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MEAN SQUARE CONVERGENCE OF STOCHASTIC θ-METHODS FOR NONLINEAR NEUTRAL STOCHASTIC DIFFERENTIAL DELAY EQUATIONS

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Abstract. This paper is devoted to the convergence analysis of stochastic θ-methods for nonlinear neutral stochastic differential delay equations (NSDDEs) in Itô sense. The basic idea is to reformulate the original problem eliminating the dependence on the differentiation of the solution in the past values, which leads to a stochastic differential algebraic system. Drift-implicit stochastic θ-methods are proposed for the coupled system. It is shown that the stochastic θ-methods are mean-square convergent with order $\frac{1}{2}$ for Lipschitz continuous coefficients of underlying NSDDEs. A nonlinear numerical example illustrates the theoretical results.

Key words. neutral stochastic differential delay equations, mean-square continuity, stochastic θ-methods, mean-square convergence, consistency

1. Introduction

Neutral delay differential equations (NDDEs) have found diverse applications in many fields such as control theory, oscillation theory, electrodynamics, biomathematics, and medical sciences. NDDEs arise in two formulations, explicit and implicit. Explicit NDDEs share the form of

$$x'(t) = F(t, x(t), x(t-\tau(t)), x'(t-\tau(t))), \quad t \in [t_0, T], \tag{1}$$

$$x(t) = \phi_0(t), x'(t) = \phi_1(t), \quad t \le t_0, \tag{2}$$

where $\tau(t) \ge 0$. Implicit NDDEs share the form of

$$(x(t) - D(t, x(t), x(t-\tau(t)))' = F(t, x(t), x(t-\tau(t))), \quad t \in [t_0, T], \tag{3}$$

$$x(t) = \phi_0(t), x'(t) = \phi_1(t), \quad t \le t_0, \tag{4}$$

which is also called "Hale's form". (3) can be formally rewritten as (1). However, under careful scrutiny, one may find that equation (3) is not necessarily equivalent to (1) even if D and τ are differentiable, noting that a non-differentiable function is probably a solution of (3). Therefore, the study of the explicit and implicit forms, and their available numerical methods differ. A stability analysis of both the exact solutions and the numerical approximations for explicit NDDEs has been presented in [3]. For the case of implicit NDDEs, Vermiglio and Torelli [23] investigated the stability of analytical solutions and the numerical approximations. The idea, which is to reformulate the original problem eliminating the dependence on the derivative of the solution in the last value, was presented both in [3] and [23]. There is an extensive literature on numerical schemes for NDDEs (see, for example, [2, 4, 8, 14]).

Many physical phenomena can be modelled by stochastic dynamical systems whose evolution in time is governed by random forces as well as intrinsic dependence of the state on a finite part of its past history. Such models may be identified as stochastic functional differential equations (SFDEs). The theory of SFDEs has been well developed and there is an extensive literature (see, for example, [18, 15]).

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The numerical methods on SFDEs have also been well established. Buckwar [6] discussed the strong convergence of drift-implicit one-step schemes to the solution of SFDEs. Fan, Liu and Cao [7] obtained sufficient conditions for the existence and uniqueness of solutions of stochastic pantograph equations and investigated the convergence of semi-implicit Euler method for the stochastic pantograph equations. There are other papers on numerical methods for SFDEs (see, for example, [5, 9, 16]). However, they are not on numerical methods for neutral SFDEs. A comprehensive introduction to numerical stochastic ordinary differential equations is given in the books by Allen [1], Kloeden and Platen [10], Kloeden et al [11], and Milstein and Tretyakov [17], and the surveys of Schurz [19], [20], [21] and Talay [22].

Motivated by chemical engineering systems and the theory of aeroelasticity, Kolmanovskii et al [12, 13] introduced a class of neutral stochastic functional differential equations. Mao [15] investigated existence and uniqueness, moment and pathwise estimates, exponential stability of neutral stochastic functional differential equation

$$d[x(t) - D(x_t)] = F(t, x_t)dt + G(t, x_t)dW(t). \tag{5}$$

and a special case of (5), that is, neutral stochastic differential delay equation

$$d[x(t) - D(x(t-\tau))] = F(t, x(t), x(t-\tau))dt + G(t, x(t), x(t-\tau))dW(t). \tag{6}$$

To our best knowledge, no results on convergence of numerical methods for (5) and (6) have been presented in the literature. This paper is devoted to the convergence analysis of the stochastic θ-methods for nonlinear neutral stochastic differential delay equations (6). The basic idea is to transfer the system (6) into a stochastic ordinary differential system plus a functional recursion and then eliminate the dependence on the differentiation of the solution in the past values, which leads to a stochastic differential algebraic system. A drift-implicit stochastic θ-method is proposed for the coupled system. It is shown that these stochastic θ-methods are mean-square convergent with order $\frac{1}{2}$ under the usual smoothness assumptions. A nonlinear numerical example illustrates the theoretical results.

2. Neutral stochastic differential delay equations

Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^d$ and $\langle x, y\rangle$ be the Euclidean inner product of vectors $x, y \in \mathbb{R}^d$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$.

Assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is increasing and right-continuous, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). $W(t) = (W_1(t), \ldots, W_m(t))^T$ is supposed to be a standard m-dimensional Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with mutually independent coordinates W_i throughout the paper.

Furthermore, let $0 \leq t_0 < T < \infty$, $\mathcal{B}^d$ be the Borel σ-algebra and

$$F : [t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \quad G : [t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m} \text{ and } \ D : \mathbb{R}^d \to \mathbb{R}^d$$

be all Borel measurable real-valued functions. Consider the d-dimensional neutral stochastic differential delay equations (NSDDEs) in Itô-sense

$$d[x(t) - D(x(t-\tau))] = F(t, x(t), x(t-\tau))dt + G(t, x(t), x(t-\tau))dW(t), \ t \in [t_0, T] \tag{7}$$

with initial data

$$x(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0]. \tag{8}$$

$\varphi(t)$ is assumed to be continuous, $(\mathcal{F}_{t_0}, \mathcal{B}^d)$-measurable with $\mathbb{E}(\sup_{t_0-\tau\le t\le t_0} |\varphi(t)|^2) < +\infty$, where $\tau > 0$ is a constant. By the definition of Itô-interpreted stochastic differential equations, equation (7) means that, for every $t_0 \le t \le T$, we have

$$x(t)-D(x(t-\tau)) = x(t_0)-D(x(t_0-\tau)) + \int_{t_0}^t F(s, x(s), x(s-\tau))ds + \int_{t_0}^t G(s, x(s), x(s-\tau))dW(s). \tag{9}$$

Throughout this paper, we assume that the functions F, G and D satisfy the following conditions: there exist positive constants K_L, K_B such that, for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ and $t \in [t_0, T]$, we have

$$|F(t,x_1,y_1) - F(t,x_2,y_2)|^2 \vee |G(t,x_1,y_1) - F(t,x_2,y_2)|^2 \le K_L(|x_1-x_2|^2 + |y_1-y_2|^2), \tag{10}$$

and, for all $x, y \in \mathbb{R}^d$ and $t \in [t_0, T]$,

$$|F(t,x,y)|^2 \vee |G(t,x,y)|^2 \le K_B(1 + |x|^2 + |y|^2) \tag{11}$$

and there is a $\kappa \in (0,1)$ such that for all $x, y_1, y_2 \in \mathbb{R}^d$

$$|D(y_1) - D(y_2)| \le \kappa |y_1 - y_2|, \tag{12}$$

$$|D(x)| \le \kappa |x|, \tag{13}$$

where $a \vee b := \max\{a, b\}$.

Proposition 2.1. *([15]) If the conditions (10), (11) and (12) are fulfilled, then there is a unique solution $x(t)$ to equation (7) with initial data (8). Moreover, the solution belongs to $\mathcal{M}^2([t_0-\tau, T]; \mathbb{R}^d)$, that is, $\mathbb{E}\int_{t_0-\tau}^T |x(t)|^2 dt < \infty$.*

Proposition 2.2. *([15]) Let $p \ge 2$. If the conditions (11) and (13) are fulfilled, then we have*

$$\mathbb{E}\left(\sup_{t_0-\tau\le s\le t} |x(t)|^p\right) \le C_B\Big(1 + \mathbb{E}\left(\sup_{t_0-\tau\le t\le t_0} |\varphi(t)|^p\right)\Big), \quad t \in [t_0, T], \tag{14}$$

where constant C_B depends on K_B, κ and T.

Lemma 2.1. *Let conditions (11) - (12) hold. Assume that the initial function $\varphi(t)$ is uniformly Lipschitz L^2-continuous, that is there is a positive constant C_1 such that*

$$\mathbb{E}|\varphi(u_2) - \varphi(u_1)|^2 \le C_1(u_2 - u_1) \quad \text{if} \;\; t_0 - \tau \le u_1 < u_2 \le t_0. \tag{15}$$

Then

$$\mathbb{E}|x(t) - x(s)|^2 \le C_2(t - s) \tag{16}$$

for all $t_0 \le s < t \le T$ with $t - m\tau \in [-\tau, 0], s - m\tau \in [-\tau, 0]$, where constant C_2 depends on constants C_1, T, initial function $\varphi(t)$ and positive integer m.

Proof. By (9), we have

$$\mathbb{E}|x(t)-x(s)|^2 \le 3\Bigg(\mathbb{E}|D(x(t-\tau)) - D(x(s-\tau))|^2 + \mathbb{E}|\int_s^t F(u, x(u), x(u-\tau))du|^2 + \mathbb{E}|\int_s^t G(u, x(u), x(u-\tau))dW(u)|^2\Bigg). \tag{17}$$

Using the Hölder inequality, the properties of Itô integral, conditions (11)- (12) and Proposition 2.2, we can then obtain that

$$\begin{aligned}
& \mathbb{E}|x(t)-x(s)|^2 \\
\le\ & 3(\kappa^2\mathbb{E}|x(t-\tau)-x(s-\tau)|^2+(t-s)\mathbb{E}(\int_s^t|F(u,x(u),x(u-\tau))|^2du) \\
& +\mathbb{E}(\int_s^t|G(u,x(u),x(u-t))|^2du)) \\
\le\ & 3(\kappa^2\mathbb{E}|x(t-\tau)-x(s-\tau)|^2+(t-s+1)K_B\int_s^t(1+\mathbb{E}|x(u)|^2+\mathbb{E}|x(u-\tau)|^2)du) \\
\le\ & 3(\kappa^2\mathbb{E}|x(t-\tau)-x(s-\tau)|^2+(t-s+1)K_B\int_s^t(1+2C_B(1+\mathbb{E}(\sup_{t_0-\tau\le t\le t_0}|\varphi(t)|^2)))du) \\
\le\ & 3(\kappa^2\mathbb{E}|x(t-\tau)-x(s-\tau)|^2+(T-t_0+1)K_B(1+2C_B)(1+\mathbb{E}(\sup_{t_0-\tau\le t\le t_0}|\varphi(t)|^2))(t-s)) \\
\le\ & C_3(\mathbb{E}|x(t-\tau)-x(s-\tau)|^2+t-s) \\
\le\ & C_3(C_3(\mathbb{E}|x(t-2\tau)-x(s-2\tau)|^2+t-s)+t-s) \\
\le\ & \ldots \\
\le\ & C_3^m(\mathbb{E}|x(t-m\tau)-x(s-m\tau)|^2)+C_3(1+C_3+\ldots+C_3^{m-1})(t-s) \\
\le\ & C_3^mC_1(t-s)+C_3(1+C_3+\ldots+C_3^{m-1})(t-s) \\
\le\ & C_2(t-s),
\end{aligned}$$

where

$$\begin{aligned}
C_3 &= 3\max\{\kappa^2,(T-t_0+1)K_B(1+2C_B)(1+\mathbb{E}(\sup_{t_0-\tau\le t\le t_0}|\varphi(t)|^2))\}, \\
C_2 &= C_3^mC_1+C_3(1+C_3+\ldots+C_3^{m-1}).
\end{aligned}$$

Therefore, this lemma is proved. □

3. Stochastic θ-methods

By defining the function

$$y(t)=x(t)-D(x(t-\tau)),\quad t\in[t_0,T], \tag{18}$$

equation (7) can be rewritten to as

$$dy(t)=f(t,y(t),x(t-\tau))dt+g(t,y(t),x(t-\tau))dW(t),\quad t\in[t_0,T], \tag{19}$$

$$x(t)=y(t)+D(x(t-\tau)),\quad t\in[t_0,T] \tag{20}$$

with initial value

$$y(t_0)=x(t_0)-D(x(t_0-\tau)), \tag{21}$$

where

$$f(t,y,z)=F(t,y+D(z),z),\quad g(t,y,z)=G(t,y+D(z),z). \tag{22}$$

Let $h>0$ be a given step size satisfying

$$hN_\tau=\tau\quad\text{for some positive integer } N_\tau>\tau.$$

For convenience we can assume $N_\tau\ge 2$. We define a family of meshes with a uniform step size h on the interval $[t_0,T]$ by $t_n=t_0+nh, n=0,1,\ldots,N, t_0+Nh\le T$. Consider a stochastic θ-method for initial value problems (19)- (22)

$$\begin{aligned}
y_{n+1} &= y_n+h((1-\theta)f(t_n,y_n,x_{n-N_\tau})+\theta f(t_{n+1},y_{n+1},x_{n+1-N_\tau})) \\
&\quad +g(t_n,y_n,x_{n-N_\tau})\Delta W_n,\quad n=0,1,\ldots,N-1, && (23)\\
x_n &= y_n+D(x_{n-N_\tau}),\quad n=0,1,\ldots,N, && (24)
\end{aligned}$$

where y_n, x_n are strong approximations to $y(t_n)$ and $x(t_n)$, respectively, $\Delta W_n = W_{n+1} - W_n$, and the initial values are given by $y_0 = y(t_0), x_{n-N_\tau} = \varphi(t_0 + t_n - \tau)$ for $n - N_\tau \leq 0$.

We will provide estimates of the local error, which is defined as the defect that is obtained when the exact solution values are inserted into the numerical scheme (23), that is,

$$\begin{aligned} \delta_h(t_n) &= y(t_{n+1}) - y(t_n) - h(1-\theta)f(t_n, y(t_n), x(t_n - \tau)) \\ &\quad -h\theta f(t_{n+1}, y(t_{n+1}), x(t_{n+1} - \tau)) - g(t_n, y(t_n), x(t_n - \tau))\Delta W_n. \end{aligned} \tag{25}$$

Definition 3.1. *The method (23)-(24) is said to be mean-square consistent with order p ($p > 0$) if the following estimates hold:*

$$\max_{0\leq n\leq N-1} \|\mathbb{E}(\delta_h(t_n)|\mathcal{F}_{t_n})\|_{L_2} \leq \bar{C}h^{p+1} \quad \text{as } h = \frac{\tau}{N_\tau} \to 0, \tag{26}$$

and

$$\max_{0\leq n\leq N-1} \|\delta_h(t_n)\|_{L_2} \leq \bar{C}h^{p+\frac{1}{2}} \quad \text{as } h = \frac{\tau}{N_\tau} \to 0, \tag{27}$$

where the constant $\bar{C}$ does not depend on h, but may depend on T and the initial data. Here $\|z\|_{L_2} := (\mathbb{E}|z|^2)^{1/2}$.

Definition 3.2. *The method (23)-(24) is said to be mean-square convergent, with order p ($p > 0$), on the mesh-points, when*

$$\max_{0\leq n\leq N} \|x(t_n) - x_n\|_{L_2} \leq \tilde{C}h^p \quad \text{as } h = \frac{\tau}{N_\tau} \to 0, \tag{28}$$

where the constant $\tilde{C}$ does not depend on h, but may depend on T and the initial data.

Theorem 3.1. *Assume that the conditions (10)- (13) and (15) hold and there exists a positive constant $\bar{K}$ such that for any $s, t \in [t_0, T], x, y \in R^d$*

$$|F(s,x,y) - F(t,x,y)|^2 \vee |G(s,x,y) - G(t,x,y)|^2 \leq \bar{K}(1 + |x|^2 + |y|^2)|s-t|, \tag{29}$$

then the method (23) is mean-square consistent with order $p = \frac{1}{2}$.

Proof. It follows from (25) that

$$\begin{aligned} \delta_h(t_n) &= \int_{t_n}^{t_{n+1}} (f(s, y(s), x(s-\tau)) - f(t_n, y(t_n), x(t_n - \tau)))ds \\ &\quad + \int_{t_n}^{t_{n+1}} (g(s, y(s), x(s-\tau)) - g(t_n, y(t_n), x(t_n - \tau)))dW(s) \\ &\quad -\theta h(f(t_{n+1}, y(t_{n+1}), x(t_{n+1} - \tau)) - f(t_n, y(t_n), x(t_n - \tau))) \\ &= \int_{t_n}^{t_{n+1}} (F(s, x(s), x(s-\tau)) - F(t_n, x(t_n), x(t_n - \tau)))ds \\ &\quad + \int_{t_n}^{t_{n+1}} (G(s, x(s), x(s-\tau)) - G(t_n, x(t_n), x(t_n - \tau)))dW(s) \\ &\quad -\theta h(F(t_{n+1}, x(t_{n+1}), x(t_{n+1} - \tau)) - F(t_n, x(t_n), x(t_n - \tau))). \end{aligned} \tag{30}$$

Using the properties of the Itô integral and conditional expectations, and the Hölder inequality, we obtain

$$\begin{aligned}
&|\mathbb{E}(\delta_h(t_n)|\mathcal{F}_{t_n})|^2 \\
&= |\mathbb{E}(\int_{t_n}^{t_{n+1}}(F(s,x(s),x(s-\tau))-F(t_n,x(t_n),x(t_n-\tau)))ds|\mathcal{F}_{t_n}) \\
&\quad -\theta h\mathbb{E}(F(t_{n+1},x(t_{n+1}),x(t_{n+1}-\tau))-F(t_n,x(t_n),x(t_n-\tau))|\mathcal{F}_{t_n})|^2 \\
&\le 2|\mathbb{E}(\int_{t_n}^{t_{n+1}}(F(s,x(s),x(s-\tau))-F(t_n,x(t_n),x(t_n-\tau)))ds|\mathcal{F}_{t_n})|^2 \\
&\quad +2\theta^2h^2|\mathbb{E}(F(t_{n+1},x(t_{n+1}),x(t_{n+1}-\tau))-F(t_n,x(t_n),x(t_n-\tau))|\mathcal{F}_{t_n})|^2 \\
&\le 2\mathbb{E}(|\int_{t_n}^{t_{n+1}}F(s,x(s),x(s-\tau))-F(t_n,x(t_n),x(t_n-\tau))ds|^2|\mathcal{F}_{t_n}) \\
&\quad +2\theta^2h^2\mathbb{E}(|F(t_{n+1},x(t_{n+1}),x(t_{n+1}-\tau))-F(t_n,x(t_n),x(t_n-\tau))|^2|\mathcal{F}_{t_n})) \\
&\le 2h\mathbb{E}(\int_{t_n}^{t_{n+1}}|F(s,x(s),x(s-\tau))-F(t_n,x(t_n),x(t_n-\tau))|^2ds|\mathcal{F}_{t_n}) \\
&\quad +2\theta^2h^2\mathbb{E}(|F(t_{n+1},x(t_{n+1}),x(t_{n+1}-\tau))-F(t_n,x(t_n),x(t_n-\tau))|^2|\mathcal{F}_{t_n}))
\end{aligned}$$

By condition (10) and (29), we have

$$\begin{aligned}
&|\mathbb{E}(\delta_h(t_n)|\mathcal{F}_{t_n})|^2 \\
&\le 4h\mathbb{E}(\int_{t_n}^{t_{n+1}}|F(s,x(s),x(s-\tau))-F(t_n,x(s),x(s-\tau))|^2ds|\mathcal{F}_{t_n}) \\
&\quad +4h\mathbb{E}(\int_{t_n}^{t_{n+1}}|F(t_n,x(s),x(s-\tau))-F(t_n,x(t_n),x(t_n-\tau))|^2ds|\mathcal{F}_{t_n}) \\
&\quad +4\theta^2h^2\mathbb{E}(|F(t_{n+1},x(t_{n+1}),x(t_{n+1}-\tau))-F(t_n,x(t_{n+1}),x(t_{n+1}-\tau))|^2|\mathcal{F}_{t_n})) \\
&\quad +4\theta^2h^2\mathbb{E}(|F(t_n,x(t_{n+1}),x(t_{n+1}-\tau))-F(t_n,x(t_n),x(t_n-\tau))|^2|\mathcal{F}_{t_n})) \\
&\le 4\bar{K}h\mathbb{E}(\int_{t_n}^{t_{n+1}}(1+|x(s)|^2+|x(s-\tau)|^2)(s-t_n)ds|\mathcal{F}_{t_n}) \\
&\quad +4K_Lh\mathbb{E}(\int_{t_n}^{t_{n+1}}(|x(s)-x(t_n)|^2+|x(s-\tau)-x(t_n-\tau)|^2)ds|\mathcal{F}_{t_n}) \\
&\quad +4\bar{K}\theta^2h^3\mathbb{E}(1+|x(t_{n+1})|^2+|x(t_{n+1}-\tau)|^2)|\mathcal{F}_{t_n})) \\
&\quad +4K_L\theta^2h^2\mathbb{E}(|x(t_{n+1})-x(t_n)|^2+|x(t_{n+1}-\tau)-x(t_n-\tau)|^2)|\mathcal{F}_{t_n})). \qquad (31)
\end{aligned}$$

Using Proposition 2.2 and Lemma 2.1, we arrive at

$$\begin{aligned}
\mathbb{E}|\mathbb{E}(\delta_h(t_n)|\mathcal{F}_{t_n})|^2 &\le 2\bar{K}(1+2C_B(1+\mathbb{E}(\sup_{t_0-\tau\le t\le t_0}|\varphi(t)|^2)))h^3+4C_2K_Lh^3 \\
&\quad +4\bar{K}\theta^2(1+2C_B(1+\mathbb{E}(\sup_{t_0-\tau\le t\le t_0}|\varphi(t)|^2)))h^3+8C_2K_L\theta^2h^3, \\
&= C_4h^3, \qquad (32)
\end{aligned}$$

where $C_4 = 2\bar{K}(1+2\theta^2)(1+2C_B(1+\mathbb{E}(\sup_{t_0-\tau\le t\le t_0}|\varphi(t)|^2)))+4C_2K_L(1+2\theta^2)$.

Now we estimate $\|\delta_h(t_n)\|_{L_2}$. Using the properties of the Itô integral, the Hölder inequality, we get

$$\begin{aligned}
\mathbb{E}|\delta_h(t_n)|^2 \quad \leq \quad & 3\mathbb{E}|\int_{t_n}^{t_{n+1}}(F(s,x(s),x(s-\tau))-F(t_n,x(t_n),x(t_n-\tau)))ds|^2 \\
&+3\mathbb{E}|\int_{t_n}^{t_{n+1}}(G(s,x(s),x(s-\tau))-G(t_n,x(t_n),x(t_n-\tau)))dW(s)|^2 \\
&+3\theta^2h^2\mathbb{E}|F(t_{n+1},x(t_{n+1}),x(t_{n+1}-\tau))-F(t_n,x(t_n),x(t_n-\tau))|^2 \\
\leq \quad & 3h\mathbb{E}\int_{t_n}^{t_{n+1}}|F(s,x(s),x(s-\tau))-F(t_n,x(t_n),x(t_n-\tau)))|^2ds \\
&+3\mathbb{E}\int_{t_n}^{t_{n+1}}|G(s,x(s),x(s-\tau))-G(t_n,x(t_n),x(t_n-\tau)))|^2ds \\
&+3\theta^2h^2\mathbb{E}|F(t_{n+1},x(t_{n+1}),x(t_{n+1}-\tau))-F(t_n,x(t_n),x(t_n-\tau))|^2
\end{aligned} \tag{33}$$

which yields that

$$\begin{aligned}
\mathbb{E}|\delta_h(t_n)|^2 \;\leq\; & 6h\mathbb{E}\int_{t_n}^{t_{n+1}}|F(s,x(s),x(s-\tau))-F(t_n,x(s),x(s-\tau))|^2ds \\
&+6h\mathbb{E}\int_{t_n}^{t_{n+1}}|F(t_n,x(s),x(s-\tau)-F(t_n,x(t_n),x(t_n-\tau))|^2ds \\
&+6\mathbb{E}\int_{t_n}^{t_{n+1}}|G(s,x(s),x(s-\tau))-G(t_n,x(s),x(s-\tau))|^2ds \\
&+6\mathbb{E}\int_{t_n}^{t_{n+1}}|G(t_n,x(s),x(s-\tau))-G(t_n,x(t_n),x(t_n-\tau))|^2ds \\
&+6\theta^2h^2\mathbb{E}|F(t_{n+1},x(t_{n+1}),x(t_{n+1}-\tau))-F(t_n,x(t_{n+1}),x(t_{n+1}-\tau))|^2 \\
&+6\theta^2h^2\mathbb{E}|F(t_n,x(t_{n+1}),x(t_{n+1}-\tau))-F(t_n,x(t_n),x(t_n-\tau))|^2.
\end{aligned} \tag{34}$$

By condition (10) and (29), we have

$$\begin{aligned}
\mathbb{E}|\delta_h(t_n)|^2 \quad \leq \quad & 6(h+1)\bar{K}\mathbb{E}(\int_{t_n}^{t_{n+1}}(1+|x(s)|^2+|x(s-\tau)|^2)(s-t_n)ds) \\
&+6(h+1)K_L\mathbb{E}(\int_{t_n}^{t_{n+1}}(|x(s)-x(t_n)|^2+|x(s-\tau)-x(t_n-\tau)|^2)ds) \\
&+6\theta^2\bar{K}h^3(1+\mathbb{E}|x(t_{n+1})|^2+\mathbb{E}|x(t_{n+1}-\tau)|^2) \\
&+6\theta^2K_Lh^2(\mathbb{E}|x(t_{n+1})-x(t_n)|^2+\mathbb{E}|x(t_{n+1}-\tau)-x(t_n-\tau)|^2).
\end{aligned} \tag{35}$$

It follows from Proposition 2.2, Lemma 2.1 and inequality (35) that

$$\mathbb{E}|\delta_h(t_n)|^2 \leq C_7h^2, \tag{36}$$

where $C_7 = 6\bar{K}(1+\theta^2)(1+2C_B(1+\mathbb{E}(\sup_{t_0-\tau\leq t\leq t_0}|\varphi(t)|^2)))+12C_2K_L(1+\theta^2)$. Here we have also used the fact $h<1$ as $h\to 0$.

¿From inequalities (32) and (36) we may extract mean square consistency with rate $p=\frac{1}{2}$ while step size $h<1$. Thus, the proof of Theorem 3.1 is completed. □

Theorem 3.2. *Assume that the conditions (10)-(13), (15) and (29) hold. Then the method (23)-(24) is mean-square convergent with order* $p=\frac{1}{2}$.

Proof. Define $e(t_n) := y(t_n) - y_n$. Let $\mathcal{B}^d$ be the Borel σ-algebra generated by the Borel sets of $\mathbb{R}^d$. Note that the error $e(t_n)$ is $(\mathcal{F}_{t_n}, \mathcal{B}^d)$-measurable, since both $y(t_n)$ and y_n are $(\mathcal{F}_{t_n}, \mathcal{B}^d)$-measurable random variables. By (23) and (25), we have

$$e(t_{n+1}) = e(t_n) + \delta_h(t_n) + R(t_n), \tag{37}$$

where

$$\begin{aligned} R(t_n) &= (1-\theta)h(f(t_n, y(t_n), x(t_n - \tau)) - f(t_n, y_n, x_{n-N_\tau})) \\ &\quad +\theta h(f(t_{n+1}, y(t_{n+1}), x(t_{n+1} - \tau)) - f(t_{n+1}, y_{n+1}, x_{n+1-N_\tau})) \\ &\quad +(g(t_n, y(t_n), x(t_n - \tau)) - g(t_n, y_n, x_{n-N_\tau}))\Delta W_n. \end{aligned} \tag{38}$$

In order to estimate the global error $e(t_n)$, we will frequently use the Hölder's inequality, the inequality $(a_1 + a_2 + \cdots + a_l)^2 \le l(a_1^2 + a_2^2 + \cdots + a_l^2)$, where $a_1, a_2, \ldots, a_l$ are real numbers, and the properties of Itô integral. It follows from (37) that

$$\begin{aligned} \mathbb{E}|e(t_{n+1})|^2 &= \mathbb{E}|e(t_n)|^2 + \mathbb{E}|\delta_h(t_n)|^2 + \mathbb{E}|R(t_n)|^2 \\ &\quad +2\mathbb{E}\langle e(t_n), \delta_h(t_n)\rangle + 2\mathbb{E}\langle e(t_n), R(t_n)\rangle + 2\mathbb{E}\langle \delta_h(t_n), R(t_n)\rangle \\ &\le \mathbb{E}|e(t_n)|^2 + 2\mathbb{E}|\delta_h(t_n)|^2 + 2\mathbb{E}|R(t_n)|^2 \\ &\quad +2\mathbb{E}\langle e(t_n), \delta_h(t_n)\rangle + 2\mathbb{E}\langle e(t_n), R(t_n)\rangle. \end{aligned} \tag{39}$$

By (38), we have

$$\begin{aligned} 2\mathbb{E}|R(t_n)|^2 &\le 6(1-\theta)^2h^2\mathbb{E}|f(t_n, y(t_n), x(t_n - \tau)) - f(t_n, y_n, x_{n-N_\tau})|^2 \\ &\quad +6\theta^2h^2\mathbb{E}|f(t_{n+1}, y(t_{n+1}), x(t_{n+1} - \tau)) - f(t_{n+1}, y_{n+1}, x_{n+1-N_\tau})|^2 \\ &\quad +6\mathbb{E}(|g(t_n, y(t_n), x(t_n - \tau)) - g(t_n, y_n, x_{n-N_\tau})|^2|\Delta W_n|^2) \\ &\le 6(1-\theta)^2h^2\mathbb{E}|F(t_n, x(t_n), x(t_n - \tau)) - F(t_n, x_n, x_{n-N_\tau})|^2 \\ &\quad +6\theta^2h^2\mathbb{E}|F(t_{n+1}, x(t_{n+1}), x(t_{n+1}-\tau)) - F(t_{n+1}, x_{n+1}, x_{n+1-N_\tau})|^2 \\ &\quad +6mh\mathbb{E}|G(t_n, x(t_n), x(t_n - \tau)) - G(t_n, x_n, x_{n-N_\tau})|^2 \\ &\le 6K_L(1-\theta)^2h^2\mathbb{E}(|x(t_n) - x_n|^2 + |x(t_n - \tau) - x_{n-N_\tau}|^2) \\ &\quad +6K_L\theta^2h^2\mathbb{E}(|x(t_{n+1}) - x_{n+1}|^2 + |x(t_{n+1} - \tau) - x_{n+1-N_\tau}|^2) \\ &\quad +6mK_Lh\mathbb{E}(|x(t_n) - x_n|^2 + |x(t_n - \tau) - x_{n-N_\tau}|^2), \end{aligned} \tag{40}$$

where we used the condition (10) and the fact $\mathbb{E}|\Delta W_n|^2 = mh$. A combination of (40) and the fact that

$$\begin{aligned} &|x(t_n) - x_n|^2 = |y(t_n) + D(x(t_n - \tau)) - y_n - D(x_{n-N_\tau})|^2 \\ &\le 2|y(t_n) - y_n|^2 + 2|D(x(t_n - \tau)) - D(x_{n-N_\tau})|^2 \\ &\le 2|e(t_n)|^2 + 2\kappa^2|x(t_n - \tau) - x_{n-N_\tau}|^2 \end{aligned} \tag{41}$$

leads to the estimate

$$\begin{aligned} &2\mathbb{E}|R(t_n)|^2 \\ &\le 6K_L((1-\theta)^2h + m)h(2\mathbb{E}|e(t_n)|^2 + (1+2\kappa^2)\mathbb{E}|x(t_n - \tau) - x_{n-N_\tau}|^2) \\ &\quad +6K_L\theta^2h^2(2\mathbb{E}|e(t_{n+1})|^2 + (1+2\kappa^2)\mathbb{E}|x(t_{n+1} - \tau) - x_{n+1-N_\tau}|^2) \\ &\le C_5h(\mathbb{E}|e(t_{n+1})|^2 + \mathbb{E}|e(t_n)|^2 + \mathbb{E}|x(t_{n+1} - \tau) - x_{n+1-N_\tau}|^2 \\ &\quad +\mathbb{E}|x(t_n - \tau) - x_{n-N_\tau}|^2). \end{aligned} \tag{42}$$

where the constant C_5 is equal to the expression

$$C_5 = \max\{12C_L, 6C_L(1+2\kappa^2), 12K_L\theta^2u, 6K_L\theta^2u(1+2\kappa^2)\}|_{u=\frac{\tau}{2}}$$

with $C_L = K_L((1-\theta)^2u + m)$. Here we have also used the fact that $h = \frac{\tau}{N_\tau} \le \frac{\tau}{2}$.

It is not difficult to obtain the following inequalities

$$\begin{aligned} 2\mathbb{E}\langle e(t_n), \delta_h(t_n)\rangle &= 2\mathbb{E}\mathbb{E}(\langle e(t_n), \delta_h(t_n)\rangle|\mathcal{F}_{t_n}) \\ &\le 2\mathbb{E}|\mathbb{E}(\langle e(t_n), \delta_h(t_n)\rangle|\mathcal{F}_{t_n})| = 2\mathbb{E}|\langle e(t_n), \mathbb{E}(\delta_h(t_n)|\mathcal{F}_{t_n})\rangle| \\ &\le 2(h\mathbb{E}|e(t_n)|^2)^{\frac{1}{2}}(h^{-1}\mathbb{E}|\mathbb{E}(\delta_h(t_n)|\mathcal{F}_{t_n})|^2)^{\frac{1}{2}} \\ &\le h\mathbb{E}|e(t_n)|^2 + h^{-1}\mathbb{E}|\mathbb{E}(\delta_h(t_n)|\mathcal{F}_{t_n})|^2. \end{aligned} \tag{43}$$

It follows from (38), (10) and (41) that

$$\begin{aligned} &|\mathbb{E}(R(t_n)|\mathcal{F}_{t_n})|^2 \\ &= |(1-\theta)h\mathbb{E}(f(t_n, y(t_n), x(t_n-\tau)) - f(t_n, y_n, x_{n-N_\tau})|\mathcal{F}_{t_n}) \\ &\quad +\theta h\mathbb{E}(f(t_{n+1}, y(t_{n+1}), x(t_{n+1}-\tau)) - f(t_{n+1}, y_{n+1}, x_{n+1-N_\tau})|\mathcal{F}_{t_n})|^2 \\ &\le 2(1-\theta)^2h^2\mathbb{E}(|F(t_n, x(t_n), x(t_n-\tau)) - F(t_n, x_n, x_{n-N_\tau})|^2|\mathcal{F}_{t_n}) \\ &\quad +2\theta^2h^2\mathbb{E}(|F(t_{n+1}, x(t_{n+1}), x(t_{n+1}-\tau)) - F(t_{n+1}, x_{n+1}, x_{n+1-N_\tau})|^2|\mathcal{F}_{t_n}) \\ &\le 2K_L(1-\theta)^2h^2\mathbb{E}(|x(t_n)-x_n|^2 + |x(t_n-\tau)-x_{n-N_\tau}|^2|\mathcal{F}_{t_n}) \\ &\quad +2K_L\theta^2h^2\mathbb{E}(|x(t_{n+1})-x_{n+1}|^2 + |x(t_{n+1}-\tau)-x_{n+1-N_\tau}|^2|\mathcal{F}_{t_n}) \\ &\le 2K_L(1-\theta)^2h^2\mathbb{E}(2|e(t_n)|^2 + (1+2\kappa^2)|x(t_n-\tau)-x_{n-N_\tau}|^2|\mathcal{F}_{t_n}) \\ &\quad +2K_L\theta^2h^2\mathbb{E}(2|e(t_{n+1})|^2 + (1+2\kappa^2)|x(t_{n+1}-\tau)-x_{n+1-N_\tau}|^2|\mathcal{F}_{t_n}) \\ &\le C_6h^2(\mathbb{E}(|e(t_{n+1})|^2|\mathcal{F}_{t_n}) + \mathbb{E}|e(t_n)|^2 \\ &\quad +\mathbb{E}|x(t_{n+1}-\tau)-x_{n+1-N_\tau}|^2 + \mathbb{E}|x(t_n-\tau)-x_{n-N_\tau}|^2), \end{aligned} \tag{44}$$

where

$$C_6 = \max\{4K_L(1-\theta)^2, 2K_L(1-\theta)^2(1+2\kappa^2), 4K_L\theta^2, 2K_L\theta^2(1+2\kappa^2)\}.$$

By (44), we have

$$\begin{aligned} 2\mathbb{E}\langle e(t_n), R(t_n)\rangle &= 2\mathbb{E}\mathbb{E}(\langle e(t_n), R(t_n)\rangle|\mathcal{F}_{t_n}) \\ &\le 2\mathbb{E}|\mathbb{E}(\langle e(t_n), R(t_n)\rangle|\mathcal{F}_{t_n})| \\ &= 2\mathbb{E}|\langle e(t_n), \mathbb{E}(R(t_n)|\mathcal{F}_{t_n})\rangle| \\ &\le 2(h\mathbb{E}|e(t_n)|^2)^{\frac{1}{2}}(h^{-1}\mathbb{E}|\mathbb{E}(R(t_n)|\mathcal{F}_{t_n})|^2)^{\frac{1}{2}} \\ &\le h\mathbb{E}|e(t_n)|^2 + h^{-1}\mathbb{E}|\mathbb{E}(R(t_n)|\mathcal{F}_{t_n})|^2 \\ &\le h\mathbb{E}|e(t_n)|^2 + C_6h(\mathbb{E}|e(t_{n+1})|^2 + \mathbb{E}|e(t_n)|^2 \\ &\quad +\mathbb{E}|x(t_{n+1}-\tau)-x_{n+1-N_\tau}|^2 + \mathbb{E}|x(t_n-\tau)-x_{n-N_\tau}|^2) \\ &\le C_7h(\mathbb{E}|e(t_{n+1})|^2 + \mathbb{E}|e(t_n)|^2 \\ &\quad +\mathbb{E}|x(t_{n+1}-\tau)-x_{n+1-N_\tau}|^2 + \mathbb{E}|x(t_n-\tau)-x_{n-N_\tau}|^2), \end{aligned} \tag{45}$$

where $C_7 = 1 + C_6$. Inserting (42), (43) and (45) into (39) yields that

$$\begin{aligned} \mathbb{E}|e(t_{n+1})|^2 \le\ &(C_5+C_7)h\mathbb{E}|e(t_{n+1})|^2 + (1+(1+C_5+C_7)h)\mathbb{E}|e(t_n)|^2 \\ &+(C_5+C_7)h(\mathbb{E}|x(t_{n+1}-\tau)-x_{n+1-N_\tau}|^2 + \mathbb{E}|x(t_n-\tau)-x_{n-N_\tau}|^2) \\ &+2\mathbb{E}|\delta_h(t_n)|^2 + h^{-1}\mathbb{E}|\mathbb{E}(\delta_h(t_n)|\mathcal{F}_{t_n})|^2. \end{aligned} \tag{46}$$

Let $m_T = [\frac{T-t_0}{\tau}] + 1$, where $[a]$ is the integer with $a - 1 < [a] \le a$. Now to treat inequality (46) further, repeat the application of recursive form (41) until there is some positive integer i such that $t_{n+1} - i\tau, t_n - i\tau \in [t_0 - \tau, t_0]$. Thus, we obtain

$$\begin{aligned} \mathbb{E}|e(t_{n+1})|^2 &\le (C_5+C_7)h\mathbb{E}|e(t_{n+1})|^2 + (1+(1+C_5+C_7)h)\mathbb{E}|e(t_n)|^2 \\ &\quad +4(C_5+C_7)h(1+2\kappa^2+\ldots+(2\kappa^2)^{m_T-1})\max_{0\le j\le n}\mathbb{E}|e(t_j)|^2 \\ &\quad +2\mathbb{E}|\delta_h(t_n)|^2 + h^{-1}\mathbb{E}|\mathbb{E}(\delta_h(t_n)|\mathcal{F}_{t_n})|^2, \end{aligned}$$

which yields that

$$
\begin{aligned}
(1-C_8h)\mathbb{E}|e(t_{n+1})|^2 \quad &\leq \quad (1+C_9h)\mathbb{E}|e(t_n)|^2 + C_{10}h \max_{0\leq j\leq n} \mathbb{E}|e(t_j)|^2 \\
&\quad +2h\left(h^{-\frac{1}{2}}(\mathbb{E}|\delta_h(t_n)|^2)^{\frac{1}{2}} + h^{-1}(\mathbb{E}|\mathbb{E}(\delta_h(t_n)|\mathcal{F}_{t_n})|^2)^{\frac{1}{2}}\right)^2, \quad (47)
\end{aligned}
$$

where $C_8 = C_5+C_7, C_9 = 1+C_5+C_7, C_{10} = 4(1+2\kappa^2+\ldots+(2\kappa^2)^{m_T-1})(C_5+C_7)$. Now, Theorem 3.1 implies that there exists a constant C_{11} such that

$$
2h\left(h^{-\frac{1}{2}}(\mathbb{E}|\delta_h(t_n)|^2)^{\frac{1}{2}} + h^{-1}(\mathbb{E}|\mathbb{E}(\delta_h(t_n)|\mathcal{F}_{t_n})|^2)^{\frac{1}{2}}\right)^2 \leq C_{11}h^2. \quad (48)
$$

For the y-component, define the total mean square errors

$$
Q_0 := 0, \quad Q_k := \max_{0\leq j\leq k} \mathbb{E}|e(t_j)|^2. \quad (49)
$$

Let $h_0 < \min\{\frac{1}{C_8}, \frac{\tau}{2}\}$. Note that

$$
\frac{1+(C_9+C_{10})h}{1-C_8h} = 1 + C_{12}h, \quad \text{if } 0 < h = \frac{\tau}{N_\tau} < h_0, \quad (50)
$$

where $C_{12} = (C_8+C_9+C_{10})/(1-C_8h_0)$. It follows from (47), (48) and (50) that the total mean square errors for the y-component satisfy the estimation

$$
\begin{aligned}
Q_{n+1} \quad &\leq \quad (1+C_{12}h)Q_n + C_{13}h^2 \leq \ldots \\
&\leq \quad C_{13}h^2(1+(1+C_{12}h)+\ldots+(1+C_{12}h)^n) \\
&= \quad C_{13}h\frac{(1+C_{12}h)^{n+1}-1}{C_{12}} \leq C_{13}\frac{e^{T-t_0}-1}{C_{12}}h, \quad (51)\\
&\qquad n = 1,\ldots,N-1, \ 0 < h = \frac{\tau}{N_\tau} < h_0,
\end{aligned}
$$

where $C_{13} = C_{11}/(1-C_8h_0)$. As a consequence, inequality (51) implies that

$$
\max_{0\leq n\leq N} \|y(t_n) - y_n\|_{L_2} \leq C_{14}h^{\frac{1}{2}}, \quad 0 < h = \frac{\tau}{N_\tau} < h_0, \quad (52)
$$

where $C_{14} = \sqrt{C_{13}(e^{T-t_0}-1)/C_{12}}$.

Furthermore, for the x-component, we can estimate the mean square errors by

$$
\begin{aligned}
\mathbb{E}|x(t_n) - x_n|^2 \quad &= \quad \mathbb{E}|y(t_n) + D(x(t_n-\tau)) - y_n - D(x_{n-N_\tau})|^2 \\
&\leq \quad \mathbb{E}(|e(t_n)| + \kappa|x(t_n-\tau) - x_{n-N_\tau}|)^2 \\
&\leq \quad \mathbb{E}(|e(t_n)| + |x(t_n-\tau) - x_{n-N_\tau}|)^2 \\
&\leq \quad \mathbb{E}(|e(t_n)| + |e(t_{n-N_\tau})| + |x(t_n-2\tau) - x_{n-2N_\tau}|)^2 \\
&\leq \quad \ldots \\
&\leq \quad m_T^2 \max_{0\leq j\leq n} \mathbb{E}|e(t_j)|^2,
\end{aligned}
$$

which also gives the estimate

$$
\max_{0\leq n\leq N} \|x(t_n) - x_n\|_{L_2} \leq C_{15}h^{\frac{1}{2}} \quad (53)
$$

with appropriate constant C_{15}. Therefore, Theorem 3.2 is proved. □

4. Numerical example

The purpose of this section is to illustrate our theoretical estimates obtained in Section 3 by a numerical experiment.

Example 4.1. *Consider the following nonlinear neutral stochastic differential delay equations (SDDEs)*

$$d[x(t)-\frac{1}{2}x(t-1)] = \sin(x(t)+x(t-1))dt + \cos(x(t)-x(t-1))dW(t),\ \ t\in[0,2]$$
$$x(t) = cos(t), \quad t\in[-1,0]. \tag{54}$$

It is easy to verify that SDDE (54) satisfies the conditions of Theorem 3.2. Because it is difficult to find the analytic form of the exact solution $x(t)$ of (54) (in fact, it is not known to us so far), we need to solve (54) by stochastic θ-methods

$$\begin{aligned} y_{n+1} &= y_n + h\Big[(1-\theta)\sin(y_n+\frac{3}{2}x_{n-N_\tau}) + \theta\sin(y_{n+1}+\frac{3}{2}x_{n+1-N_\tau})\Big] \\ &\quad + \cos(y_n - \frac{1}{2}x_{n-N_\tau})\Delta W_n; \quad n=0,1,...,N-1, \\ x_n &= y_n + \frac{1}{2}x_{n-N_\tau}; \quad n=0,1,...,N; \quad N_\tau = \frac{1}{h} \end{aligned} \tag{55}$$

with a sufficiently small mesh size (here $h=\Delta t = 2^{-14}$) and identify their outcomes as the "exact solution" $x(t)$ for the error-comparison. The error-analysis below is based on a comparison to this "exact solution" as a kind of "reference solution". Note that, due to implicit algebraic equations at each step for $\theta\neq 0$, we implement the Newton-Raphson method to resolve locally these implicit relations. For practical convenience, the Newton-Raphson iteration is carried out at least 5 times in order to establish a more practical stopping rule for the local iteration.

To illustrate the convergence of the θ-methods, 6000 sample trajectories are simulated for the step size $h = 16\Delta t, 32\Delta t, 64\Delta t, 128\Delta t, 256\Delta t, 512\Delta t$. To construct the confidence intervals for the absolute mean square errors $\epsilon^r = \epsilon_{2^{3+r}\Delta t}, r = 1,2,3,4,5,6$, we arrange the simulations into $M=60$ batches of $N=100$ simulations each and estimate the variance of $\hat{\epsilon}^r$ in the following way. We denote by $\tilde{x}^r_{T,k,j}$ the value of the kth generated trajectory of related numerical approximation with $h = 2^{3+r}\Delta t, r=1,2,3,4,5,6$, in the jth batch at time $T=2$ and by $x_{T,k,j}$ the corresponding value of the "exact solution" as the "reference solution".

The absolute mean square errors

$$\hat{\epsilon}^r_j = \frac{1}{N}\sum_{k=1}^{N}|x_{T,k,j}-\tilde{x}^r_{T,k,j}|^2$$

of the M batches $j=1,2,\ldots,M$ are independent and approximately Gaussian distributed for large N. We have arranged the errors into batches because we can use the Student t-distribution to construct confidence intervals by a sum of independent Gaussian or approximately Gaussian distributed random variables with unknown variance. In particular, we estimate the mean of the batch averages

$$\hat{\epsilon}^r = \frac{1}{M}\sum_{j=1}^{M}\hat{\epsilon}^r_j = \frac{1}{NM}\sum_{j=1}^{M}\sum_{k=1}^{N}|x_{T,k,j}-\tilde{x}^r_{T,k,j}|^2$$

and use the formula

$$\hat{\sigma}^2_{\epsilon^r} = \frac{1}{M-1}\sum_{j=1}^{M}(\hat{\epsilon}^r_j - \hat{\epsilon}^r)^2$$

to estimate the variance $\hat{\sigma}^2_{\epsilon^r}$ of the batch averages. For the Student t-distribution with $M-1$ degrees of freedom an $100(1-\alpha)\%$ confidence interval for ϵ^r has the form

$$(\hat{\epsilon}^r - \Delta\hat{\epsilon}^r, \hat{\epsilon}^r + \Delta\hat{\epsilon}^r)$$

with

$$\Delta\hat{\epsilon}^r = t_{1-\alpha,M-1}\sqrt{\frac{\hat{\sigma}^2_{\epsilon^r}}{M}},$$

where $t_{1-\alpha,M-1}$ is determined from the Student t-distribution with $M-1$ degrees of freedom. For $M = 60$ and $\alpha = 0.1$ we have $t_{1-\alpha,M-1} \approx 1.672$. In this case the absolute mean square error ϵ^r will lie in the corresponding confidence interval $(\hat{\epsilon}^r - \Delta\hat{\epsilon}^r, \hat{\epsilon}^r + \Delta\hat{\epsilon}^r)$ with probability $1-\alpha = 0.9, r = 1,2,3,4,5,6$, all measured at the fixed terminal time $T = 2$.

In Table 1, we list the absolute mean square errors and their 90% confidence intervals for $\hat{\epsilon}^r = \epsilon_{2^{3+r}\Delta t}, r = 1,2,3,4,5,6$ and $\theta = 0, 0.5, 1$.

Table 1. Absolute errors and 90% confidence intervals for SDDE (54) at $T = 2$

	$\theta = 0$		$\theta = 0.5$		$\theta = 1$	
r	$\hat{\epsilon}^r$	$\Delta\hat{\epsilon}^r$	$\hat{\epsilon}^r$	$\Delta\hat{\epsilon}^r$	$\hat{\epsilon}^r$	$\Delta\hat{\epsilon}^r$
1	1.140e-4	5.533e-6	1.234e-4	9.138e-6	1.195e-4	8.411e-6
2	2.369e-4	1.223e-5	2.583e-4	2.529e-5	2.566e-4	1.524e-5
3	4.915e-4	3.617e-5	5.184e-4	4.144e-5	5.071e-4	3.254e-5
4	1.050e-3	8.306e-5	1.048e-3	7.274e-5	1.051e-3	7.616e-5
5	2.225e-3	1.371e-4	2.111e-3	1.378e-4	2.198e-3	1.578e-4
6	4.893e-3	2.725e-4	4.408e-3	3.192e-4	4.613e-3	3.077e-4

We clearly recognize that the errors and its confidence intervals get smaller with decreasing step size (i.e. decreasing parameter r). So the predicted convergence of θ-methods applied to problem (54) is empirically confirmed.

In Fig.1, we plot the results on $\log_2 \epsilon_h$ versus $\log_2 h$, where $\epsilon_h = \sqrt{\hat{\epsilon}^r}, h = 2^{3+r}\Delta t, r = 1,2,3,4,5,6$. We note that all the curves in Fig.1 appear as straight lines with slope 1. It clearly illustrates the stochastic θ-methods for problem (54) is mean-square convergent with order $\frac{1}{2}$.

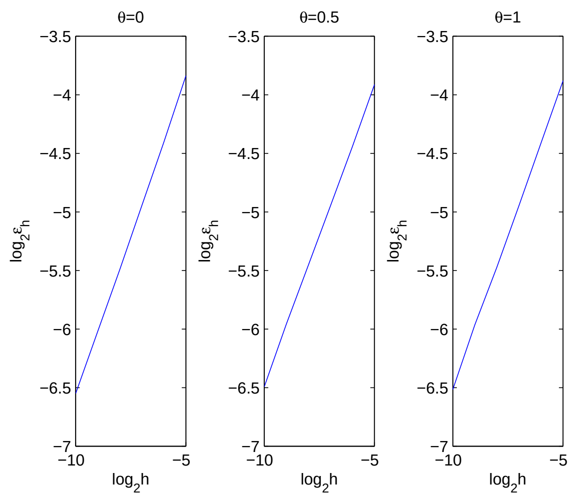

FIG. 1. $\log_2 \sqrt{\hat{\epsilon}^r}$ versus $\log_2 h$

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