(X_L)^\bullet = \lim_{U \subset X/k} (X/k \times_k U)^\bullet_{\text{Zariski open}}$$
where \((\cdots)^*\) = points of codimension \(\bullet\), and:

\[
\text{CH}^*(X_L) = \text{Coker}\left( \prod_{x \in (X_L)_{* - 1}} L(x)^\times \xrightarrow{\text{div}} \prod_{x \in (X_L)_{\bullet}} \mathbb{Z} \right),
\]

\[
\text{CH}^*(X \times U) = \text{Coker}\left( \prod_{y \in (X \times U)_{* - 1}} k(y)^\times \xrightarrow{\text{div}} \prod_{y \in (X \times U)_{\bullet}} \mathbb{Z} \right).
\]

Now take the direct limit of the latter expression to get the former.

Now consider now an embedding \(L \hookrightarrow \mathbb{C}\).

**Lemma 3.1.** The kernel of the pullback \(\text{CH}^*(X_L) \to \text{CH}^*(X_\mathbb{C})\) is torsion.

**Proof.** Step I. Suppose that \(K/L\) is a finite extension. Then one has a finite proper map \(X_K \to X_L\), hence a norm \(\text{CH}^*(X_K) \to \text{CH}^*(X_L)\). Thus the kernel of the pullback \(\text{CH}^*(X_L) \to \text{CH}^*(X_K)\) is torsion.

Step II. Let \(\overline{L}\) be the algebraic closure of \(L\) in \(\mathbb{C}\). Then the kernel of the pullback \(\text{CH}^*(X_L) \to \text{CH}^*(X_{\overline{L}})\) is torsion.

**Proof.** Suppose \(\xi \in \text{CH}^*(X_L)\) has the property that \(\xi = 0\) in \(\text{CH}^*(X_{\overline{L}})\). Then by collecting the coefficients in \(\overline{L}\) of polynomials defining \(\xi \sim_{\text{rat}} 0\), we can assume that \(\xi = 0\) in \(\text{CH}^*(X_K)\), for some finite extension \(K/L\). By step I, we are done.

Step III. The kernel of the pullback \(\text{CH}^*(X_{\overline{L}}) \to \text{CH}^*(X_\mathbb{C})\) is zero.

**Proof.** We can write:

\[
\text{CH}^*(X_\mathbb{C}) = \lim_{U/\overline{L} \text{ finite type}} \text{CH}^*(X_{\overline{L}} \times_{\overline{L}} U).
\]

But \(\overline{L}\) is algebraically closed, hence \(U(\overline{L}) \neq \emptyset\). But an \(\overline{L}\)-point of \(U\) gives a section of \(\text{CH}^*(X_{\overline{L}}) \to \text{CH}^*(X_{\overline{L}} \times_{\overline{L}} U)\). Thus \(\xi \mapsto 0 \in \text{CH}^*(X_{\mathbb{C}}) \Rightarrow \xi \mapsto 0 \in \text{CH}^*(X_{\overline{L}} \times_{\overline{L}} U)\) for some \(U/\overline{L}\). Hence \(\xi = 0\) in \(\text{CH}^*(X_{\overline{L}})\).

Now let \(D \subset X\) be a subvariety, and assume that \(\text{CH}^n(X \setminus D)_\mathbb{Q} = 0\). We can view \(X\), \(D\) as defined over a given field \(k\) of finite transcendence degree over \(\mathbb{Q}\), as mentioned earlier. Thus:

\[
\eta = \Delta(\eta) \in \text{CH}^n(X_L) \to \text{CH}^n((X \setminus D)_L)
\]

\[
\Downarrow
\]

\[
\text{CH}^n((X \setminus D)_\mathbb{C})_{\mathbb{Q}} = 0
\]
Hence \( N\{\eta\} \mapsto 0 \in \text{CH}^n((X\setminus D)_L) \) for some \( N \in \mathbb{N} \) by lemma 3.1. By lemma 3.0,
\[
\text{CH}^n((X\setminus D)_L) = \lim_{U \subset X/k} \text{CH}^n((X\setminus D)/_k \times_k U),
\]
hence \( N\Delta \mapsto 0 \in \text{CH}^n((X\setminus D)/_k \times_k U) \) for some \( U \subset X/k \). Set \( E := X\setminus U \). By using the exact sequence:
\[
\begin{align*}
\left\{ \text{Cycles supported on} \right\} & \to \text{CH}^n(X/k \times_k X/k) \\
(D \times X) \cup (X \times E) & \to \text{CH}^n(U \times_k (X\setminus D)/_k) \to 0,
\end{align*}
\]
it follows that \( N\Delta \sim_{\text{rat}} \Gamma_1 + \Gamma_2 \), where \( \Gamma_1 \) is supported on \( D \times X \) and \( \Gamma_2 \) is supported on \( X \times E \).

**Corollary 3.3.** ([B-S] (op. cit.)) Given the notation above.

1. If \( \dim D \leq 3 \), then \( \text{Hodge}^{2,2}(X, \mathbb{Q}) \) holds.

2. If \( \dim D \leq 2 \), then the Abel-Jacobi map (see Appendix I to lecture 1) \( \Phi_2 : z^2_{\text{hom}}(X) \to J^2(X) \) is surjective; and more specifically, \( J^2_2(X) = J^2_2(X), \) i.e. \( \Phi_2(z^2_{\text{alg}}(X)) = J^2(X) \). In particular, \( \text{GHC}(1, 3, X) \) holds.

**Proof.** We have
\[
H^\bullet(X, \mathbb{C}) = N[\Delta], H^\bullet(X, \mathbb{C}) = [\Gamma], H^\bullet(X, \mathbb{C}) + [\Gamma_2], H^\bullet(X, \mathbb{C}).
\]
We work with the diagrams below, where
\[
(E \xrightarrow{\sigma} E \hookrightarrow X, \ D \xrightarrow{\sigma} D \hookrightarrow X),
\]
are desingularizations, together with the projection formula.
\[
\begin{array}{ccc}
\text{\hat{D}} \times X & \xrightarrow{\sigma \times 1} & X \times X \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
\hat{D} & \xrightarrow{j} & X
\end{array}
\]
THREE LECTURES ON THE HODGE CONJECTURE

\[ X \times \tilde{E} \]

\[ \downarrow^{1 \times \sigma} \]

\[ \pi_1 \leftarrow X \times X \rightarrow \pi_2 \]

\[ \text{Pr}_1 \leftarrow \quad \text{Pr}_2 \rightarrow \]

\[ \left[ \tilde{\Gamma}_1 \stackrel{(\sigma \times 1)*}{\rightarrow} \Gamma_1 ; \quad \tilde{\Gamma}_2 \stackrel{(1 \times \sigma)*}{\rightarrow} \Gamma_2 \right] \]

(I)

\[ [\Gamma_1]_* H^\bullet(X) = \text{Pr}_{2,*} \left( [\Gamma_1] \bullet \text{Pr}_1^* H^\bullet(X) \right) \]

\[ = \text{Pr}_{2,*} \left( (\sigma \times 1)_* [\Gamma_1] \bullet \text{Pr}_1^* H^\bullet(X) \right) \]

\[ = \text{Pr}_{2,*} \circ (\sigma \times 1)_*(\tilde{\Gamma}_1) \bullet (\sigma \times 1)^* \circ \text{Pr}_1^* H^\bullet(X) \]

\[ = \pi_{2,*}((\tilde{\Gamma}_1) \bullet \pi_1^* \circ j^* H^\bullet(X)), \text{ use } \left\{ \begin{array}{l} j \circ \pi_1 = \text{Pr}_1 \circ (\sigma \times 1), \\ \pi_2 = \text{Pr}_2 \circ (\sigma \times 1) \end{array} \right. \]

\[ \Rightarrow [\tilde{\Gamma}_1]_* (j^* H^\bullet(X)). \text{ But } j^* H^\bullet(X) \subset H^\bullet(\tilde{D}), \]

hence \[ [\Gamma_1]_* H^\bullet(X) \subset [\tilde{\Gamma}_1]_* H^\bullet(\tilde{D}). \]

[For (II) below, we can assume that \( E \subset X \) is a divisor.]

(II)

\[ [\Gamma_2]_* H^\bullet(X) = \text{Pr}_{2,*} \left( [\Gamma_2] \bullet \text{Pr}_1^* H^\bullet(X) \right) \]

\[ = \text{Pr}_{2,*} \left( (1 \times \sigma)_* [\Gamma_2] \bullet \text{Pr}_1^* H^\bullet(X) \right) \]

\[ = \text{Pr}_{2,*} \circ (1 \times \sigma)_*([\Gamma_2] \bullet (1 \times \sigma)^* \circ \text{Pr}_1^* H^\bullet(X)) \]

\[ = j_* \circ \pi_{2,*}((\tilde{\Gamma}_2) \bullet \pi_1^* H^\bullet(X)), \text{ use } \left\{ \begin{array}{l} j \circ \pi_2 = \text{Pr}_2 \circ (1 \times \sigma), \\ \pi_1 = \text{Pr}_1 \circ (1 \times \sigma) \end{array} \right. \]

\[ \Rightarrow j_* [\tilde{\Gamma}_2]_* H^\bullet(X) \subset j_* H^{\bullet-2}(\tilde{E}), \]

hence \[ [\Gamma_2]_* H^\bullet(X) \subset j_* H^{\bullet-2}(\tilde{E}). \]

Now set \( \bullet = 4 \) in (I) and (II) above. Since the Hodge conjecture holds for projective algebraic manifolds of dimension \( \leq 3 \), together with the Lefschetz (1, 1) theorem, it follows that Hodge\(^2\)(\(X, \mathbb{Q}\)) holds, viz., (3.3)(1) holds. For (3.3)(2), it is a consequence of the above and some functorial properties of the Abel-Jacobi mapping, together with some classical results on divisors and zero-cycles. This will be left to the reader.

Exercise. Let \( X \) be a projective algebraic manifold of dimension \( n \). Suppose that \( \text{CH}^n(X \setminus D)_\mathbb{Q} = 0 \) for some divisor \( D \subset X \). Show
that GHC(1, n, X) holds. Hence using the remark (i) following (3.4) in the Appendix I below, deduce GHC(1, n, X) for any smooth complete intersection \( X \subset \mathbb{P}^{n+r} \) of dimension \( n \), which satisfies \( H^{r,0}(X) = 0 \).

**APPENDIX I**

**Proposition 3.4.** (see [Ro] (§4)) Let \( X \subset \mathbb{P}^{n+1} \) be a smooth hypersurface of degree \( d \leq n + 1 \). [Note: \( d \leq n + 1 \Leftrightarrow H^{n,0}(X) = 0 \).] Then \( \text{CH}^n(X \setminus \{ p \})_Q = 0 \), for any \( p \in X \).

Remarks. (i) A similar story holds for smooth complete intersections \( X \) of dimension \( n \) in \( \mathbb{P}^{n+r} \) with \( H^{n,0}(X) = 0 \) ([Ro] (§4)) and in fact Roitman proves in this case that \( \text{CH}^n_{\text{alg}}(X) = 0 \).

(ii) Proposition (3.4) can also be deduced from Appendix I to lecture 2, so long as \( X \) is general.

**Proof of (3.4).** It suffices to show that \( N \cdot (\{ p - q \}) = 0 \) for any points \( p, q \in X \) and some integer \( N \neq 0 \). First fix a point \( p \in X \). By a PGL action, we may assume that \( p = [1, 0, \ldots, 0] \) in the homogeneous coordinates \([z_0, \ldots, z_{n+1}] \in \mathbb{P}^{n+1}\). Write \( X = \{ F = 0 \} \subset \mathbb{P}^{n+1} \), \( F = F(z_0, \ldots, z_{n+1}) \) homogenous of degree \( d \). Since \( p \in X \), it follows that \( F(z_0, \ldots, z_{n+1}) = \sum_{j=1}^d z_0^{d-j} F_j(z_1, \ldots, z_{n+1}) \), where \( F_j \) is homogenous of degree \( j \) in \( (z_1, \ldots, z_{n+1}) \). Now consider the local affine coordinates \((x_1, \ldots, x_{n+1}) = (\frac{z_1}{z_0}, \ldots, \frac{z_{n+1}}{z_0})\), where \( p \) now corresponds to \((0, \ldots, 0)\).

Write \( f = f(x_1, \ldots, x_{n+1}) = F = \sum_{j=1}^d f_j(x_1, \ldots, x_{n+1}) \), \( f_j = F_j / z_0^j \) homogenous of degree \( j \) in \((x_1, \ldots, x_{n+1})\). Any line \( \ell := \mathbb{P}^1 \subset \mathbb{P}^{n+1} \) passing through \( p \in X \), corresponds in affine coordinates to a line \( \mathbb{C} \subset \mathbb{C}^{n+1} \) passing through \((0, \ldots, 0) \in \{ f = 0 \} \subset \mathbb{C}^{n+1} \). Thus \( \ell_{[v]} \) through \((0, \ldots, 0) \) looks like \( \ell(t) = \{tv | t \in \mathbb{C}\} \), where \( v \in \mathbb{C}^{n+1} \setminus \{0\} \), viz. \([v] \in \mathbb{P}^n\). Note that \( f(tv) = tf_1(v) + \cdots + t^d f_d(v) \). Thus \( \ell \subset X \Leftrightarrow [v] \in \{ f_1 = \cdots = f_d = 0 \} \subset \mathbb{P}^n \), and either \( \ell \subset X \) or \( \ell \cap X = dp\). Hence \( \text{dim} \Sigma_p \geq n - (d - 1) = n + 1 - d \geq 0 \), since \( d \leq n + 1 \); moreover, \( \text{deg} \Sigma_p \leq M \), where \( M = (d-1)! \) [Bezout’s theorem]. By taking hyperplane sections, we can assume that \( \text{dim} \Sigma_p = 0 \). Set \( Z_p = \bigcup_{[v] \in \Sigma_p} \ell_{[v]} \). Then \( Z_p \) is a cone with vertex \( p \), and any point \( p' \in Z_p \), has the property that \( p' \sim \text{rat} p \). Let \( M(p) = \text{deg} Z_p \), \( M(q) = \text{deg} Z_q \), for any given \( p, q \in X \). Then \( M(p), M(q) \leq (d-1)! \) and \( M(p)Z_q - M(q)Z_p \sim 0 = CH_1(\mathbb{P}^{n+1})_{\text{deg}0} \). Thus \( N(\{ p - q \}) = 0 \), for some \( N \leq d((d-1)!)^2 \). [Note: Since \( N \) is bounded, it likewise follows that \( \text{CH}^n_{\text{alg}}(X) = 0 \), using the well-known divisibility of \( \text{CH}^n_{\text{alg}}(X) \).]
APPENDIX II

The method of [B-S] generalizes as follows (see [Pa] for details):

**Proposition 3.5.** Let $X$ be a projective algebraic manifold, $\dim X = n$, and assume given subvarieties $\bar{Y}_j \subset X$ for each $j \in \{0, \ldots, m\}$ such that $\text{CH}_j(X \setminus \bar{Y}_j)_\mathbb{Q} = 0$ for each $j$. Then for each $j$, we have cycles $\Gamma_j \in \text{CH}^n(X \times X)_\mathbb{Q}$ such that $|\Gamma_j| \subset Y_j \times X$, and a cycle $\Gamma^{m+1} \in \text{CH}^n(X \times X)_\mathbb{Q}$ such that $|\Gamma^{m+1}| \subset X \times W$, where $W \subset X$ has pure codimension $m + 1$, so that the diagonal class $\Delta_X \in \text{CH}^n(X \times X)_\mathbb{Q}$ decomposes into:

$$\Delta_X = \Gamma_0 + \cdots + \Gamma_m + \Gamma^{m+1}.$$ 

**Proof.** By induction on $m \geq -1$, the statement is clearly true for $m = -1$, where $\Delta_X = \Gamma^0$. By induction, we can assume given

$$\Delta_X = \Gamma_0 + \cdots + \Gamma_{m-1} + \Gamma^m,$$

where accordingly, $|\Gamma^m| \subset X \times W'$, $W'$ having pure codimension $m$ on $X$. Further, we assume that $\text{CH}_m((X \setminus Y_m)_\mathbb{Q}) = 0$ for some subvariety $Y_m \subset X$. There is no loss of generality in assuming that $W'$ is irreducible, and that $X$, $W'$, $\Gamma^m$ are defined over a subfield $k \subset \mathbb{C}$ with $\text{trdeg}_\mathbb{Q}k < \infty$. Note that $\Gamma^m$, being supported on $X \times W'$, determines a corresponding cycle class $\Gamma^m \in \text{CH}^{n-m}(X \times W')$, and thus a class in $\text{CH}^{n-m}(X_L)$, by restriction to the generic point of $W'/k$, where $L = k(W')$. Next, $\text{CH}^{n-m}(X_L) = \text{CH}_m(X_L)$ and $\text{CH}_m((X \setminus Y_m)_\mathbb{C})_\mathbb{Q} = 0$. Now fix an embedding $L \subset \mathbb{C}$. Using the injection $\text{CH}_m((X \setminus Y_m)_\mathbb{C})_\mathbb{C} \hookrightarrow \text{CH}_m((X \setminus Y_m)_\mathbb{C})_\mathbb{Q}$ (Lemma 3.1), it follows by the same arguments in lecture 3, that in $\text{CH}^n(X \times X)_\mathbb{Q}$, $\Gamma^m = \Gamma_m + \Gamma^{m+1}$ where $|\Gamma_m| \subset Y_m \times X$, and $|\Gamma^{m+1}| \subset X \times W$, where $W$ has pure codimension $m + 1$ in $X$. \hfill $\Box$

**Exercise.** Using Proposition 3.5, deduce the following:

**Corollary 3.6.** Assume given a projective algebraic manifold $X$. Suppose that $\text{CH}_\ell(X)_\mathbb{Q} \simeq \mathbb{Q}$ for all $\ell \in \{0, \ldots, m\}$. Then $H^p(X, \mathbb{Q}) = N^{m+1}H^p(X, \mathbb{Q})$ for all $p \geq 2m + 1$. In particular, $\text{GHC}(m + 1, p, X)$ holds for all $p \geq 2m + 1$.

**Appendix A: The Tate conjecture, absolute Hodge cycles, and some recent developments**

We recall the statement of the Hodge conjecture (classical version), namely: For a smooth projective variety of dimension $n$ over $\mathbb{C}$, the cycle class map

$$\text{cl}_k : \text{CH}^k(X) \otimes \mathbb{Q} \to H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X),$$

holds.
is surjective. By Poincaré duality, the space $H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$ is identified with

\[
\left\{ \gamma \in H_{2n-2k}(X, \mathbb{Q}) \mid \int_{\gamma} \omega = 0, \forall \omega \in F^{n-k+1} H^{2n-2k}(X, \mathbb{C}) \right\}.
\]

In other words, the space of analytic = algebraic (co)-cycles can be "computed" using Hodge theory. Actually as indicated earlier (in a footnote, regarding the exponential short exact sequence), it is more natural to throw in twists by introducing the pure Hodge structure \( \mathbb{Q}(m) := \mathbb{Q}(2\pi\sqrt{-1})^m \) of type \((-m, -m)\), for any given integer \( m \), and in which case the cycle map becomes

\[(*) \quad \text{cl}_k : \text{CH}^k(X) \otimes \mathbb{Q} \to H^{2k}(X, \mathbb{Q}(k)) \cap H^{k,k}(X),\]

which is \((2\pi\sqrt{-1})^k\) times the fundamental class of an algebraic cycle. Thus the image of the cycle class map in \((*)\) lies in the \( \mathbb{Q}(k)\)-classes of Hodge type \((0, 0)\). (Note \( \mathbb{Q}(1) \) is called the Tate twist, but more generally see below.) The Hodge conjecture has an arithmetical analog, namely the Tate conjecture, which we will briefly state. First, we will assume that the reader has some familiarity with \( \ell \)-adic cohomology, albeit only the formal properties of this cohomology are needed here. A good reference for this is the book by Milne [Mil]. As only the formal properties of \( \ell \)-adic cohomology are needed, and for a description of these properties, the reader can also consult Hartshorne’s book [Har](Appendix C). Let \( X \) be a smooth projective variety over a field \( L \), with algebraic closure \( \overline{L} \), \( G = \text{Gal}(\overline{L}/L) \) the Galois group, and \( \overline{X} = X \times_L \overline{L} \). For a prime \( \ell \neq \text{char}(L) \), we consider the \( \ell \)-adic field \( \mathbb{Q}_\ell \), namely the quotient field of the \( \ell \)-adic integers

\[ \mathbb{Z}_\ell := \lim_{\leftarrow} \mathbb{Z}/\ell^n \mathbb{Z}, \]

and the \( \ell \)-adic cohomology

\[ H^i_{\text{et}}(\overline{X}, \mathbb{Q}_\ell) = \left( \lim_{\leftarrow} H^i_{\text{et}}(\overline{X}, \mathbb{Z}/\ell^n \mathbb{Z}) \right) \otimes \mathbb{Q}_\ell. \]

As in the transcendental case, there is a cycle class map:

\[ \text{cl}_k : \text{CH}^k(X) \otimes \mathbb{Q}_\ell \to H^{2k}_{\text{et}}(\overline{X}, \mathbb{Q}_\ell), \]
and likewise, it is more natural to modify the weights by introducing the Tate twist\(^{11}\) (see [Ta1]), viz.,

\[
\text{cl}_k : \text{CH}^k(X) \otimes \mathbb{Q}_\ell \rightarrow H^{2k}_{\text{et}}(\overline{X}, \mathbb{Q}_\ell(k)),
\]

whose image lies in the subspace \(H^{2k}_{\text{et}}(\overline{X}, \mathbb{Q}_\ell(k))^G\) of classes invariant under \(G\). The celebrated Tate conjecture [Ta1] asserts that for a finitely generated field \(L\) (i.e. finitely generated over the prime field), the cycle class map

\[
\text{cl}_k : \text{CH}^k(X) \otimes \mathbb{Q}_\ell \rightarrow H^{2k}_{\text{et}}(\overline{X}, \mathbb{Q}_\ell(k))^G,
\]

is surjective.

**Exercise.** Let \(X \subset \mathbb{P}^5\) be a general quintic fourfold over \(\mathbb{C}\). Note that \(X\) can be defined over a field of finite transcendence degree over \(\mathbb{Q}\). Formulate a version of Tate conjecture for \(X\) in terms of the Fano variety of lines on \(X\). (Warning: Unlike the Hodge conjecture, the Tate conjecture is not known in general for surfaces.)

A survey of the status of the Tate conjecture as of 1994 can be found in [Ta2]. Some special cases mentioned there are the following. In the case of *divisors* (i.e. codimension 1 cycles), the conjecture has been verified in a number of instances, such as for Abelian varieties (Tate, Zarhin in characteristic \(p > 0\); Faltings in characteristic 0, as part of his proof of the Mordell conjecture), for \(K3\) surfaces in characteristic 0 (and with some restrictions in characteristic \(p > 0\)), and for various types of modular surfaces and threefolds. For higher codimensional cycles, the conjecture has been verified for a number of Fermat hypersurfaces (due to Tate, Shioda) as well as for many classes of Abelian varieties (due to Tate, Tankeev, Murty, Shioda, et al). The reader is encouraged to consult the references cited there ([Ta2]).

It is general "yoga" that the Hodge and Tate conjectures are like opposite sides of the same coin. That yoga abounds, for indeed there are a number of examples in the literature where the Hodge and Tate conjectures can be verified simultaneously. For example, from the work of [P], it is known that for an Abelian variety \(X\) defined over a finitely generated subfield \(L \subset \mathbb{C}\), the Tate conjecture implies the Hodge conjecture. Also, as another example, it is shown in [Mi3] that if the Hodge conjecture holds for all Abelian varieties of CM type over \(\mathbb{C}\), then the

\[^{11}\text{Set } \mathbb{Q}_\ell(1) := \mathbb{Z}_\ell(1) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \text{ where } \mathbb{Z}_\ell(1) := \lim_{\leftarrow} \mu_{\ell^n}, \]

and where \(\mu_{\ell^n}\) are the \(\ell^n\)-th roots of unity in \(\overline{\mathbb{L}}\), and which has a natural \(G\) action. Then the definition of twisted \(\ell\)-adic cohomology can be taken to be: \(H^{2k}_{\text{et}}(\overline{X}, \mathbb{Q}_\ell(k)) = H^{2k}_{\text{et}}(\overline{X}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(1)^{\otimes k}\). Note: \(\mathbb{Q}_\ell(m)\) for \(m < 0\) is also defined ([Ta1]); albeit we do not need it here.
Tate conjecture holds for all Abelian varieties defined over the algebraic closure of a finite field. Another case at point is the comparison of both conjectures in [Hu](section 8.9), making use of the absolute Hodge cycles, which we will now describe.

One way of incorporating the analytic and arithmetic points of view is via Deligne’s notion of absolute Hodge cycles. Let us now assume that char$(L) = 0$, with $X$ defined over $L$, and from now on, we will assume that $L = \overline{L}$ is algebraically closed and of finite transcendence degree over $\mathbb{Q}$. Thus there is an embedding $\sigma : L \hookrightarrow \mathbb{C}$. The comparison isomorphism theorems give

\begin{equation}
(A.0) \quad H^i(X \times_{L,\sigma} \mathbb{C}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong H^i_{\text{et}}(X \times_{L,\sigma} \mathbb{C}, \mathbb{Q}_\ell) \cong H^i_{\text{et}}(X, \mathbb{Q}_\ell),
\end{equation}

and

\begin{equation}
(A.1) \quad H^i(X \times_{L,\sigma} \mathbb{C}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^i_{\text{DR}}(X \times_{L,\sigma} \mathbb{C}, \mathbb{C}) \cong H^i_{\text{DR}}(X/L) \otimes_{L,\sigma} \mathbb{C},
\end{equation}

where $H^i(X \times_{L,\sigma} \mathbb{C}, \mathbb{Q})$ is singular cohomology, and $H^i_{\text{DR}}(X/L)$ is algebraic de Rham cohomology (see [DMOS] for a definition), and where $H^i_{\text{DR}}(X/\mathbb{Q}) := H^i_{\text{DR}}(X/\mathbb{Q}, \mathbb{C})$ agrees with ordinary de Rham cohomology. Now consider the products

\begin{equation}
(A.2) \quad H^i_{\text{DR}}(X \times_{L,\sigma} \mathbb{C}) \times \left( \prod_\ell H^i_{\text{et}}(X \times_{L,\sigma} \mathbb{C}, \mathbb{Q}_\ell) \right),
\end{equation}

\begin{equation}
(A.3) \quad H^i_{\text{DR}}(X/L) \otimes_{L,\sigma} \mathbb{C} \times \left( \prod_\ell H^i_{\text{et}}(X, \mathbb{Q}_\ell) \right).
\end{equation}

From (A.0) and (A.1), one has the following data:

(i) An isomorphism which we will denote by $\sigma^* = \sigma^*_{\text{DR}} \times \sigma^*_{\text{et}}$:

$$\sigma^* : H^i_{\text{DR}}(X/L) \otimes_{L,\sigma} \mathbb{C} \times \left( \prod_\ell H^i_{\text{et}}(X, \mathbb{Q}_\ell) \right) \cong H^i_{\text{DR}}(X \times_{L,\sigma} \mathbb{C}) \times \left( \prod_\ell H^i_{\text{et}}(X \times_{L,\sigma} \mathbb{C}, \mathbb{Q}_\ell) \right),$$

and

(ii) a diagonal embedding

$$H^i(X \times_{L,\sigma} \mathbb{C}, \mathbb{Q}) \hookrightarrow H^i_{\text{DR}}(X \times_{L,\sigma} \mathbb{C}) \times \left( \prod_\ell H^i_{\text{et}}(X \times_{L,\sigma} \mathbb{C}, \mathbb{Q}_\ell) \right).$$

Now roughly speaking, an absolute Hodge cycle relative to $\sigma$ is a class $\xi \in H^{2k}_{\text{DR}}(X/L) \times \left( \prod_\ell H^{2k}_{\text{et}}(X, \mathbb{Q}_\ell) \right)$, for which $\sigma^*(\xi)$ belongs to the
rational subspace $H^{2k}(X_k, \mathbb{C}, \mathbb{Q})$ and for which the de Rham component of $\sigma^*(\xi)$ lies in $F^{k}H^{2k}_{\text{DR}}(X \times_{L, \sigma} \mathbb{C})$. A precise statement of an absolute Hodge cycle relative to $\sigma$, involves twists. Thus to state this precisely, we briefly introduce these twists. For an integer $m \geq 1$, let $\mu_m(L) = \{ \zeta \in L \mid \zeta^m = 1 \}$, and put
$$A^f(1) = (\lim_{\leftarrow} \mu_m(L)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$ One sets $A^f(m) = A^f(1)^{\otimes m}$. There is a natural extension of $A^f(m)$ for negative $m$, but we will omit this here ([MDOS] (p. 19)). (Here $A^f(m)$ is a module over the so-called ring $A^f$ of finite adèles for $\mathbb{Q}$.) Now put
$$H^i_{\text{et}, f}(X) = \prod_{\ell} H^i_{\text{et}}(X, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} \mathbb{Q},$$
and
$$H^i_{\text{et}, f}(X)(m) = H^i_{\text{et}, f}(X) \otimes_{A^f} A^f(m).$$ Now with twists thrown into the picture, we have the following data:

(i) An isomorphism
$$\sigma^*: (H^i_{\text{DR}}(X/ L)(m) \otimes_{L, \sigma} \mathbb{C}) \times H^i_{\text{et}, f}(X)(m) \cong (H^i_{\text{DR}}(X \times_{L, \sigma} \mathbb{C})(m) \otimes_{L, \sigma} \mathbb{C}) \times H^i_{\text{et}, f}(X \times_{L, \sigma} \mathbb{C})(m).$$

(ii) An embedding
$$H^i(X \times_{k, \sigma} \mathbb{C}, \mathbb{Q}(m)) \hookrightarrow (H^i_{\text{DR}}(X \times_{L, \sigma} \mathbb{C})(m) \otimes_{L, \sigma} \mathbb{C}) \times H^i_{\text{et}, f}(X \times_{L, \sigma} \mathbb{C})(m).$$

**Definition A.4.** A class
$$\xi \in (H^i_{\text{DR}}(X/ L)(m)) \times H^i_{\text{et}, f}(X)(m)$$
is called a Hodge cycle relative to $\sigma$, if $\sigma^*(\xi)$ is lies in the rational subspace $H^{2k}(X \times_{L, \sigma} \mathbb{C}, \mathbb{Q}(k))$, and the de Rham component of $\xi$ is contained in $F^{k}H^{2k}_{\text{DR}}(X \times_{L, \sigma} \mathbb{C})(k) = F^{k}H^{2k}_{\text{DR}}(X \times_{L, \sigma} \mathbb{C})$. If $\xi$ is a Hodge cycle relative to all embeddings $\sigma : L \hookrightarrow \mathbb{C}$, then it is called an absolute Hodge cycle.

Note that any algebraic cycle on $X/ L$ is an absolute Hodge cycle and conversely the Hodge conjecture implies that all absolute Hodge cycles are algebraic cycles.

It is anticipated that every Hodge cycle is an absolute Hodge cycle, i.e. if $\xi$ is a Hodge cycle relative to one embedding $\sigma : L \hookrightarrow \mathbb{C}$, then it is a Hodge cycle relative to all embeddings $\sigma$. So far, this has been proven by Deligne in the case where $X$ an Abelian variety [MDOS].

Next, we want to discuss some recent developments on the Hodge conjecture since the appearance of the survey book [Lew2]. For recent works on Abelian varieties, the reader can consult for example [Mur],
[Ge], [Ge-V], [Mi2], [Mi3], [M-Z1], [M-Z2], [Ha] and [Ab1] - [Ab6], one of the more interesting developments that we wish to explain is the recent work of Abdulali [Ab6]. To explain his result, we introduce the following terminology. Let $V = V_\mathbb{Q}$ be a finite dimensional $\mathbb{Q}$-vector space with Hodge structure

$$V_{\mathbb{C}} := V_\mathbb{Q} \otimes \mathbb{C} = \bigoplus_{p+q = N} V^{p,q},$$

of weight $N$. The Hodge structure $V$ is said to be geometric if it is isomorphic to a Hodge substructure of the cohomology of a smooth projective variety over $\mathbb{C}$, and effective if $V^{p,q} = 0$ unless $p, q \geq 0$. Let $m$ be an integer. The twisted Hodge structure $V(m) = V_\mathbb{Q} \otimes \mathbb{Q}(2\pi \sqrt{-1})^m$ is the Hodge structure of weight $N - 2m$ with $V(m)^{p,q} = V^{p-m,q-m}$. The Grothendieck amended Hodge conjecture implies that the twist $V(m)$ of any geometric Hodge structure $V$, is still geometric provided that $V(m)$ is effective. We say that a smooth complex projective variety $X$ is dominated by a class of varieties $\mathcal{Y}$ if, for every irreducible Hodge structure $W$ occurring in the cohomology of $X$, the twist $W(r)$ occurs in the cohomology of some member of $\mathcal{Y}$, where if $N$ is the weight of the Hodge structure $W$, then

$$r := \min \left\{ p \mid W^{p,N-p} \neq 0 \right\},$$

is the height of the Hodge structure. As observed by Grothendieck [Gro], the (always assumed amended) general Hodge conjecture for $X$ is implied by the existence of a class of varieties $\mathcal{Y}$ which dominate $X$, together with the classical Hodge conjecture for $Y \times X$ for all $Y \in \mathcal{Y}$.

Exercise. Prove this!

Abdulali shows in a series of papers ([Ab2], [Ab4], [Ab5]) that certain Abelian varieties are dominated by subclasses of the class of all Abelian varieties, and he used this to deduce the general Hodge conjecture in some cases. This raises the question as to whether every Abelian variety is dominated by the class of all Abelian varieties. Abdulali [Ab6] shows that the answer to this question is no. As indicated in his paper, this suggests that those extraordinary Hodge structures that he constructs can be used as a testing ground for the general Hodge conjecture.

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THREE LECTURES ON THE HODGE CONJECTURE

REFERENCES


THREE LECTURES ON THE HODGE CONJECTURE


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