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INEXACT SOLVERS FOR SADDLE-POINT SYSTEM ARISING FROM DOMAIN DECOMPOSITION OF LINEAR ELASTICITY PROBLEMS IN THREE DIMENSIONS

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Abstract. In this paper, we propose a domain decomposition method with Lagrange multipliers for three-dimensional linear elasticity, based on geometrically non-conforming subdomain partitions. Some appropriate multiplier spaces are presented to deal with the geometrically non-conforming partitions, resulting in a discrete saddle-point system. An augmented technique is introduced, such that the resulting new saddle-point system can be solved by the existing iterative methods. Two simple inexact preconditioners are constructed for the saddle-point system, one for the displacement variable, and the other for the Schur complement associated with the multiplier variable. It is shown that the global preconditioned system has a nearly optimal condition number, which is independent of the large variations of the material parameters across the local interfaces.

Key Words. Domain decomposition, geometrically non-conforming, Lagrange multiplier, saddle-point system, preconditioners, condition number.

1. Introduction

In recent years, there has been a fast growing interest in the domain decomposition methods (DDMs) with Lagrange multipliers, which were studied early in [6], [7], and [22]. Such DDMs have many advantages over the traditional DDMs in applications (cf. [1], [5], [21]). In this paper, we will develop a domain decomposition method with Lagrange multipliers to solve compressible elasticity problems in three dimensions. We consider certain geometrically non-conforming subdomain partitions with meshes that are nonmatching across the subdomain interfaces.

The Lagrange multiplier DDM has been developed as a non-conforming discretization method, such that the resulting approximation possesses the optimal accuracy, see [4], [20], [26]. For this purpose, the jumps of the solutions across the subdomain interfaces would be orthogonal to a certain Lagrange multiplier space, which should be appropriately chosen. This weak continuity condition leads to a saddle-point system for the displacement variable and the multiplier variable. It is known that the displacement variable corresponds to a singular problem on each floating subdomain. There exist many techniques to deal with such singularity, for example, the FETI-type methods [7, 8, 9, 19], regularized method [12] and

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augmented method [15]. After handling the singularity, we can eliminate the displacement variable to build an interface equation, or solve the saddle-point system directly by some preconditioned iterative methods.

A domain decomposition with Lagrange multipliers for solving linear elasticity problems in two dimensions was introduced in [19], in which inexact solvers were considered. A recent work on mortar discretization with geometrically nonconforming partitions for solving linear elasticity problems is a FETI-DP algorithm designed in [18]. To resolve the singularity associated with the displacement variable, a certain set of primal constraints was selected in [18] from the subdomain faces by some rules. After building the saddle-point system, Schur complement system was first got by eliminating the interior displacement variables in every subdomain, then an interface equation of the Lagrange multiplier was obtained by eliminating the primal constraint unknowns. Similar to other FETI-DP algorithms, a Neumann-Dirichlet preconditioner was constructed for the interface equation.

In the present paper, we study DDM with Lagrange multipliers for solving three-dimensional linear elasticity problems with jump coefficients. As in [15] (for Laplace equations), we propose a special augmented method to handle the singularity of the floating subdomains without introducing any additional constraints. But, we here introduce a different augmented term from the one considered in [15], since the original augmented term seems inefficient to elasticity problems. Since no interface equation needs to be built in the method, inexact solvers can be applied to both the primal operator and the Schur complement operator. For our method, we design a small coarse problem with the degree of freedoms equaling six times the number of the floating subdomains. We notice that the elasticity operator is spectrally equivalent to Laplace operator in every subdomain, then any existing preconditioner for the vector Laplace operator can be used directly as an inexact solver for the underlying operator. We show that the global preconditioned system has a nearly optimal condition number, which is independent of the large variations of the material parameters across the local interfaces.

The outline of the reminder of the paper is as follows. We introduce a new augmented saddle-point problem in section 2. In section 3, we construct two preconditioners for the saddle-point system and give a convergence of the preconditioned system. The main results of the paper will be shown in section 4. In section 5, we describe a class of cheap local solvers. Finally, we report some numerical results in section 6.

2. Linear elasticity and domain decomposition

In this section, we introduce a variational problem arising from the displacement formulation of compressible linear elasticity, and describe a discretization based on geometrically non-conforming domain decompositions.

2.1. The model problem. The unknown in the equations of linear elasticity is the displacement of a linear elastic material under the actions of external and internal forces. We denote the elastic body by $\Omega \subset \mathbb{R}^3$, and its boundary by $\partial\Omega$. We assume that one part of the boundary Γ_0, is clamped, i.e. with homogeneous Dirichlet boundary conditions, and that the rest, $\Gamma_1 := \partial\Omega\backslash\Gamma_0$, is subject to a surface force $\mathbf{g}$, i.e. a natural boundary condition. We can also introduce an internal volume force $\mathbf{f}$, e.g. gravity. The differential formulation is as follows (i=1, 2, 3):

$$
\left\{\begin{aligned} -\sum_{j=1}^{3}\frac{\partial\sigma_{ij}}{\partial x_j}(\mathbf{u}) &= f_i \quad in\ \Omega,\\ \mathbf{u} &= \mathbf{0} \quad on\ \Gamma_0,\\ \sum_{j=1}^{3}\sigma_{ij}(\mathbf{u})(\hat{n}_j) &= g_i \quad on\ \Gamma_1. \end{aligned}\right. \tag{2.1}
$$

The subspace $H^1_{\Gamma_0}(\Omega) \subset H^1(\Omega)$ is the set of functions having the zero trace on Γ_0. We introduce the vector valued Sobolev spaces $[H^1_{\Gamma_0}(\Omega)]^3$ and $[H^1(\Omega)]^3$, equipped with the usual product norm as follows:

$$
\| \mathbf{u} \|_{1,\Omega} := (| \mathbf{u} |^2_{H^1(\Omega)} + \| \mathbf{u} \|^2_{L_2(\Omega)})^{1/2}
$$

with $\| \mathbf{u} \|^2_{L_2(\Omega)} := \int_\Omega | \mathbf{u} |^2 \, dx$ and $| \mathbf{u} |^2_{H^1(\Omega)} := \| \nabla\mathbf{u} \|^2_{L_2(\Omega)}$.

The linear elasticity problem is: find the displacement $\mathbf{u} \in [H^1_{\Gamma_0}(\Omega)]^3$ of the elastic body Ω, such that

$$
\int_\Omega G(x)\ \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})dx + \int_\Omega G(x)\beta(x)\ div\mathbf{u}\ div\mathbf{v}dx = \langle \mathbf{F}, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in [H^1_{\Gamma_0}(\Omega)]^3. \tag{2.2}
$$

Here, $G(x) = E(x)/(1+\nu(x))$, $\beta(x) = 1/(1-2\nu(x))$ are material parameters which depend on the Young's modulus $E(x) > 0$ and the Poisson ratio $\nu(x) \in (0, 1/2]$. We assume that $\nu(x)$ is bounded away from $1/2$, excluding the case of incompressible elasticity problem. Then, we have $\beta(x) = O(1)$. The linearized strain tensor is defined by

$$
\varepsilon_{ij}(\mathbf{u}) := \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), \quad i, j = 1, 2, 3.
$$

The tensor product and the force term are given by

$$
\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) := \sum_{i,j=1}^{3} \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}), \quad \langle \mathbf{F}, \mathbf{v} \rangle := \sum_{i=1}^{3} \int_\Omega f_i v_i dx + \sum_{i=1}^{3} \int_{\Gamma_1} g_i v_i d\sigma.
$$

The associated bilinear form of linear elasticity is

$$
a(\mathbf{u}, \mathbf{v}) = \int_\Omega G(x)\ \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})dx + \int_\Omega G(x)\beta(x)\ div\mathbf{u}\ div\mathbf{v}dx, \quad \mathbf{u}, \mathbf{v} \in [H^1_{\Gamma_0}(\Omega)]^3.
$$

Let $A : [H^1_{\Gamma_0}(\Omega)]^3 \to [H^1_{\Gamma_0}(\Omega)]^3$ be the operator defined by

$$
(A\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}), \quad \mathbf{u} \in [H^1_{\Gamma_0}(\Omega)]^3, \ \forall \mathbf{v} \in [H^1_{\Gamma_0}(\Omega)]^3. \tag{2.3}
$$

It is obvious that the bilinear form $a(\cdot, \cdot)$ is continuous with respect to $\| \cdot \|_{1,\Omega}$. The ellipticity of the bilinear form can be established by the following Korn's first inequality.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. Then there exists a positive constant $c = c(\Omega, \Gamma_0)$, such that*

$$
\int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u})dx \geq c|\mathbf{u}|^2_{H^1(\Omega)}, \quad \forall \mathbf{u} \in [H^1_{\Gamma_0}(\Omega)]^3.
$$

And the problem (2.2) possesses a unique solution $\mathbf{u} \in [H^1_{\Gamma_0}(\Omega)]^3$.

2.2. The discrete problem based on domain decomposition. Since we only consider compressible elastic materials, it follows from Lemma 2.1 that the bilinear form $a(\cdot,\cdot)$ is uniformly elliptic. We can therefore successfully discretize the system (2.2) with low-order, conforming finite elements.

From the Korn's inequalities, we notice that the operator A defined in (2.3) is spectrally equivalent to Laplace operator. The ratio of the two spectrally equivalent constants depends on the variations of the material parameters $G(x)$ and $\beta(x)$. When the parameters $G(x)$ and $\beta(x)$ have large variations in the domain Ω, the ratio is large, and we can not use a preconditioner for Laplace operator to precondition the global operator A. In that case, we should adopt the geometrically non-conforming subdomain partitions, such that the ratio of the two spectrally equivalent constants is not large in each subdomain. Then, we can use a preconditioner for Laplace operator to precondition the local operator of A in each subdomain.

Let the domain Ω be decomposed into the union of non-overlapping polyhedral subdomains $\Omega_1, \cdots, \Omega_N$. Let d_k denote the size of the subdomain Ω_k. The union of the subdomain boundaries is defined by

$$\Gamma = (\bigcup_{k=1}^{N} \partial\Omega_k) \backslash \partial\Omega.$$

As usual, we assume that each Ω_k is a polyhedron, and make a quasi-uniform and regular triangulation $\mathcal{T}_{h_k}$ on Ω_k, with h_k denoting the mesh size. The grids may not match across the subdomain interfaces. Let $V_h(\Omega_k)$ denote the linear finite element space associated with $\mathcal{T}_{h_k}$. Define

$$V_h(\partial\Omega_k) = V_h(\Omega_k)|_{\partial\Omega_k} \text{ and } V_h(\Omega) = \prod_{k=1}^{N} V_h(\Omega_k).$$

We denote the interface of two subdomains Ω_i and Ω_j by F_{ij}, that can be only part of a face of Ω_i and Ω_j. Among the subdomain faces, we select multiplier faces Γ_l such that

$$\bigcup_l \bar{\Gamma}_l = \bigcup_{ij} \bar{F}_{ij}, \qquad \Gamma_l \cap \Gamma_k = \emptyset,\ l \neq k. \tag{2.4}$$

Here, each Γ_l is a full face of a subdomain. In order to understand the definition of the multiplier faces more clearly, we give a figure (Figure 1) to explain the definition for the case in two dimensions. In Figure 1, the subdomain partition is geometrically non-conforming, and the dotted lines denote the multiplier edges which satisfy the condition (2.4), and each multiplier edge Γ_l is a full edge of a subdomain.

Since a multiplier face $\Gamma_l \subset \partial\Omega_i$ may intersect several subdomain boundaries $\partial\Omega_j$, we should use the restriction of some triangulation $\mathcal{T}_{h_i}$ on Γ_l to generate the local multiplier space on Γ_l. For example, in Figure 1, the multiplier edge $\Gamma_l \subset \partial\Omega_2$ intersects the subdomain boundaries $\partial\Omega_4$ and $\partial\Omega_5$, then we choose the restriction of the triangulation $\mathcal{T}_{h_2}$ on Γ_l to generate the local multiplier space on Γ_l.

For a multiplier face Γ_l, let Ω_{i_l} denotes one of the subdomains such that Γ_l is a full edge (or face) of Ω_{i_l}. We define $W_h(\Gamma_l)$ as the vector version of the mortar multiplier space or the dual multiplier space for elliptic problems in three dimensions. These "elliptic" multiplier spaces were defined in [4] and [20] for the case of tetrahedra elements, and were defined, for the case of hexahedra elements, as the tensor product of two one-dimensional multiplier spaces introduced in [5] and [26]. As pointed out just before, we require that the multiplier space is associated

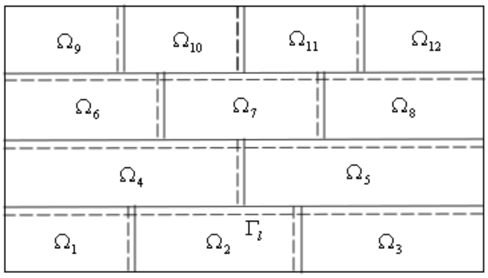

FIGURE 2.1. a geometrically non-conforming subdomain partition in two dimensions

with the triangulation $\mathcal{T}_{h_{i_l}}$. Define the global multiplier space

$$W_h(\Gamma) = \prod_{\Gamma_l \subset \Gamma} W_h(\Gamma_l).$$

Let $Q_l : L^2(\Gamma_l) \longrightarrow W_h(\Gamma_l)$ be the orthogonal projection with respect to the L^2-inner product on Γ_l. For $\mathbf{v} \in V(\Omega)$, set $\mathbf{v}|_{\Omega_k} = \mathbf{v}_k$. Define

$$\tilde{V}_h(\Omega) = \{\mathbf{v} \in V_h(\Omega) : Q_l(\mathbf{v}_i|_{\Gamma_l} - \mathbf{v}_j|_{\Gamma_l}) = 0 \ for\ each\ \Gamma_l \subset \Gamma\}. \tag{2.5}$$

Note that we do not require $\tilde{V}_h(\Omega) \subset [H^1(\Omega)]^3$.

Define the local bilinear form

$$\mathcal{A}_k(\mathbf{u},\mathbf{v}) = \int_{\Omega_k} G(x)\ \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})dx + \int_{\Omega_k} G(x)\beta(x)\ div\mathbf{u}\ div\mathbf{v}dx,$$
$$\mathbf{u},\mathbf{v} \in [H^1(\Omega_k)]^3.$$

The discrete problem of (2.2) is the following: find $\mathbf{u}_h \in \tilde{V}_h(\Omega)$ such that

$$\sum_{k=1}^{N} \mathcal{A}_k(\mathbf{u}_h^k, \mathbf{v}_h^k) = (\mathbf{F}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \tilde{V}_h(\Omega). \tag{2.6}$$

Here, $\mathbf{u}_h^k$ is the restriction of $\mathbf{u}_h$ in Ω_k. As in the case of elliptic problems, we can verify the existence and the uniqueness of the solution of (2.6), and can derive the optimal energy error estimate of the approximate solution (refer to [10]).

2.3. Augmented saddle-point systems. We will transform (2.6) into a standard saddle-point problem.

Let $A_k : V_h(\Omega_k) \longrightarrow V_h(\Omega_k)$ be the local operator defined by

$$(A_k\mathbf{u}_h^k, \mathbf{v}_h^k)_{\Omega_k} = \mathcal{A}_k(\mathbf{u}_h^k, \mathbf{v}_h^k), \quad \mathbf{u}_h^k \in V_h(\Omega_k), \quad \forall \mathbf{v}_h^k \in V_h(\Omega_k).$$

We define the operator $B_k : V_h(\Omega_k) \longrightarrow W_h(\Gamma)$ as follows:

$$(B_k\mathbf{u}_h^k)|_{\Gamma_l} = \begin{cases} \sigma_l Q_l(\mathbf{u}_h^k|_{\Gamma_l}), & \Gamma_l \subset \partial\Omega_k, \\ 0, & \Gamma_l \not\subset \partial\Omega_k. \end{cases}$$

where σ_l is the sign function, $\sigma_l = 1$ for the multiplier faces, and $\sigma_l = -1$ for the other faces.

Define the operators $A : V_h(\Omega) \longrightarrow V_h(\Omega)$ and $B : V_h(\Omega) \longrightarrow W_h(\Gamma)$ by

$$A|_{V_h(\Omega_k)} = A_k$$

and

$$B\mathbf{v}_h = \sum_{k=1}^{N} B_k \mathbf{v}_h^k, \quad \mathbf{v}_h \in V_h(\Omega),$$

respectively.

Let $\langle \cdot, \cdot \rangle$ denote the L^2-inner product on Γ, and let $B^t : W_h(\Gamma) \longrightarrow V_h(\Omega)$ denote the adjoint of B, which satisfies

$$\langle B^t \mu_h, \mathbf{v}_h \rangle = \langle \mu_h, B\mathbf{v}_h \rangle \qquad \forall \mu_h \in W_h(\Gamma),\ \mathbf{v}_h \in V_h(\Omega).$$

It is easy to see that the space $\tilde{V}_h(\Omega)$ can be written as

$$\tilde{V}_h(\Omega) = \{\mathbf{v}_h \in V_h(\Omega) : B\mathbf{v}_h = 0\}.$$

Then (2.6) is equivalent to the following saddle-point problem: find $(\mathbf{u}_h, \lambda_h) \in V_h(\Omega) \times W_h(\Gamma)$ such that

$$\left\{ \begin{aligned} A\mathbf{u}_h + B^t \lambda_h &= \mathbf{F}, \\ B\mathbf{u}_h &= \mathbf{0}. \end{aligned} \right. \tag{2.7}$$

Here, the unknown λ_h is called the Lagrange multiplier for the constraint $B\mathbf{u}_h = \mathbf{0}$.

Although the operator A is defined locally, the system (2.7) cannot be solved in the standard way. The main difficulty is that each local operator A_k corresponding to some interior subdomain Ω_k is singular on $V_h(\Omega_k)$, so the global operator A is also singular on $V_h(\Omega)$. To resolve the singularity, we introduce an augmented method, which has been discussed in [15] for the case of elliptic problems, but the situation here is quite different.

Let r be a positive number. The classical augmented multiplier framework can be written as

$$\left\{ \begin{aligned} (A + rB^t B)\mathbf{u}_h + B^t \lambda_h &= \mathbf{F}, \\ B\mathbf{u}_h &= \mathbf{0}, \end{aligned} \right. \tag{2.8}$$

which has the same solution with (2.7). The material parameters $G(x)$ and $\beta(x)$ may have large jumps across the interface Γ. To avoid the influence of the jumps, we consider another augmented Lagrange multiplier formulation instead of (2.8).

Without loss of generality, we assume that

$$G(x) = G_k \text{ (const.) and } \beta(x) = \beta_k \text{ (const.)},\ \forall x \in \Omega_k.$$

For each multiplier face Γ_l, set

$$\alpha_l = \min_{\partial\Omega_k \cap \Gamma_l \neq \emptyset} G_k.$$

We define the operator $\bar{B}_k : V_h(\Omega_k) \to W_h(\Gamma)$ by

$$(\bar{B}_k \mathbf{u}_h^k)|_{\Gamma_l} = \left\{ \begin{array}{ll} \sigma_l \alpha_l^{\frac{1}{2}} Q_l(\mathbf{u}_h^k|_{\Gamma_l}), & \Gamma_l \subset \partial\Omega_k, \\ 0, & \Gamma_l \not\subset \partial\Omega_k. \end{array} \right.$$

and the operator $\bar{B} : V_h(\Omega) \to W_h(\Gamma)$

$$\bar{B}\mathbf{v}_h = \sum_{k=1}^{N} \bar{B}_k \mathbf{v}_h^k, \quad \mathbf{v}_h \in V_h(\Omega).$$

It is easy to see that $\bar{B}\mathbf{v}_h = 0$ if and only if $B\mathbf{v}_h = 0$. Thus, the system (2.7) has the same solution with the weighted saddle-point problem

$$\left\{ \begin{aligned} (A + r\bar{B}^t \bar{B})\mathbf{u}_h + B^t \lambda_h &= \mathbf{F}, \\ B\mathbf{u}_h &= \mathbf{0}. \end{aligned} \right. \tag{2.9}$$

Note that the operator $A+r\bar{B}^t\bar{B}$ is also symmetric and positive definite on $V_h(\Omega)$. For the case of elliptic problems, we can simply choose $r=d^{-1}$ with d denotes the size of the subdomains (see [15]). However, this choice will not give the desired equivalent results for the elasticity problems.

Let $M : W_h(\Gamma) \to W_h(\Gamma)$ be a symmetric and positive definite operator, which will be defined exactly in the following section. We introduce a new saddle-point problem (compare [17])

$$\left\{ \begin{aligned} (A+\bar{B}^tM^{-1}\bar{B})\mathbf{u}_h + B^t\lambda_h &= \mathbf{F}, \\ B\mathbf{u}_h &= \mathbf{0}. \end{aligned} \right. \tag{2.10}$$

Define $A^* = A + \bar{B}^tM^{-1}\bar{B}$. Since the operator A^* is not yet block diagonal, it is not practical to eliminate directly the variable $\mathbf{u}_h$ in (2.10). Fortunately, many iterative methods have been developed for solving saddle-point problems such as (2.10), for example, the inexact Uzawa-type methods (see [3]), the preconditioned CG method based on a positive definite reformulation (see [2]). The efficiency of these iterative methods strongly depend on two preconditioners $\bar{A}$ and $\bar{S}$ which are spectrally equivalent to A^* and to Schur complement $\tilde{S} = B\bar{A}^{-1}B^t$, respectively.

3. The construction of the preconditioners

This section is devoted to the construction of the preconditioners $\bar{A}$ and $\bar{S}$.

For convenience, we will follow [27] to adopt the notations $\lesssim$, $\gtrsim$ and $\eqsim$ in the subsequent analysis of this work. $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \eqsim y_3$ mean that $x_1 \le C_1y_1$, $x_2 \ge c_2y_2$ and $c_3x_3 \le y_3 \le C_2x_3$ for some constants C_1, c_2, c_3 and C_3 that are independent of d_k, h_k and possible large jumps of the coefficients $G(x)$ and $\beta(x)$ across the interface Γ.

3.1. A preconditioner for A^*.

Define the local solver $\bar{A}_k : V_h(\Omega_k) \longrightarrow V_h(\Omega_k)$ by

$$\begin{aligned} &(\bar{A}_k\mathbf{v},\mathbf{w})_{\Omega_k} = G_k[(\nabla\mathbf{v},\nabla\mathbf{w})_{\Omega_k} + d_k^{-2}(\mathbf{v},\mathbf{w})_{\Omega_k}], \\ &\mathbf{v}\in V_h(\Omega_k),\ \forall\mathbf{w}\in V_h(\Omega_k). \end{aligned} \tag{3.1}$$

In some applications, one may be more interested in an inexact solver for $\bar{A}_k$. Let $\hat{A}_k$ be a symmetric and positive definite operator on $V_h(\Omega_k)$, such that

$$(\bar{A}_k\mathbf{v},\mathbf{v})_{\Omega_k} \lesssim (\hat{A}_k\mathbf{v},\mathbf{v})_{\Omega_k} \le \gamma_k(\bar{A}_k\mathbf{v},\mathbf{v})_{\Omega_k} \tag{3.2}$$

for any $\mathbf{v}\in V_h(\Omega_k)$. Define $\hat{A} : V_h(\Omega)\to V_h(\Omega)$ by $\hat{A}|_{V_h(\Omega_k)} = \hat{A}_k$ for each k, and set $\hat{S} = \bar{B}\hat{A}^{-1}\bar{B}^t$. For each multiplier face Γ_l, let $I_l : W_h(\Gamma) \longrightarrow W_h(\Gamma_l)$ denote the natural restriction operator, and let $I_l^t : W_h(\Gamma_l) \longrightarrow W_h(\Gamma)$ denote the zero extension operator, which is just the adjoint of I_l. For $\varphi\in W_h(\Gamma)$, set $\varphi_l = \varphi|_{\Gamma_l} = I_l\varphi$. Define $M : W_h(\Gamma)\to W_h(\Gamma)$ such that

$$\langle M\phi,\psi\rangle \eqsim \sum_{\Gamma_l}\langle \hat{S}I_l^t\phi_l, I_l^t\psi_l\rangle,\ \phi\in W_h(\Gamma),\ \forall\psi\in W_h(\Gamma).$$

To handle the nonlocal operator $\bar{B}^tM^{-1}\bar{B}$ in A^*, we need to introduce a coarse solver. Set $V_0 = ker(A)$, note that V_0 is spanned by the rigid body motions, i.e., three translations and three rotations in each subdomain. The natural choice of the coarse solver is the restriction of A^* on V_0. Define the coarse solver A_0 by

$$(A_0\mathbf{v}_0,\mathbf{v}_0) = (A^*\mathbf{v}_0,\mathbf{v}_0) = \langle M^{-1}\bar{B}\mathbf{v}_0,\bar{B}\mathbf{v}_0\rangle,\ \forall\mathbf{v}_0\in V_0. \tag{3.3}$$

Theorem 3.1. *Let $Q_0 : V_h(\Omega) \longrightarrow V_0$ denote the L^2- orthogonal projector. Define $\bar{A}^{-1} = \hat{A}^{-1} + A_0^{-1}Q_0$, then*

$$cond(\bar{A}^{-1}A^*) \lesssim \hat{\gamma}. \tag{3.4}$$

where $\hat{\gamma} = \max_{1\leq k\leq N} \gamma_k$.

Remark 3.1. *The inexact solver $\hat{A}_k$ can be chosen as any preconditioner for Laplace-type operator $\bar{A}_k$. The implementation of such inexact local solver is much cheaper than that of the exact local solver corresponding to $\bar{A}_k$.*

3.2. A preconditioner for Schur complement. Define $S = B(A^*)^{-1}B^t$ and $\tilde{S} = B\bar{A}^{-1}B^t$. In the following, we construct a preconditioner for Schur complement $\tilde{S}$ or S.

We first define a discrete dual norm $\| \cdot \|_{-*,\Gamma_l}$ on a multiplier face Γ_l, and give some assumptions.

For a multiplier face Γ_l, let Ω_{i_l} be the subdomain satisfying $\Gamma_l \subset \partial\Omega_{i_l}$ (see Subsection 2.2), and let Ω_{j_l} denote any other subdomain with its boundary intersecting Γ_l. Throughout this paper, we define $V_i(\Gamma_l) = V_h(\partial\Omega_{i_l})|_{\Gamma_l}$, and $V_j(\Gamma_l) = V(\partial\Omega_{j_l})|_{\Gamma_l}$. As before, we assume that the local multiplier space $W_h(\Gamma_l)$ is associated with the local trace space $V_i(\Gamma_l)$. Then, the space $W_h(\Gamma_l)$ has the same dimension with $V_i^0(\Gamma_l)$, where $V_i^0(\Gamma_l) = V_i(\Gamma_l) \cap H_0^1(\Gamma_l)$.

Let $\| \cdot \|_{1/2,\Gamma_l^0}$ denotes the norm on the space $H_{00}^{1/2}(\Gamma_l)$ (namely, the norm $\| \cdot \|_{H_{00}^{1/2}(\Gamma_l)}$ defined in [27]). Define the discrete dual norm $\| \cdot \|_{-*,\Gamma_l}$ by

$$\| \mu_h \|_{-*,\Gamma_l} = \sup_{\mathbf{v}_h \in V_i^0(\Gamma_l)} \frac{\langle \mu_h, \mathbf{v}_h \rangle_{\Gamma_l}}{\| \mathbf{v}_h \|_{\frac{1}{2},\Gamma_l^0}}, \quad \mu_h \in W_h(\Gamma_l).$$

For a multiplier face Γ_l, let Λ_l be a symmetric and positive definite operator defined on $W_h(\Gamma_l)$. We assume that Λ_l has the following spectral equivalence with the norm $\| \cdot \|_{-*,\Gamma_l}$

$$\langle \Lambda_l \phi_l, \ \phi_l \rangle_{\Gamma_l} \eqsim \alpha_l^{-1} \| \phi_l \|_{-*,\Gamma_l}^2, \quad \forall \phi_l \in W_h(\Gamma_l). \tag{3.5}$$

Now we define the preconditioner $\bar{S}$ by

$$\bar{S}^{-1} = \sum_{\Gamma_l} I_l^t {\Lambda_l}^{-1} I_l.$$

In section 5, we will derive the concrete form of the local solvers ${\Lambda_l}^{-1}$.

For ease of notation, set $\Phi(\frac{d}{h}) = \max_{1\leq k\leq N} [1 + \log(d_k/h_k)]^2$.

Theorem 3.2. *For the preconditioner $\bar{S}$, we have*

$$cond(\bar{S}^{-1}\tilde{S}) \lesssim C\hat{\gamma}\Phi(\frac{d}{h}) \tag{3.6}$$

and

$$cond(\bar{S}^{-1}S) \lesssim C\Phi(\frac{d}{h}), \tag{3.7}$$

where the constant C in (3.6) *and* (3.7) *is independent of the large variations of the coefficients across the local interfaces Γ_l.*

4. Analysis

This section is devoted to prove the results given in the last section.

4.1. The proof of Theorem 3.1. To prove Theorem 3.1, we need several Lemmas.

We will use repeatedly weighted norms on subdomains Ω_k. For ease of notation, let $\hat{\Omega} \subset \Omega$ denote a generic subdomain with the "size" $\hat{d}$. Define the weighted norm

$$\|\mathbf{v}\|_{1,\hat{\Omega}} = (|\mathbf{v}|^2_{1,\hat{\Omega}} + \hat{d}^{-2}\|\mathbf{v}\|^2_{0,\hat{\Omega}})^{\frac{1}{2}}, \quad \mathbf{v} \in [H^1(\hat{\Omega})]^3.$$

The following result can be derived by Korn's second inequality (cf., [23]) and the standard scaling argument.

Lemma 4.1. *There exists a positive constant c such that*

$$\int_{\hat{\Omega}} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) + \hat{d}^{-2} \parallel \mathbf{u} \parallel^2_{0,\hat{\Omega}} \geq c \parallel \mathbf{u} \parallel^2_{1,\hat{\Omega}}, \quad \forall \mathbf{u} \in [H^1(\hat{\Omega})]^3.$$

We can now derive a Korn's inequality on the space

$$\{\mathbf{u} \in [H^1(\hat{\Omega})]^3 : \mathbf{u} \perp ker(\varepsilon)\}.$$

The null space $ker(\varepsilon)$ is the space of rigid body motions. In three dimensions, the corresponding space is spanned by three translations

$$\mathbf{r}_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{r}_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{r}_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and three rotations

$$\mathbf{r}_4 := \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix}, \ \mathbf{r}_5 := \begin{bmatrix} -x_3 \\ 0 \\ x_1 \end{bmatrix}, \ \mathbf{r}_6 := \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}.$$

Define norms

$$\|\mathbf{v}\|_{E_1,\hat{\Omega}} := \|\varepsilon(\mathbf{v})\|^2_{0,\hat{\Omega}} + \hat{d}^{-2}\|\mathbf{v}\|^2_{0,\hat{\Omega}}, \quad \mathbf{v} \in [H^1(\hat{\Omega})]^3$$

and

$$\|\mathbf{v}\|_{E_2,\hat{\Omega}} := \|\varepsilon(\mathbf{v})\|^2_{0,\hat{\Omega}} + \hat{d}^{-2}\sum_{i=1}^{6} |\int_{\hat{\Omega}} (\mathbf{r}_i)^t \mathbf{v} dx|^2, \quad \mathbf{v} \in [H^1(\hat{\Omega})]^3.$$

The following result can be found in [19] and Nećas [24]

Lemma 4.2. *There exist constants $0 < c \leq C < \infty$, such that*

$$c \parallel \mathbf{u} \parallel_{E_1,\hat{\Omega}} \leq \parallel \mathbf{u} \parallel_{E_2,\hat{\Omega}} \leq C \parallel \mathbf{u} \parallel_{E_1,\hat{\Omega}}, \quad \forall \mathbf{u} \in [H^1(\Omega)]^3.$$

We obviously have

$$\parallel \varepsilon(\mathbf{u}) \parallel_{0,\hat{\Omega}} \leq \parallel \nabla \mathbf{u} \parallel_{0,\hat{\Omega}}, \quad \forall \mathbf{u} \in [H^1(\Omega)]^3. \tag{4.1}$$

Using (4.1), Lemma 4.1 and Lemma 4.2, we obtain

Lemma 4.3. *There exist constants $0 < c \leq C < \infty$, such that*

$$c \parallel \nabla \mathbf{u} \parallel_{0,\hat{\Omega}} \leq \parallel \varepsilon(\mathbf{u}) \parallel_{0,\hat{\Omega}} \leq C \parallel \nabla \mathbf{u} \parallel_{0,\hat{\Omega}}, \quad \forall \mathbf{u} \in [H^1(\hat{\Omega})]^3, \mathbf{u} \perp ker(\varepsilon).$$

Lemma 4.4. *The following inequality holds:*

$$(\bar{B}^t M^{-1} \bar{B} \mathbf{v}_h, \mathbf{v}_h) \lesssim (\hat{A} \mathbf{v}_h, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h(\Omega). \tag{4.2}$$

PROOF. It is clear that

$$(4.3)\quad \begin{aligned}(\bar{B}^t M^{-1}\bar{B}\mathbf{v}_h, \mathbf{v}_h) &= (\hat{A}^{-\frac{1}{2}}\bar{B}^t M^{-1}\bar{B}\mathbf{v}_h, \hat{A}^{\frac{1}{2}}\mathbf{v}_h)\\ &\leq \|\hat{A}^{-\frac{1}{2}}\bar{B}^t M^{-1}\bar{B}\mathbf{v}_h\|_{0,\Omega}\cdot\|\hat{A}^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega}.\end{aligned}$$

It suffices to estimate $\|\hat{A}^{-\frac{1}{2}}\bar{B}^t M^{-1}\bar{B}\mathbf{v}_h\|_{0,\Omega}$. In fact, we have

$$(4.4)\quad \begin{aligned}\|\hat{A}^{-\frac{1}{2}}\bar{B}^t M^{-1}\bar{B}\mathbf{v}_h\|_{0,\Omega}^2 &= (\bar{B}\hat{A}^{-1}\bar{B}^t(M^{-1}\bar{B}\mathbf{v}_h), M^{-1}\bar{B}\mathbf{v}_h)\\ &= (\hat{S}(M^{-1}\bar{B}\mathbf{v}_h), M^{-1}\bar{B}\mathbf{v}_h).\end{aligned}$$

Set $\phi = M^{-1}\bar{B}\mathbf{v}_h$. Then, $\phi = \sum_{\Gamma_l} I_l^t\phi_l$ with $\phi_l = \phi|_{\Gamma_l}$. Therefore,

$$(\hat{S}(M^{-1}\bar{B}\mathbf{v}_h), M^{-1}\bar{B}\mathbf{v}_h) = (\hat{S}(\sum_{\Gamma_l} I_l^t\phi_l), \sum_{\Gamma_l} I_l^t\phi_l) \lesssim \sum_{\Gamma_l}(\hat{S}I_l^t\phi_l, I_l^t\phi_l).$$

By the definition of M, we further deduce

$$(\hat{S}(M^{-1}\bar{B}\mathbf{v}_h), M^{-1}\bar{B}\mathbf{v}_h) \lesssim (M\phi,\phi) = (\bar{B}^t M^{-1}\bar{B}\mathbf{v}_h, \mathbf{v}_h).$$

Plugging this in (4.4) leads to

$$\|\hat{A}^{-\frac{1}{2}}\bar{B}^t M^{-1}\bar{B}\mathbf{v}_h\|_{0,\Omega}^2 \lesssim (\bar{B}^t M^{-1}\bar{B}\mathbf{v}_h, \mathbf{v}_h).$$

Combining the above inequality with (4.3) yields

$$(\bar{B}^t M^{-1}\bar{B}\mathbf{v}_h, \mathbf{v}_h)^{\frac{1}{2}} \lesssim \|\hat{A}^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega},$$

which gives (4.2).

Proof of Theorem 3.1. The inequality (3.4) can be derived by

$$\hat{\gamma}^{-1}(\mathbf{v}_h, A^*\mathbf{v}_h) \lesssim ((\hat{A}^{-1} + A_0^{-1}Q_0)A^*\mathbf{v}_h, A^*\mathbf{v}_h) \lesssim (\mathbf{v}_h, A^*\mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h(\Omega).$$

Consider the space decomposition $V_h(\Omega) = V_0 + \bar{V}$, with $V_0 = ker(A)$ and $\bar{V} \subset V_h(\Omega)$. By the abstract Schwarz theory, we need only to prove that

(a) for any $\varphi_{\mathbf{0}} \in V_0$ and $\bar{\varphi} \in \bar{V}$, we have

$$(4.5)\quad (A^*(\varphi_{\mathbf{0}} + \bar{\varphi}), \varphi_{\mathbf{0}} + \bar{\varphi}) \lesssim (A_0\varphi_{\mathbf{0}}, \varphi_{\mathbf{0}}) + (\hat{A}\bar{\varphi}, \bar{\varphi});$$

(b) for any $\mathbf{v}_h \in V_h(\Omega)$, there is a decomposition $\mathbf{v}_h = \mathbf{v}_0 + \bar{\mathbf{v}}$ with $\mathbf{v}_0 \in V_0$ and $\bar{\mathbf{v}} \in \bar{V}$ such that

$$(4.6)\quad (A_0\mathbf{v}_0, \mathbf{v}_0) + (\hat{A}\bar{\mathbf{v}}, \bar{\mathbf{v}}) \lesssim \hat{\gamma}(A^*\mathbf{v}_h, \mathbf{v}_h).$$

We first consider (a). By the triangle inequality and (3.3), we deduce

$$(4.7)\quad \begin{aligned}(A^*(\varphi_{\mathbf{0}} + \bar{\varphi}), \varphi_{\mathbf{0}} + \bar{\varphi}) &\leq 2[(A^*\varphi_{\mathbf{0}}, \varphi_{\mathbf{0}}) + (A^*\bar{\varphi}, \bar{\varphi})]\\ &\lesssim (A_0\varphi_{\mathbf{0}}, \varphi_{\mathbf{0}}) + (A\bar{\varphi}, \bar{\varphi})\\ &+ (\bar{B}^t M^{-1}\bar{B}\bar{\varphi}, \bar{\varphi}).\end{aligned}$$

From the definitions of A and $\bar{A}_k$, we have

$$\begin{aligned}(A\bar{\varphi}, \bar{\varphi}) &= \int_\Omega G(x)\ \varepsilon(\bar{\varphi}) : \varepsilon(\bar{\varphi})dx + \int_\Omega G(x)\beta(x)|div\ \bar{\varphi}|^2 dx\\ &\lesssim \sum_{k=1}^N G_k\|\bar{\varphi}\|_{1,\Omega_k}^2 \lesssim \sum_{k=1}^N (\bar{A}_k(\bar{\varphi}|_{\Omega_k}), \bar{\varphi}|_{\Omega_k})_{\Omega_k}.\end{aligned}$$

Here, we have used the assumption $\beta_k = O(1)$. Then, it follows from (3.2) that

$$(A\bar{\varphi},\bar{\varphi}) \lesssim \sum_{k=1}^{N} (\hat{A}_k(\bar{\varphi}|_{\Omega_k}),\bar{\varphi}|_{\Omega_k})_{\Omega_k} = (\hat{A}\bar{\varphi},\bar{\varphi}).$$

Substituting this inequality into (4.7) and using (4.2) yield (4.5).

Now, we consider (b). For $\mathbf{v}_h \in V_h(\Omega)$, define $\mathbf{v}_0 \in V_0$ as follows: for the interior subdomains Ω_k, define $\mathbf{v}_0|_{\Omega_k} \in ker(A_k)$ by $(\mathbf{v}_h - \mathbf{v}_0, \mathbf{r}_i)_{\Omega_k} = 0$ $(i = 1, 2, \cdots, 6)$; otherwise, $\mathbf{v}_0|_{\Omega_k} = 0$. In other words, $\mathbf{v}_0$ is just the L^2 projection of $\mathbf{v}_h$ into the null space V_0. Moreover, we define $\bar{\mathbf{v}} = \mathbf{v}_h - \mathbf{v}_0$. Then, we have $\bar{\mathbf{v}} \perp ker(\varepsilon)$ and $\mathbf{v}_h = \mathbf{v}_0 + \bar{\mathbf{v}}$. At first, we prove

$$(\hat{A}\bar{\mathbf{v}},\bar{\mathbf{v}}) \lesssim \hat{\gamma}(A^*\mathbf{v}_h,\mathbf{v}_h), \quad \bar{\mathbf{v}} = \mathbf{v}_h - \mathbf{v}_0. \tag{4.8}$$

For convenience, set $\bar{\mathbf{v}}|_{\Omega_k} = \bar{\mathbf{v}}_k$ and $\mathbf{v}_h|_{\Omega_k} = \mathbf{v}_k$. Then, we get by (3.2) and Lemma 4.1 (or Lemma 4.3)

$$\begin{aligned}(\hat{A}\bar{\mathbf{v}},\bar{\mathbf{v}}) &= \textstyle\sum_{k=1}^{N} (\hat{A}_k\bar{\mathbf{v}}_k,\bar{\mathbf{v}}_k)_{\Omega_k} \le \sum_{k=1}^{N} \gamma_k(\bar{A}_k\bar{\mathbf{v}}_k,\bar{\mathbf{v}}_k)_{\Omega_k} \\ &\lesssim \textstyle\sum_{k=1}^{N} \gamma_k G_k(\|\varepsilon(\bar{\mathbf{v}}_k)\|^2_{0,\Omega_k} + d_k^{-2}\|\bar{\mathbf{v}}_k\|^2_{0,\Omega_k}).\end{aligned} \tag{4.9}$$

For the interior subdomains Ω_k, we have by Lemma 4.2 and the definition of $\mathbf{v}_0$

$$\begin{aligned}\|\varepsilon(\bar{\mathbf{v}}_k)\|^2_{0,\Omega_k} + d_k^{-2}\|\bar{\mathbf{v}}_k\|^2_{0,\Omega_k} &\lesssim \|\varepsilon(\bar{\mathbf{v}}_k)\|^2_{0,\Omega_k} + d_k^{-2}\sum_{i=1}^{6}(\bar{\mathbf{v}}_k,\mathbf{r}_i)^2_{0,\Omega_k} \\ &= \|\varepsilon(\mathbf{v}_k - \mathbf{v}_0|_{\Omega_k})\|^2_{0,\Omega_k} \\ &+ d_k^{-2}\sum_{i=1}^{6}(\mathbf{v}_k - \mathbf{v}_0|_{\Omega_k},\mathbf{r}_i)^2_{\Omega_k} \\ &= \|\,\varepsilon(\mathbf{v}_k)\,\|^2_{0,\Omega_k} \le (A_k\mathbf{v}_k,\mathbf{v}_k).\end{aligned} \tag{4.10}$$

For the subdomains Ω_k closing $\partial\Omega$, we have $\bar{\mathbf{v}}_k = \mathbf{v}_k = 0$ on $\partial\Omega_k \cap \partial\Omega$. By Friedrich's inequality and Lemma 2.1, we deduce

$$\begin{aligned}\|\varepsilon(\bar{\mathbf{v}}_k)\|^2_{0,\Omega_k} + d_k^{-2}\|\bar{\mathbf{v}}_k\|^2_{0,\Omega_k} &= \|\varepsilon(\mathbf{v}_k)\|^2_{0,\Omega_k} + d_k^{-2}\|\mathbf{v}_k\|^2_{0,\Omega_k} \\ &\lesssim \|\varepsilon(\mathbf{v}_k)\|^2_{0,\Omega_k} \le (A_k\mathbf{v}_k,\mathbf{v}_k).\end{aligned} \tag{4.11}$$

Substituting (4.10) and (4.11) into (4.9), yields (4.8).

On the other hand, we have

$$\begin{aligned}(A_0\mathbf{v}_0,\mathbf{v}_0) &\eqsim (A^*\mathbf{v}_0,\mathbf{v}_0) = (A^*(\mathbf{v}_h - \bar{\mathbf{v}}),\mathbf{v}_h - \bar{\mathbf{v}}) \\ &\le 2[(A^*\mathbf{v}_h,\mathbf{v}_h) + (A^*\bar{\mathbf{v}},\bar{\mathbf{v}})].\end{aligned} \tag{4.12}$$

It follows from (4.2) and (4.8) that

$$(A^*\bar{\mathbf{v}},\bar{\mathbf{v}}) \lesssim (\hat{A}\bar{\mathbf{v}},\bar{\mathbf{v}}) \lesssim \hat{\gamma}(A^*\mathbf{v}_h,\mathbf{v}_h).$$

Substituting the above inequality into (4.12) yields

$$(A_0\mathbf{v}_0,\mathbf{v}_0) \lesssim \hat{\gamma}(A^*\mathbf{v}_h,\mathbf{v}_h).$$

This, together with (4.8), gives (4.6).

4.2. The proof of Theorem 3.2. In this subsection we prove Theorem 3.2. The proof is similar to that of Theorem 3.2 in [15]. But, for reader's convenience, we still give the outline of the proof. We consider only the inequality (3.6), since the inequality (3.7) can be proved in the same way. To prove (3.6), we need some auxiliary results.

Consider the natural space decomposition $W_h(\Gamma) = \sum_{\Gamma_l} I_l^t W_h(\Gamma_l)$. The following result can be derived by Theorem 2.1 of [13]. This result can be regarded as a variant of the abstract Schwarz theory.

Lemma 4.5. *Assume that the following conditions are satisfied:*
(i) for each $\mu_h \in W_h(\Gamma)$, we have

$$\sum_{\Gamma_l} \langle \Lambda_l I_l \mu_h, I_l \mu_h \rangle_{\Gamma_l} \lesssim C_1 \hat{\gamma} \langle \tilde{S} \mu_h, \mu_h \rangle; \tag{4.13}$$

(ii) for any $\phi_l \in W_h(\Gamma_l)$, we have

$$\langle \tilde{S}(\sum_{\Gamma_l} I_l^t \phi_l), \sum_{\Gamma_l} I_l^t \phi_l \rangle \lesssim C_2 \Phi(\frac{d}{h}) \sum_{\Gamma_l} \langle \Lambda_l \phi_l, \phi_l \rangle_{\Gamma_l}. \tag{4.14}$$

Then the inequality (3.6) holds with $C \lesssim C_1 C_2$.

The above Lemma gives a convenient way to estimate the condition number of the preconditioned Schur complement. To estimate the constants C_1 and C_2 in Lemma 4.5, we need to study carefully a discrete dual norm on the local boundary $\partial\Omega_k (k = 1, 2, ..., N)$.

Define

$$W_h(\partial\Omega_k) = \{\phi_h|_{\partial\Omega_k} : \ \phi_h \in W_h(\Gamma)\}.$$

Let $\| \cdot \|_{-*,\partial\Omega_k}$ be the discrete dual norm defined by

$$\|\mu_h\|_{-*,\partial\Omega_k} = \sup_{\mathbf{v}_h \in V_h(\partial\Omega_k)} \frac{\langle \mu_h, \mathbf{v}_h \rangle_{\partial\Omega_k}}{\|\mathbf{v}_h\|_{\frac{1}{2},\partial\Omega_k}}, \quad \mu_h \in W_h(\partial\Omega_k)$$

with

$$\|\mathbf{v}_h\|_{\frac{1}{2},\partial\Omega_k} = (|\mathbf{v}_h|^2_{\frac{1}{2},\partial\Omega_k} + d_k^{-1} \|\mathbf{v}_h\|^2_{0,\partial\Omega_k})^{\frac{1}{2}}.$$

For ease of notation, we define $\pm\mu_h \in W_h(\Gamma)$ for $\mu_h \in W_h(\Gamma)$ as follows:

$$(\pm\mu_h)|_{\Gamma_l} = \sigma_l(\mu_h|_{\Gamma_l}), \quad for\ each\ \Gamma_l \subset \Gamma.$$

Let $\bar{A}_k$ be the local solver defined in (3.2). Define the operator $R_k : W_h(\Gamma) \to V_h(\Omega_k)$ by $R_k = \bar{A}_k^{-1} B_k^t$. The following result can be derived as in Lemma 4.3 of [15]

Lemma 4.6. *For any index k, we have*

$$G_k^{-1} \| \pm \mu_h\|^2_{-*,\partial\Omega_k} \lesssim (\bar{A}_k R_k \mu_h, R_k \mu_h)_{\Omega_k} \lesssim G_k^{-1} \| \pm \mu_h\|^2_{-*,\partial\Omega_k} \quad \forall \mu_h \in W_h(\Gamma). \tag{4.15}$$

In the following we give several extension results of the discrete dual norm $\| \cdot \|_{-*,\partial\Omega_k}$, which will be used when estimating the constant C_2. For a multiplier face Γ_l, we always use Ω_{i_l} to denote the subdomain which contains Γ_l as a full face, and use $V_i(\Gamma_l)$ to denote the trace space defining $W_h(\Gamma_l)$ (see Subsection 3.2 for the details).

It is known that the local multiplier space $W_h(\Gamma_l)$ satisfies the inverse estimate and the approximation property (i.e., H_1 and H_3 in [15]). By these, we can prove the following two results as in [15].

Lemma 4.7. *For each $\Gamma_l \subset \Gamma$, we have*

$$\sup_{\mathbf{v}_h \in V_i(\Gamma_l)} \frac{\langle \mu_l, \mathbf{v}_h \rangle_{\Gamma_l}}{\|\mathbf{v}_h\|_{\frac{1}{2},\Gamma_l}} \lesssim [1 + log(d_{i_l}/h_{i_l})] \|\mu_l\|_{-*,\Gamma_l}, \quad \forall \mu_l \in W_h(\Gamma_l). \tag{4.16}$$

Lemma 4.8. *For each $\Gamma_l \subset \Gamma$, we have*

$$\|I_l^t \mu_l\|_{-*,\partial\Omega_j} \lesssim [1 + log(d_{i_l}/h_{i_l})] \|\mu_l\|_{-*,\Gamma_l}, \quad \forall \mu_l \in W_h(\Gamma_l). \tag{4.17}$$

Here, Ω_j is any subdomain (including Ω_{i_l}) that intersects Γ_l.

The following lemma is a direct consequence of the following relations:

$$I_l^t(V_i^0(\Gamma_l)) \subset V_h(\partial\Omega_{i_l}) \text{ and } I_l^t(V_j^0(\Gamma_l)) \subset V(\partial\Omega_j)).$$

Lemma 4.9. *For each face $\Gamma_l \subset \Gamma$ we have*

$$\|\mu_h\|_{-*,\Gamma_l} \lesssim \| \pm \mu_h\|_{-*,\partial\Omega_{i_l}}, \quad \forall \mu_h \in W_h(\Gamma) \tag{4.18}$$

and

$$\sup_{\mathbf{v}_h \in V_j^0(\Gamma_l)} \frac{\langle \mu_h, \mathbf{v}_h \rangle_{\Gamma_l}}{\|\mathbf{v}_h\|_{\frac{1}{2},\Gamma_l^0}} \lesssim \| \pm \mu_h\|_{-*,\partial\Omega_j}, \quad \forall \mu_h \in W(\Gamma). \tag{4.19}$$

For ease of notation, define $S_0 = BA_0^{-1}Q_0B^t$.

Lemma 4.10. *For $\mu_h \in W_h(\Gamma)$, define $\mu_l = I_l\mu_h \in W_h(\Gamma_l)$. Then*

$$\langle S_0\mu_h, \mu_h\rangle \lesssim \Phi(\frac{d}{h}) \sum_{\Gamma_l} \langle \Lambda_l \mu_l, \mu_l\rangle_{\Gamma_l}. \tag{4.20}$$

PROOF. Define $\mathbf{u}_0 = A_0^{-1}Q_0B^t\mu_h (\in V_0)$. Then

$$\begin{aligned} \langle S_0\mu_h, \mu_h\rangle &= \langle B\mathbf{u}_0, \mu_h\rangle = (Q_0B^t\mu_h, \mathbf{u}_0) \\ &= (A_0\mathbf{u}_0, \mathbf{u}_0) \eqsim (\bar{B}^t M^{-1}\bar{B}\mathbf{u}_0, \mathbf{u}_0). \end{aligned} \tag{4.21}$$

From the definitions of B and $\bar{B}$, we know that

$$(\bar{B}\mathbf{u}_0)|_{\Gamma_l} = \alpha_l^{\frac{1}{2}}(B\mathbf{u}_0)|_{\Gamma_l}$$

using the relation above and Lemma 4.7, we have

$$\begin{aligned} (M^{-1}\bar{B}\mathbf{u}_0, \bar{B}\mathbf{u}_0) &\eqsim \langle \mu_h, B\mathbf{u}_0\rangle = \sum_{\Gamma_l} \alpha_l^{-\frac{1}{2}} \langle \mu_h, \bar{B}\mathbf{u}_0\rangle_{0,\Gamma_l} \\ &\lesssim \sum_{\Gamma_l} \sup_{\mathbf{v}_h \in V_i(\Gamma_l)} \frac{|\langle \mu_l, \mathbf{v}_h\rangle_{\Gamma_l}|}{\|\mathbf{v}_h\|_{\frac{1}{2},\Gamma_l}} \cdot \alpha_l^{-\frac{1}{2}} \|\bar{B}\mathbf{u}_0\|_{\frac{1}{2},\Gamma_l} \\ &\lesssim \Phi^{\frac{1}{2}}(\frac{d}{h}) (\sum_{\Gamma_l} \alpha_l^{-1} \|\mu_l\|^2_{-*,\Gamma_l})^{\frac{1}{2}} \cdot (\sum_{\Gamma_l} \|\bar{B}\mathbf{u}_0\|^2_{\frac{1}{2},\Gamma_l})^{\frac{1}{2}} \end{aligned}$$

By the definitions of M, the restriction $M|_{W_h(\Gamma_l)}$ is spectrally equivalent to the vector version of S_{ij} in [11] (with $\Gamma_{ij} = \Gamma_l$). Then, as in Theorem 3.1 of [11], we can verify that

$$\langle M^{-1}\phi_h, \phi_h\rangle \eqsim \sum_{\Gamma_l} \|\phi_h\|^2_{\frac{1}{2},\Gamma_l}, \quad \forall \phi_h \in W_h(\Gamma).$$

Then, we have

$$\sum_{\Gamma_l} \| \bar{B}\mathbf{u}_0 \|^2_{\frac{1}{2},\Gamma_l} \eqsim (M^{-1}\bar{B}\mathbf{u}_0, \bar{B}\mathbf{u}_0).$$

Thus, we get by (3.5)

$$\begin{aligned} (M^{-1}\bar{B}\mathbf{u}_0, \bar{B}\mathbf{u}_0) &\lesssim \Phi(\frac{d}{h}) \sum_{\Gamma_l} \alpha_l^{-1} \|\mu_l\|^2_{-*,\Gamma_l} \\ &\lesssim \Phi(\frac{d}{h}) \sum_{\Gamma_l} \langle \Lambda_l\mu_l, \mu_l\rangle_{\Gamma_l}. \end{aligned}$$

Proof of Theorem 3.2. By Lemma 4.5, we need only to estimate the constants C_1 and C_2 in (4.13) and (4.14).

It is easy to see that

$$
\begin{aligned}
\langle \tilde{S}\mu_h, \mu_h\rangle &= \sum_{k=1}^{N} \langle B_k \hat{A}_k^{-1} B_k^t \mu_h, \mu_h\rangle_{\partial\Omega_k} + \langle BA_0^{-1} Q_0 B^t \mu_h, \mu_h\rangle \\
&\eqsim \sum_{k=1}^{N} \langle \bar{A}_k^{-1} B_k^t \mu_h, B_k^t \mu_h\rangle_{\Omega_k} + \langle S_0 \mu_h, \mu_h\rangle \\
&= \sum_{k=1}^{N} (\bar{A}_k R_k \mu_h, R_k \mu_h)_{\Omega_k} + \langle S_0 \mu_h, \mu_h\rangle.
\end{aligned} \tag{4.22}
$$

We first estimate the constant C_1.

By (3.5), Lemma 4.9 and Lemma 4.6, we have

$$
\begin{aligned}
\langle \Lambda_l I_l \mu_h, I_l \mu_h\rangle_{\Gamma_l} &\lesssim \alpha_l^{-1} \|I_l \mu_h\|_{-*,\Gamma_l}^2 \lesssim \alpha_l^{-1} \| \pm I_l \mu_h\|_{-*,\partial\Omega_{i_l}}^2 \\
&\lesssim \alpha_l^{-1} G_{i_l} (\bar{A}_{i_l} R_{i_l} \mu_h, R_{i_l} \mu_h)_{\Omega_{i_l}}.
\end{aligned}
$$

Summing over Γ_l to the above inequality and using (4.22) yield (4.13) with $C_1 \lesssim \max\limits_{\Gamma_l}(G_{i_l}/\alpha_l)$.

We next estimate the constant C_2. For ease of notation, define $\phi = \sum_{\Gamma_l} I_l^t \phi_l$. Then, from (4.22), we have

$$
\begin{aligned}
&\langle \tilde{S}(\sum_{\Gamma_l} I_l^t \phi_l), \sum_{\Gamma_l} I_l^t \phi_l\rangle = \langle \tilde{S}\phi, \phi\rangle \\
&\eqsim \sum_{k=1}^{N} (\bar{A}_k R_k \phi, R_k \phi)_{\Omega_k} + \langle S_0 \phi, \phi\rangle.
\end{aligned} \tag{4.23}
$$

It follows from Lemma 4.6 that

$$
\begin{aligned}
(\bar{A}_k R_k \phi, R_k \phi)_{\Omega_k} &\lesssim G_k^{-1} \| \pm \phi\|_{-*,\partial\Omega_k}^2 \\
&\lesssim G_k^{-1} \sum_{\Gamma_l \subset \partial\Omega_k} \| \pm I_l^t (\phi|_{\Gamma_l})\|_{-*,\partial\Omega_k}^2.
\end{aligned}
$$

Substituting the above inequality into (4.23) and noting that $\phi|_{\Gamma_l} = \phi_l$ yield

$$
\begin{aligned}
&\langle \tilde{S}(\sum_{\Gamma_l} I_l^t \phi_l), \sum_{\Gamma_l} I_l^t \phi_l\rangle \lesssim \langle S_0 \phi, \phi\rangle \\
&+ \sum_{k=1}^{N} \sum_{\Gamma_l \subset \partial\Omega_k} G_k^{-1} \|I_l^t \phi_l\|_{-*,\partial\Omega_k}^2.
\end{aligned}
$$

We notice that $G_k^{-1} \le \alpha_l^{-1}$ for $\Gamma_l \subset \partial\Omega_k$. Then, the above inequality, together with Lemma 4.8, leads to

$$
\langle \tilde{S}(\sum_{\Gamma_l} I_l^t \phi_l), \sum_{\Gamma_l} I_l^t \phi_l\rangle \lesssim \langle S_0 \phi, \phi\rangle + \Phi(\frac{d}{h}) \sum_{\Gamma_l} \alpha_l^{-1} \|\phi_l\|_{-*,\Gamma_l}^2. \tag{4.24}
$$

Furthermore, Lemma 4.10 and (3.5) yield (4.14) with $C_2 \lesssim 1$.

5. Implementations

In this section, we describe the matrix form of the preconditioner $\bar{A}$, and give the definition of the local interface operator Λ_l.

5.1. On the preconditioner $\bar{A}$. The preconditioner $\bar{A}$ defined in section 3 has the form $\bar{A}^{-1} = \hat{A}^{-1} + A_0^{-1}Q_0$. In the following, we will discuss the implementation of $\bar{A}^{-1}$.

We first explain the matrix form of the operator $\hat{A}^{-1}$. From (3.2), the local operator $\hat{A}_k$ can be chosen as a preconditioner of Laplace-type operator $\bar{A}_k$ defined by (3.1). Let $\hat{\mathcal{A}}_k$ be the matrix form of $\hat{A}_k$, i.e., a preconditioner for the stiffness matrix of $\bar{A}_k$. Then, the matrix form $\hat{\mathcal{A}}$ of $\hat{A}$ is block-diagonal

$$\hat{\mathcal{A}} = diag(\hat{\mathcal{A}}_1\ \hat{\mathcal{A}}_2 \cdots \hat{\mathcal{A}}_N).$$

The action of $\hat{\mathcal{A}}$ can be implemented in parallel.

Let $\mathcal{A}^*, \mathcal{A}_0, \bar{\mathcal{B}}^t, \bar{\mathcal{B}}, \mathcal{M}, \hat{\mathcal{S}}$ denote the matrix forms of the operators $A^*, A_0, \bar{B}^t$, $\bar{B}, M$ and $\hat{S}$, respectively. By the definition, the coarse solver A_0 is just the natural restriction of A^* on the null space V_0. Let $\mathcal{C}^t : V_h(\Omega) \to V_0$ be the transformation matrix, which has non-zero elements only in floating subdomains, and $\mathcal{C}$ denotes its adjoint matrix. Then, the matrix $\mathcal{A}_0$ can be written as

$$\mathcal{A}_0 = \mathcal{C}^t\mathcal{A}^*\mathcal{C} = \mathcal{C}^t\bar{\mathcal{B}}\mathcal{M}^{-1}\bar{\mathcal{B}}^t\mathcal{C}.$$

By the definition of M, $\mathcal{M}$ is the block-diagonal matrix

$$\mathcal{M} = diag(\mathcal{M}_1\ \mathcal{M}_2 \cdots \mathcal{M}_L),$$

where L denotes the number of the multiplier faces Γ_l, and $\mathcal{M}_l (l = 1, 2, ..., L)$ is a preconditioner for the l-th diagonal block of $\hat{\mathcal{S}} = \bar{\mathcal{B}}\hat{\mathcal{A}}^{-1}\bar{\mathcal{B}}^t$. Then,

$$\mathcal{M}^{-1} = diag(\mathcal{M}_1^{-1}\ \mathcal{M}_2^{-1} \cdots \mathcal{M}_L^{-1}).$$

A simple choice of $\mathcal{M}_l^{-1}$ is to define $\mathcal{M}_l^{-1}$ as the matrix expression of Λ_l^{-1} (see the next subsection).

It is clear that the transformation matrix $\mathcal{C}$ is a block-diagonal matrix. Besides, the matrix $\bar{\mathcal{B}}$ possesses a block and sparse structure. Then, coarse solver $\mathcal{A}_0$, which is only related to the floating subdomains, is a low order and sparse matrix. The action of $\mathcal{A}_0^{-1}$ can be implemented in a cheap manner.

5.2. On the preconditioner of Schur complement. It is clear that the preconditioner $\bar{S}$ is determined by the local solvers Λ_l ($\Gamma_l \subset \Gamma$), which should satisfy the assumption (3.5). To define Λ_l, we need an auxiliary result.

Let $K_l : W_h(\Gamma_l) \to V_i^0(\Gamma_l)$ denotes the L^2 projection on $V_i^0(\Gamma_l)$ and K_l^t : $V_i^0(\Gamma_l) \to W_h(\Gamma_l)$ denotes its adjoint operator with respect to the $L^2(\Gamma_l)$ inner product. The following result can be proved as in [15]

Theorem 5.1. *Let $\hat{\Lambda}_l : V_i^0(\Gamma_l) \to V_i^0(\Gamma_l)$ be a symmetric and positive definite operator satisfying $\langle\hat{\Lambda}_l\cdot,\cdot\rangle_{\Gamma_l} \eqsim \|\cdot\|^2_{\frac{1}{2},\Gamma_l^0}$. Define $\Lambda_l = \alpha_l^{-1}K_l^t\hat{\Lambda}_l^{-1}K_l$, then the operator satisfies $(\langle\Lambda_l\cdot,\cdot\rangle_{\Gamma_l})^{1/2} \eqsim \alpha_l^{-\frac{1}{2}}\ \|\cdot\|_{-*,\Gamma_l}$.*

Since the space $W_h(\Gamma_l)$ has the same dimension with $V_i^0(\Gamma_l)$, and satisfies $inf-sup$ condition, the operator K_l is non-singular. Then,

$$\Lambda_l^{-1} = \alpha_l K_l^{-1}\hat{\Lambda}_l(K_l^t)^{-1}.$$

The operator $\hat{\Lambda}_l$ can be defined by the discrete $H^{\frac{1}{2}}$-norm on $V_i^0(\Gamma_l)$ (refer to [15]). The matrix form of Λ_l^{-1} can be built just as the second author did in Subsection 5.3 of [15] (the only difference is that a vector field is involved here). Since the matrix of K_l is sparse, the implementation of action of Λ_l^{-1} is cheap.

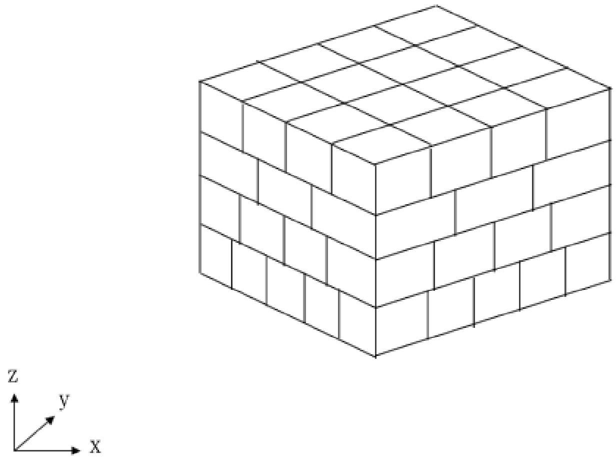

FIGURE 6.2. the geometrically non-conforming subdomain partition with 4 / 5, 4, 3, 4 in three dimensions

6. Numerical experiments

In this section, we use DDM developed in this paper to solve the compressible linear elasticity problem (2.2). Here, the domain Ω is the unit cube: $\Omega = [0,1]^3$, the Poisson ratio $\upsilon = 0.3$, and the elasticity module $E = 200$.

Let Ω be decomposed into N cubes, which are numbered as $\Omega_1, \Omega_2, \cdots, \Omega_N$. We use the uniform triangulation $\mathcal{T}_{h_k}$ and the standard Q_1 finite elements for each subdomain Ω_k. For convenience, we assume that d_k/h_k are same for every k. In our numerical experiments, the partition 4 / 5,4,3,4 means that the domain Ω is well-distributed into 4 parts along Z-axe, and is well-distributed into 5, 4, 3, 4 parts along the X-axe and Y-axe (see Figure 2). The other partitions have the similar meanings.

As demonstrated in section 2, our method results in the augmented saddle-point system (2.10). The corresponding algebraic system will be solved by the Uzawa-type method described by Algorithm 3.1 in [16]. In the inner iteration of this method, we use the inexact local solvers $\hat{A}_k$ and the coarse solver A_0 described in section 5.1; in the outer iteration, we use the inexact solvers introduced in subsection 5.2.

The initial guess is chosen as the zero vector, and the termination criterion ε is defined to be the relative residual norm. The iteration terminates when $\varepsilon \leq 1.0e-5$.

case(1). the parameter G_k is

$$G_k = \begin{cases} 10^5 & k = 1, N, \\ 1 & \texttt{else}. \end{cases}$$

the numerical results are in table 6.1.

case(2). the parameter G_k is

$$G_k = \begin{cases} 10^5 & k \text{ is odd}, \\ 1 & k \text{ is even}. \end{cases}$$

the numerical results are in table 6.2.

d_k/h_k	4 / 5,4,3,4 (N=66)	5 / 6,5,4,5,5 (N=127)	6 / 7,6,5,6,5,7 (N=220)
4	39	40	39
6	42	43	42
8	44	45	45
12	46	47	47

TABLE 6.1. Iteration counts for case(1)

d_k/h_k	4 / 5,4,3,4 (N=66)	5 / 6,5,4,5,5 (N=127)	6 / 7,6,5,6,5,7 (N=220)
4	41	43	42
6	43	45	44
8	45	46	46
12	47	48	47

TABLE 6.2. Iteration counts for case(2)

The above numerical results indicate that the iteration counts depend slightly on the ratio $\Phi(\frac{d}{h})$ and are almost independent of the number of subdomains and the large variations of the material parameters across the local interfaces. The numerical results demonstrate the efficiency of our method.

7. Conclusions

In this paper, we have proposed a DDM with Lagrange multipliers based on geometrically non-conforming subdomain partitions to solve the compressible linear elasticity problems in three dimensions. In this method, we have defined a new augmented system to handle the singularity on floating subdomains. Then, we have constructed two efficient preconditioners for the saddle-point system. Both the theoretical results and the numerical experiments show that our method is efficient.

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