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A POSTERIORI ERROR ANALYSIS FOR FEM OF THERMISTOR PROBLEMS

WALTER ALLEGRETTO AND NINGNING YAN

Abstract. In this paper, we present what we believe is the first a posteriori finite element error analysis for the system of equations that governs micromachined microsensors. Our main result is the establishment of an efficient and reliable a posteriori error estimator Θ.

Key Words. Thermistor problems, adaptive finite element methods, a posteriori error analysis.

1. Introduction

The system of equations that govern thermistor behavior has a long history. Recently, it has been the subject of intensive investigations, see e.g. [7] for a survey of the subject. We are interested in this paper in the version of the system that has been recently proposed as a model for micromachined microsensors, [4], [5], [6], [7]. The equations incorporate terms that account for heat losses to the surrounding gas and radiation effects. Some of these are expressed as nonlocal terms, and to avoid physically contradictory effects at high gas pressures, the system of equations is expressed as an obstacle problem, [3]. In this article, the existence of solutions and their long time behaviors was considered. The error analysis of numerical approximations, based on finite volume methods, were considered in [2]. In this paper, we present what we believe to be the first a posteriori error analysis for the finite element approximation of the obstacle system introduced in [3]. In particular we obtain an efficient and reliable a posteriori error estimator Θ. Our analysis involves, in part, the adaptation of results earlier obtained for elliptic equations. To the best of our knowledge, these results are new even for the classical thermistor systems, as described in [7].

2. Finite Element Approximation of Thermistor problems

Let $\Omega \subset R^2$ be a polygonal domain, $J = (0,T)$, $J_t = (0,t)$. Let $(\cdot,\cdot)$ be the inner product on Ω. In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,q,\Omega}$ and seminorm $|\cdot|_{m,q,\Omega}$. Denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$. Set

$$H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}.$$

We denote by $L^s(0,T;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from $(0,T)$ into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(0,T;W^{m,p}(\Omega))} =$

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$(\int_0^T \|v\|^s_{W^{m,p}(\Omega)}dt)^{\frac{1}{s}}$ for $s \in [1,\infty)$ and the standard modification for $s=\infty$. Similarly, one define the spaces $H^1(0,T;W^{m,p}(\Omega))$ and $C^l(0,T;W^{m,p}(\Omega))$. The details can be found in [16]. In addition c or C denotes a general positive constant independent of h. Let $\|v\|_{-1,W}$ represent the negative norm of v defined by

$$\|v\|_{-1,\Omega} = \sup_{w\in H^1_0(\Omega), w\neq 0} \frac{(v,w)}{\|w\|_{1,\Omega}},$$

$$\|v\|_{-1,\Omega\times J_t} = \sup_{w\in H^1(0,t;H^1_0(\Omega)), w\neq 0} \frac{\int_0^t (v,w)dt}{\|w\|_{1,\Omega\times J_t}}.$$

Let $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$. Set

$$H^1_D(\Omega) \equiv \{v \in H^1(\Omega) : v|_{\partial\Omega_D} = 0\}, \qquad H^1_{\phi_0}(\Omega) \equiv \{v \in H^1(\Omega) : v|_{\partial\Omega_D} = \phi_0\}.$$

Consider the model problem of thermistor problems: find $(u,\phi) \in K \times H^1_{\phi_0}(\Omega)$ with $u_t \in H^{-1}(\Omega)$ for each t such that

$$(u_t, u-v) + (k(u)\nabla u, \nabla(u-v)) + \eta(\int_\Omega G(x,y)u(y)dy, u-v)$$

$$+\alpha(u^4, u-v) \le (\sigma(u)|\nabla\phi|^2, u-v) \qquad \forall v \in K, \tag{2.1}$$

$$(\sigma(u)\nabla\phi, \nabla w) = 0 \qquad \forall w \in H^1_D(\Omega), \tag{2.2}$$

$$u(x,0) = u_0(x) \ge 0, \tag{2.3}$$

where $(\cdot,\cdot)$ denotes the standard $L^2(\Omega)$ inner product, and

$$K = \{v \in H^1_0(\Omega) : \ v \ge 0\}.$$

Throughout this paper, we assume that $u_0 \in H^1_0(\Omega)$, $\phi_0 \in C^\infty(\partial\Omega_D)$, η, α are constants, $0 \le G(x,y) < \infty$. In the physically significant case, ϕ_0 is piecewise constant function (in space) on the components of $\partial\Omega_D$ which represent the contacts. Moreover, it is assumed that $0 < c \le \sigma(s), k(s) \le C < \infty$, and there exists a constant $C_0 > 0$ such that

$$|\sigma(s) - \sigma(s')| + |k(s) - k(s')| \le C_0|s - s'|, \quad s, s' \in R.$$

Using (2.2) and Green's formula, (2.1)-(2.3) can be rewritten to be

$$(u_t, u-v) + (k(u)\nabla u, \nabla(u-v)) + \eta(\int_\Omega G(x,y)u(y)dy, u-v)$$

$$+\alpha(u^4, u-v) \le (\sigma(u)\phi\nabla\phi, \nabla(v-u)) \qquad \forall v \in K, \tag{2.4}$$

$$(\sigma(u)\nabla\phi, \nabla w) = 0 \qquad \forall w \in H^1_D(\Omega), \tag{2.5}$$

$$u(x,0) = u_0(x). \tag{2.6}$$

Let us consider the finite element approximation of problem (2.4)-(2.6). Let T^h_u be a regular partition of Ω. Let h_{τ_u} be the size of the element τ_u in T^h_u, $h = \max_{\tau_u \in T^h_u}\{h_{\tau_u}\}$. Set the finite element space S^h_u to be the standard conforming piecewise linear finite element space on T^h_u. Let $V^h_u = S^h_u \cap H^1_0(\Omega)$, $K^h = \{v \in V^h_u : v \ge 0\}$. Then, it is easy to see that $K^h \subset K$. Similarly, let T^h_ϕ be another regular partition of Ω. Let h_{τ_ϕ} be the size of the element τ_ϕ in T^h_ϕ. Set the finite element space S_ϕ^h to be the standard conforming piecewise linear finite element space on T_ϕ^h. Let $V_\phi^h = S_\phi^h \cap H_D^1(\Omega)$, and

$$V_{\phi_0}^h = \{v \in S_\phi^h : \; v|_{\partial\Omega_D} = \phi_0^I\},$$

where ϕ_0^I is the standard Lagrange interpolation of ϕ_0 on the boundary $\partial\Omega_D$.

Then the semi-discrete finite element scheme of the problem (2.4)-(2.6) is to look for $(u^h, \phi^h) \in K^h \times V_{\phi_0}^h$, for any $t \in J$, such that

$$\begin{aligned} &(u_t^h, u^h - v^h) + (k(u^h)\nabla u^h, \nabla(u^h - v^h)) \\ &+\eta(\int_\Omega G(x,y)u^h(y)dy, u^h - v^h) + \alpha((u^h)^4, u^h - v^h) \\ &\quad \le (\sigma(u^h)\phi^h\nabla\phi^h, \nabla(v^h - u^h)) \quad \forall v^h \in K^h, \end{aligned} \tag{2.7}$$

$$(\sigma(u^h)\nabla\phi^h, \nabla w^h) = 0 \qquad \forall w^h \in V_\phi^h, \tag{2.8}$$

$$u^h(x,0) = u_0^I(x),$$

where u_0^I is a piecewise linear interpolation of u_0 on T_u^h.

For the finite element space V_u^h, introduce a special piecewise linear interpolation (see [9], for details) as follows:

$$\pi_h v = \sum_{z\in\mathcal{N}_h} v_z \varphi_z, \tag{2.9}$$

where $\mathcal{N}_h$ is the set of inner nodes, φ_z is the base function on the node z,

$$v_z = \frac{\int_{\omega_z} v\psi_z}{\int_{\omega_z}\psi_z}, \tag{2.10}$$

where ω_z is the support of φ_z,

$$\psi_z = \frac{\varphi_z}{\sum_{z\in\mathcal{N}_h}\varphi_z}. \tag{2.11}$$

Then, it is easy to see that

$$\sum_{z\in\mathcal{N}_h} \psi_z = 1. \tag{2.12}$$

Lemma 2.1. *Let $\pi_h v$ be defined by (2.9). Then, for any $v \in H^1(\Omega)$ and element τ,*

$$\begin{aligned} &\|v - \pi_h v\|_{0,\tau}^2 + h_\tau^2 |v - \pi_h v|_{1,\tau}^2 \\ &\quad \le Ch_\tau^2 \sum_{\bar\tau\cap\bar\tau'\ne\emptyset} |v|_{1,\tau'}^2 + Ch_\tau \sum_{\bar\tau\cap\bar\tau'\ne\emptyset} \|v\|_{0,\bar\tau'\cap\partial\Omega}^2, \end{aligned} \tag{2.13}$$

and for any $f \in L^2(\Omega)$,

$$\begin{aligned} \int_\Omega f(v - \pi_h v) \le C\Big(&\sum_{\bar\omega_z\cap\partial\Omega=\emptyset} h_z^2 \int_{\omega_z}(f - \bar f_z)^2 + \sum_{\bar\omega_z\cap\partial\Omega\ne\emptyset} h_z^2\int_{\omega_z} f^2\Big)^{\frac12} \\ &\times\Big(|v|_{1,\Omega}^2 + \sum_{\bar\tau\cap\partial\Omega\ne\emptyset} h_\tau^{-1}\|v\|_{0,\bar\tau\cap\partial\Omega}^2\Big)^{\frac12}, \end{aligned} \tag{2.14}$$

where

$$\bar f_z = \frac{\int_{\omega_z} f}{\int_{\omega_z} 1}.$$

Moreover, assume that $v \in H_0^1(\Omega)$*. We have*

$$\|v-\pi_h v\|_{0,\tau}^2 + h_\tau^2 |v-\pi_h v|_{1,\tau}^2 \le C h_\tau^2 \sum_{\bar\tau\cap\bar\tau'\neq\emptyset} |v|_{1,\tau'}^2, \tag{2.15}$$

and for any $f \in L^2(\Omega)$*,*

$$\begin{aligned} \int_\Omega f(v-\pi_h v) \quad &\le \quad C\Big(\sum_{\bar\omega_z\cap\partial\Omega=\emptyset} h_z^2 \int_{\omega_z} (f-\bar f_z)^2 \\ &\qquad + \sum_{\bar\omega_z\cap\partial\Omega\neq\emptyset} h_z^2 \int_{\omega_z} f^2\Big)^{\frac12} |v|_{1,\Omega}. \end{aligned} \tag{2.16}$$

Proof. When no nodes of the element τ are located on the boundary $\partial\Omega$, we have that $\pi_h v = v$ if $v \in P_0$ is a constant on ω_τ, where $\omega_\tau = \{\cup\tau' :\ \bar\tau\cap\bar\tau' \neq \emptyset\}$. Let $\hat\tau$ and $\hat\omega_\tau$ be the reference domain with unit sizes mapped from τ and ω_τ, respectively. It follows from the standard theorem of quotient spaces (see, e.g., [11]) that

$$\inf_{p\in P_0} \{\|v+p\|_{1,\hat\omega_\tau}\} \le C|v|_{1,\hat\omega_\tau}.$$

Therefore,

$$\begin{aligned} &\|v-\pi_h v\|_{0,\tau}^2 + h_\tau^2|v-\pi_h v|_{1,\tau}^2 \le Ch_\tau^2(\|v-\pi_h v\|_{0,\hat\tau}^2 + |v-\pi_h v|_{1,\hat\tau}^2) \\ = \quad & Ch_\tau^2\|v-\pi_h v\|_{1,\hat\tau}^2 = Ch_\tau^2 \inf_{p\in P_0}\{\|(I-\pi_h)(v+p)\|_{1,\hat\tau}^2\} \\ \le \quad & Ch_\tau^2 \inf_{p\in P_0}\{\|v+p\|_{1,\hat\omega_\tau}^2\} \le Ch_\tau^2|v|_{1,\hat\omega_\tau}^2 \le Ch_\tau^2|v|_{1,\omega_\tau}^2. \end{aligned}$$

This proves (2.13) for the case where there is not any node of the element τ located on the boundary $\partial\Omega$.

When there is at least one node of the element τ located on the boundary $\partial\Omega$, we have that $\partial\omega_\tau \cap \partial\Omega \neq \emptyset$. It follows from Poincare's inequality (see [11], for example) that

$$\|v\|_{1,\hat\omega_\tau}^2 \le C|v|_{1,\hat\omega_\tau}^2 + C\|v\|_{0,\partial\hat\omega_\tau\cap\partial\Omega}^2.$$

It follows that

$$\begin{aligned} &\|v-\pi_h v\|_{0,\tau}^2 + h_\tau^2|v-\pi_h v|_{1,\tau}^2 \\ \le \quad & Ch_\tau^2(\|v-\pi_h v\|_{0,\hat\tau}^2 + |v-\pi_h v|_{1,\hat\tau}^2) \\ \le \quad & Ch_\tau^2\|v\|_{1,\hat\omega_\tau}^2 \le Ch_\tau^2|v|_{1,\hat\omega_\tau}^2 + Ch_\tau^2\|v\|_{0,\hat\omega_\tau\cap\partial\Omega}^2 \\ \le \quad & Ch_\tau^2|v|_{1,\omega_\tau}^2 + Ch_\tau\|v\|_{0,\omega_\tau\cap\partial\Omega}^2. \end{aligned}$$

This proves (2.13) for the case of $\bar\omega_\tau\cap\partial\Omega \neq \emptyset$, and hence (2.13) follows for all cases.

Consider (2.14). Note that $\sum_{z\in\mathcal{N}_h}\psi_z = 1$, $\int_\Omega \psi_z(v-v_z) = 0$, and $\varphi_z = \psi_z$ if $\bar\omega_z \cap \partial\Omega = \emptyset$. We have that

$$\begin{aligned}
&\int_\Omega f(v-\pi_h v) = \sum_{z\in\mathcal{N}_h}\int_\Omega f(v\psi_z - v_z\varphi_z)\\
= \ & \sum_{\bar\omega_z\cap\partial\Omega=\emptyset}\int_\Omega (f-\bar f_z)\psi_z(v-v_z) + \sum_{\bar\omega_z\cap\partial\Omega\neq\emptyset}\int_\Omega f(v\psi_z - v_z\varphi_z)\\
\leq \ & C\Big(\sum_{\bar\omega_z\cap\partial\Omega=\emptyset} h_z^2\int_{\omega_z}(f-\bar f_z)^2 + \sum_{\bar\omega_z\cap\partial\Omega\neq\emptyset} h_z^2\int_{\omega_z} f^2\Big)^{\frac12}\\
&\times\Big(\sum_{\bar\omega_z\cap\partial\Omega=\emptyset} h_z^{-2}\int_{\omega_z}(v-v_z)^2 + \sum_{\bar\omega_z\cap\partial\Omega\neq\emptyset} h_z^{-2}\int_{\omega_z}(v\psi_z - v_z\varphi_z)^2\Big)^{\frac12}.
\end{aligned}$$

Then (2.14) follows from that

$$\sum_{\bar\omega_z\cap\partial\Omega=\emptyset} h_z^{-2}\int_{\omega_z}(v-v_z)^2 \leq C|v|_{1,\Omega}^2,$$

and

$$\sum_{\bar\omega_z\cap\partial\Omega\neq\emptyset} h_z^{-2}\int_{\omega_z}(v^2+v_z^2) \leq C\Big(|v|_{1,\Omega}^2 + \sum_{\bar\tau\cap\partial\Omega\neq\emptyset} h_\tau^{-1}\|v\|_{0,\bar\tau\cap\partial\Omega}^2\Big),$$

which can be proved in a similar way to the proof of (2.13).

Note that $v|_{\partial\Omega} = 0$ when $v \in H_0^1(\Omega)$. Then (2.15) and (2.16) follow from (2.13) and (2.14) directly. □

Now, consider the interpolation in V_ϕ^h. Let $\mathcal{N}_h^D$ be the set of nodes which are not located on $\partial\Omega_D$. Similar as (2.9), define the interpolation

$$\pi_h^* v = \sum_{z\in\mathcal{N}_h^D} v_z^*\varphi_z, \tag{2.17}$$

where φ_z is the base function on the node z,

$$v_z^* = \frac{\int_{\omega_z} v\psi_z^*}{\int_{\omega_z}\psi_z^*}, \tag{2.18}$$

where ω_z is the support of φ_z,

$$\psi_z^* = \frac{\varphi_z}{\sum_{z\in\mathcal{N}_h^D}\varphi_z}. \tag{2.19}$$

Then, it is easy to see that

$$\sum_{z\in\mathcal{N}_h^D}\psi_z^* = 1. \tag{2.20}$$

And the following Lemma can be proved in a similar way to the proof of Lemma 2.1.

Lemma 2.2. *Let $\pi_h^* v$ be defined by (2.17). Then, for any $v \in H^1(\Omega)$ and element τ,*

$$\begin{aligned}
&\|v-\pi_h^* v\|_{0,\tau}^2 + h_\tau^2|v-\pi_h^* v|_{1,\tau}^2\\
&\leq Ch_\tau^2\sum_{\bar\tau\cap\bar\tau'\neq\emptyset}|v|_{1,\tau'}^2 + Ch_\tau\sum_{\bar\tau\cap\bar\tau'\neq\emptyset}\|v\|_{0,\bar\tau'\cap\partial\Omega_D}^2,
\end{aligned}\tag{2.21}$$

and for any $f \in L^2(\Omega)$,

$$\int_\Omega f(v-\pi_h^* v) \leq C\Big(\sum_{\bar\omega_z\cap\partial\Omega_D=\emptyset} h_z^2\int_{\omega_z}(f-\bar f_z)^2+\sum_{\bar\omega_z\cap\partial\Omega_D\neq\emptyset} h_z^2\int_{\omega_z} f^2\Big)^{\frac12}$$

$$\times\Big(|v|_{1,\Omega}^2+\sum_{\bar\tau\cap\partial\Omega_D\neq\emptyset} h_\tau^{-1}\|v\|_{0,\bar\tau\cap\partial\Omega_D}^2\Big)^{\frac12}, \tag{2.22}$$

where

$$\bar f_z=\frac{\int_{\omega_z} f}{\int_{\omega_z} 1}.$$

Moreover, assume that $v \in H_D^1(\Omega)$. *We have*

$$\|v-\pi_h^* v\|_{0,\tau}^2+h_\tau^2|v-\pi_h^* v|_{1,\tau}^2 \leq Ch_\tau^2\sum_{\bar\tau\cap\bar\tau'\neq\emptyset}|v|_{1,\tau'}^2, \tag{2.23}$$

and for any $f \in L^2(\Omega)$,

$$\begin{aligned}\int_\Omega f(v-\pi_h^* v) \quad &\leq \quad C\Big(\sum_{\bar\omega_z\cap\partial\Omega_D=\emptyset} h_z^2\int_{\omega_z}(f-\bar f_z)^2\\ &\qquad +\sum_{\bar\omega_z\cap\partial\Omega_D\neq\emptyset} h_z^2\int_{\omega_z} f^2\Big)^{\frac12}|v|_{1,\Omega}.\end{aligned} \tag{2.24}$$

We introduce another well known Lemma (see, e.g., [15]), which will be used in our proof of a posteriori error estimates.

Lemma 2.3. *For all* $v \in W^{1,q}(\tau)$, $1 \leq q < \infty$,

$$\|v\|_{0,q,\partial\tau} \leq C(h_\tau^{-\frac1q}\|v\|_{0,q,\tau}+h_\tau^{1-\frac1q}|v|_{1,q,\tau}). \tag{2.25}$$

3. A posteriori Error Estimate

In order to obtain a numerical solution of acceptable accuracy for the problem, the finite element meshes have to be refined according to a mesh refinement scheme. A widely used approach in engineering is the adaptive finite element approximation. At the heart of any adaptive finite element method is an a posteriori error indicator. Adaptive finite element approximation only refines the area where the error indicator is larger so that a higher density of nodes is distributed over the area where the solution is difficult to approximate. In this section, we provide the a posteriori error estimate for the thermistor problem and its finite element approximation.

3.1. Upper bound.

Firstly, let us consider the a posteriori upper bound for the error of ϕ.

Lemma 3.1. *Let* ϕ *and* ϕ^h *be the solutions of the problem (2.5) and (2.8), respectively. Assume that* $\phi \in W^{1,p}(\Omega)$, $3 < p < 4$, *for any* $t \in J$. *Then, for any* $t \in J$,

$$|\phi-\phi^h|_{1,\Omega}^2 \leq C(\eta_1^2+\eta_2^2+\eta_3^2)+C\|u-u^h\|_{1,\Omega}^2, \tag{3.1}$$

where

$$
\begin{aligned}
\eta_1^2 &= \sum_{z\in\mathcal{N}_h^D} h_z^2 \int_{\omega_z} (\nabla\cdot(\sigma(u^h)\nabla\phi^h) - (\overline{\nabla\cdot(\sigma(u^h)\nabla\phi^h)})_z)^2 \\
&\quad + \sum_{\bar\omega_z\cap\partial\Omega_D\neq\emptyset} h_z^2 \int_{\omega_z} (\nabla\cdot(\sigma(u^h)\nabla\phi^h))^2, \\
\eta_2^2 &= \sum_{l\cap\partial\Omega_D=\emptyset} h_l \int_l [\sigma(u^h)\frac{\partial\phi^h}{\partial n}]^2, \\
\eta_3^2 &= \sum_{\bar\tau_\phi\cap\partial\Omega_D\neq\emptyset} h_{\tau_\phi}^{-1}\|\phi_0-\phi_0^I\|^2_{0,\bar\tau_\phi\cap\partial\Omega_D} + \sum_{\bar\tau_\phi\cap\partial\Omega_D\neq\emptyset} h_{\tau_\phi}|\phi_0-\phi_0^I|^2_{1,\bar\tau_\phi\cap\partial\Omega_D},
\end{aligned}
$$

where l is the edge of the element, h_l is the length of l, $[v]_l$ is the jump of v on the edge l, $[\sigma(u^h)\frac{\partial\phi^h}{\partial n}]_l = 0$ if $l\subset\partial\Omega_N$, $\bar v_z$ is the integral average of v defined in Lemma 2.1.

Proof. Let $\Phi = \phi - \phi^h$, let $\Phi_I = \pi_h^*\Phi$ be defined by (2.17). Note that $\Phi|_{\partial\Omega} = (\phi_0-\phi_0^I)|_{\partial\Omega_D} \neq 0$, and thus $\Phi\bar\in H_D^1(\Omega)$, but $\Phi_I|_{\partial\Omega_D}=0$, and hence $\Phi_I\in H_D^1(\Omega)$. Let $\rho\in H^1(\Omega)$ such that the trace of ρ on $\partial\Omega$:

$$
\gamma_0\rho = \begin{cases} \Phi-\Phi_I & \text{on } \partial\Omega_D \\ 0 & \text{on } \partial\Omega_N \end{cases},
$$

and

$$
\|\rho\|_{1,\Omega} \le C\|\gamma_0\rho\|_{\frac12,\partial\Omega} \le C\|\Phi-\Phi_I\|_{\frac12,\partial\Omega_D} = C\|\phi_0-\phi_0^I\|_{\frac12,\partial\Omega_D}.
$$

Then $\Phi-\Phi_I-\rho\in H_D^1$. It follows from (2.5), (2.8) that

$$
\begin{aligned}
& c|\Phi|^2_{1,\Omega} \le (\sigma(u^h)\nabla(\phi-\phi^h),\nabla\Phi) \\
= {}& (\sigma(u)\nabla\phi,\nabla(\Phi-\Phi_I)) - (\sigma(u^h)\nabla\phi^h,\nabla(\Phi-\Phi_I)) \\
& +((\sigma(u^h)-\sigma(u))\nabla\phi,\nabla\Phi) \\
= {}& (\sigma(u)\nabla\phi,\nabla\rho) + (\sigma(u)\nabla\phi,\nabla(\Phi-\Phi_I-\rho)) \\
& + \sum_{\tau_\phi\in T_\phi^h}\int_{\tau_\phi} \nabla\cdot(\sigma(u^h)\nabla\phi^h)(\Phi-\Phi_I) \\
& - \sum_{l\cap\partial\Omega_D=\emptyset}\int_l [\sigma(u^h)\frac{\partial\phi^h}{\partial n}](\Phi-\Phi_I) \\
& - \int_{\partial\Omega_D} \sigma(u^h)\frac{\partial\phi^h}{\partial n}(\Phi-\Phi_I) + ((\sigma(u^h)-\sigma(u))\nabla\phi,\nabla\Phi) \\
= {}& \sum_{\tau_\phi\in T_\phi^h}\int_{\tau_\phi} \nabla\cdot(\sigma(u^h)\nabla\phi^h)(\Phi-\Phi_I) \\
& - \sum_{l\cap\partial\Omega_D=\emptyset}\int_l [\sigma(u^h)\frac{\partial\phi^h}{\partial n}](\Phi-\Phi_I) + (\sigma(u)\nabla\phi,\nabla\rho) \\
& - \int_{\partial\Omega_D} \sigma(u^h)\frac{\partial\phi^h}{\partial n}(\Phi-\Phi_I) + ((\sigma(u^h)-\sigma(u))\nabla\phi,\nabla\Phi).
\end{aligned} \tag{3.2}
$$

It follows from (2.22) that

$$\begin{aligned} (3.3)\qquad & \sum_{\tau_\phi \in T_\phi^h} \int_{\tau_\phi} \nabla \cdot (\sigma(u^h) \nabla \phi^h)(\Phi - \Phi_I) \\ \leq\ & C\Big(\sum_{\bar{\omega}_z \cap \partial\Omega_D = \emptyset} h_z^2 \int_{\omega_z} (\nabla \cdot (\sigma(u^h)\nabla\phi^h) - (\overline{\nabla \cdot (\sigma(u^h)\nabla\phi^h)})_z)^2 \\ & + \sum_{\bar{\omega}_z \cap \partial\Omega_D \neq \emptyset} h_z^2 \int_{\omega_z} (\nabla \cdot (\sigma(u^h)\nabla\phi^h))^2 \Big) \\ & + C\delta\Big(|\Phi|_{1,\Omega}^2 + \sum_{\bar{\tau}_\phi \cap \partial\Omega_D \neq \emptyset} h_{\tau_\phi}^{-1} \|\Phi\|_{0,\bar{\tau}_\phi \cap \partial\Omega_D}^2 \Big) \\ \leq\ & C\eta_1^2 + C\delta|\Phi|_{1,\Omega}^2 + C \sum_{\bar{\tau}_\phi \cap \partial\Omega_D \neq \emptyset} h_{\tau_\phi}^{-1} \|\phi_0 - \phi_0^I\|_{0,\bar{\tau}_\phi \cap \partial\Omega_D}^2 \\ \leq\ & C\eta_1^2 + C\eta_3^2 + C\delta|\Phi|_{1,\Omega}^2, \end{aligned}$$

where and later δ presents a small positive constant. Similarly, it follows from (2.21) and Lemma 2.3 that

$$\begin{aligned} & \sum_{l \cap \partial\Omega_D = \emptyset} \int_l [\sigma(u^h) \frac{\partial \phi^h}{\partial n}](\Phi - \Phi_I) \\ \leq\ & C \sum_{l \cap \partial\Omega_D = \emptyset} h_l \int_l [\sigma(u^h) \frac{\partial \phi^h}{\partial n}]^2 + C\delta \sum_{l \cap \partial\Omega_D = \emptyset} h_l^{-1} \int_l (\Phi - \Phi_I)^2 \\ (3.4)\qquad \leq\ & C\eta_2^2 + C\delta \sum_{l \cap \partial\Omega_D = \emptyset} \Big(h_l^{-2} \|\Phi - \Phi_I\|_{0,\tau_l^1 \cup \tau_l^2}^2 + |\Phi - \Phi_I|_{1,\tau_l^1 \cup \tau_l^2}^2 \Big) \\ \leq\ & C\eta_2^2 + C\delta\Big(\sum_{\tau_\phi} |\Phi|_{1,\tau_\phi}^2 + C \sum_{\bar{\tau}_\phi \cap \partial\Omega_D \neq \emptyset} h_{\tau_\phi}^{-1} \|\Phi\|_{0,\bar{\tau}_\phi \cap \partial\Omega_D}^2 \Big) \\ \leq\ & C\eta_2^2 + C \sum_{\bar{\tau}_\phi \cap \partial\Omega_D \neq \emptyset} h_{\tau_\phi}^{-1} \|\phi_0 - \phi_0^I\|_{0,\bar{\tau}_\phi \cap \partial\Omega_D}^2 + C\delta|\Phi|_{1,\Omega}^2 \\ \leq\ & C\eta_2^2 + C\eta_3^2 + C\delta|\Phi|_{1,\Omega}^2. \end{aligned}$$

Since $\rho_I|_{\partial\Omega} = 0$ and $\rho_I \in V_\phi^h \subset H_D^1$, equation (2.8) implies

$$- \sum_{\tau_\phi \in T_\phi^h} \int_{\tau_\phi} \nabla \cdot (\sigma(u^h)\nabla\phi^h)\rho_I + \sum_{l \cap \partial\Omega_D = \emptyset} \int_l [\sigma(u^h) \frac{\partial \phi^h}{\partial n}]\rho_I$$

$$= (\sigma(u^h)\nabla\phi^h, \nabla\rho_I) = 0.$$

Therefore,

$$
\begin{aligned}
& (\sigma(u)\nabla\phi,\nabla\rho)-\int_{\partial\Omega_D}\sigma(u^h)\frac{\partial\phi^h}{\partial n}(\Phi-\Phi_I)\\
= \ & (\sigma(u)\nabla\phi,\nabla\rho)-\int_{\partial\Omega_D}\sigma(u^h)\frac{\partial\phi^h}{\partial n}\rho-\sum_{l\cap\partial\Omega_D=\emptyset}\int_l[\sigma(u^h)\frac{\partial\phi^h}{\partial n}]\rho\\
& +\sum_{l\cap\partial\Omega_D=\emptyset}\int_l[\sigma(u^h)\frac{\partial\phi^h}{\partial n}]\rho\\
= \ & (\sigma(u)\nabla\phi-\sigma(u^h)\nabla\phi^h,\nabla\rho)-\sum_{\tau_\phi\in T_\phi^h}\int_{\tau_\phi}\nabla\cdot(\sigma(u^h)\nabla\phi^h)\rho\\
& +\sum_{l\cap\partial\Omega_D=\emptyset}\int_l[\sigma(u^h)\frac{\partial\phi^h}{\partial n}]\rho\\
= \ & (\sigma(u)\nabla\phi-\sigma(u^h)\nabla\phi^h,\nabla\rho)-\sum_{\tau_\phi\in T_\phi^h}\int_{\tau_\phi}\nabla\cdot(\sigma(u^h)\nabla\phi^h)(\rho-\rho_I)\\
& +\sum_{l\cap\partial\Omega_D=\emptyset}\int_l[\sigma(u^h)\frac{\partial\phi^h}{\partial n}](\rho-\rho_I) \qquad (3.5)\\
\le \ & C\delta\|\sigma(u)\nabla\phi-\sigma(u^h)\nabla\phi^h\|_{0,\Omega}^2+C\|\rho\|_{1,\Omega}^2+C(\eta_1^2+\eta_2^2+\eta_3^2)\\
& +C\sum_{\bar\tau_\phi\cap\partial\Omega_D\neq\emptyset}h_{\tau_\phi}^{-1}\|\rho\|_{0,\bar\tau_\phi\cap\partial\Omega_D}^2+C\delta|\rho|_{1,\Omega}^2\\
\le \ & C\delta\|\sigma(u)\nabla\phi-\sigma(u^h)\nabla\phi^h\|_{0,\Omega}^2+C\|\phi_0-\phi_0^I\|_{\frac{1}{2},\partial\Omega_D}^2\\
& +C(\eta_1^2+\eta_2^2+\eta_3^2)+C\sum_{\bar\tau_\phi\cap\partial\Omega_D\neq\emptyset}h_{\tau_\phi}^{-1}\|\phi_0-\phi_0^I\|_{0,\bar\tau_\phi\cap\partial\Omega_D}^2\\
\le \ & C(\delta|\phi-\phi^h|_{1,\Omega}^2+|\phi|_{1,p,\Omega}^2\|u-u^h\|_{0,\frac{2p}{p-2},\Omega}^2)+C(\eta_1^2+\eta_2^2+\eta_3^2)\\
& +C\Big(\sum_{\bar\tau_\phi\cap\partial\Omega_D\neq\emptyset}h_{\tau_\phi}^{-1}\|\phi_0-\phi_0^I\|_{0,\bar\tau_\phi\cap\partial\Omega_D}^2\\
& +\sum_{\bar\tau_\phi\cap\partial\Omega_D\neq\emptyset}h_{\tau_\phi}|\phi_0-\phi_0^I|_{1,\bar\tau_\phi\cap\partial\Omega_D}^2\Big)\\
\le \ & C\delta|\Phi|_{1,\Omega}^2+C\|u-u^h\|_{1,\Omega}^2+C(\eta_1^2+\eta_2^2+\eta_3^2),
\end{aligned}
$$

where we used the assumption $\phi\in W^{1,p}(\Omega)$, $3<p<4$, the embed theorem: $\|v\|_{0,q,\Omega}\le C\|v\|_{1,\Omega}$ for $1\le q<\infty$, and the interpolation theorem:

$$\|v\|_{\frac{1}{2},\gamma}^2\le C\|v\|_{0,\gamma}\|v\|_{1,\gamma}\le\frac{C}{2}(h^{-1}\|v\|_{0,\gamma}^2+h\|v\|_{1,\gamma}^2).$$

Moreover, it is easy to see that

$$
\begin{aligned}
((\sigma(u^h)-\sigma(u))\nabla\phi,\nabla\Phi) & \le C\|u-u^h\|_{0,\frac{2p}{p-2},\Omega}|\phi|_{1,p,\Omega}|\Phi|_{1,\Omega}\\
& \le C\|u-u^h\|_{1,\Omega}^2+C\delta|\Phi|_{1,\Omega}^2. \qquad (3.6)
\end{aligned}
$$

Summing up, (3.1) follows from (3.2)-(3.6). □

In order to discuss the a posteriori error estimate, we first introduce some definitions and notations. The principal idea is an adaptation of that employed for elliptic obstacle problems, e.g., [13], [19], [20]. Note that we to deal with the time dependent problem. The following functional or functions are all dependent on the time t, and thus are different from the ones introduced in [13], [19], [20]. However, for notational simplicity, we suppress reference to t when it is not relevant to the argument in question.

Introduce a functional β such that for all $t \in J$,

$$\begin{aligned} <\beta, \psi> &= -(\sigma(u)\phi\nabla\phi, \nabla\psi) - (u_t, \psi) - (k(u)\nabla u, \nabla\psi) \\ &\quad -\eta(\int_\Omega G(x,y)u(y)dy, \psi) - \alpha(u^4, \psi), \quad \forall\psi \in H_0^1(\Omega). \end{aligned} \tag{3.7}$$

Then $\beta = 0$ when $u > 0$. Introduce an approximation to β as follows:

$$\beta_h = \sum_{z \in \mathcal{N}_h} \beta_{hz}\psi_z, \tag{3.8}$$

where ψ_z is defined by (2.11),

$$\beta_{hz}(x,t) := \begin{cases} P_z(x,t) & z \in \mathcal{C}_h \\ Q_z(t) & z \in \mathcal{N}_h \setminus \mathcal{C}_h \end{cases}, \tag{3.9}$$

$$P_z(x,t) = \nabla \cdot (\sigma(u^h)\phi^h\nabla\phi^h) - \eta \int_\Omega G(x,y)u^h(y)dy - u_t^h, \tag{3.10}$$

$$\begin{aligned} Q_z(t) &= \frac{1}{\int_{\omega_z}\psi_z}\Big(- (\sigma(u^h)\phi^h\nabla\phi^h, \nabla\varphi_z) - (u_t^h, \varphi_z) - (k(u^h)\nabla u^h, \nabla\varphi_z) \\ &\quad -\eta(\int_\Omega G(x,y)u^h(y)dy, \varphi_z) - \alpha((u^h)^4, \varphi_z)\Big), \end{aligned} \tag{3.11}$$

where

$$\mathcal{C}_h := \{z \in \mathcal{N}_h : \ u^h = 0 \text{ and } P_z \le 0 \text{ on } \omega_z\}, \tag{3.12}$$

φ_z is the base function on the node z, ω_z is the support of φ_z. Introduce a residual $\mathcal{G}_h$ by

$$\begin{aligned} <\mathcal{G}_h, \rho> &= (u_t^h, \psi) + (k(u^h)\nabla u^h, \nabla\rho) + \eta(\int_\Omega G(x,y)u^h(y)dy, \rho) \\ &\quad +\alpha((u^h)^4, \rho) + (\sigma(u^h)\phi^h\nabla\phi^h, \nabla\rho) + (\beta_h, \rho) \quad \forall\rho \in H_0^1(\Omega). \end{aligned} \tag{3.13}$$

Let $e = u^h - u$. Equations (3.7) and (3.13) imply that $\forall\rho \in H_0^1(\Omega)$,

$$\begin{aligned} <\mathcal{G}_h, \rho> &= (e_t, \rho) + (k(u^h)\nabla u^h - k(u)\nabla u, \nabla\rho) \\ &\quad +\eta(\int_\Omega G(x,y)e(y)dy, \rho) + \alpha((u^h)^4 - u^4, \rho) \\ &\quad +(\sigma(u^h)\phi^h\nabla\phi^h - \sigma(u)\phi\nabla\phi, \nabla\rho) + <\beta_h - \beta, \rho>, \end{aligned} \tag{3.14}$$

where

$$<\beta_h, \rho> := (\beta_h, \rho).$$

Lemma 3.2. *Let $\mathcal{G}_h$ be defined by (3.13). Then,*

$$\|\mathcal{G}_h\|_{-1,\Omega}^2 \le C(\eta_2^2 + \eta_4^2 + \eta_5^2), \tag{3.15}$$

where η_2 is defined in Lemma 3.1,

$$\eta_4^2 = \sum_{z \in \mathcal{N}_h \setminus \mathcal{C}_h} \int_{\omega_z} h_z^2(F - \bar{F}_z)^2 + \sum_{z \in \mathcal{N}_h \setminus \mathcal{C}_h, \bar{\omega}_z \cap \partial\Omega \neq \emptyset} h_z^2 \int_{\omega_z} F^2,$$

$$\eta_5^2 = \sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [k(u^h)\frac{\partial u^h}{\partial n}]^2,$$

where

$$\begin{aligned} F &= u_t^h - \nabla\cdot(k(u^h)\nabla u^h) + \eta\int_\Omega G(x,y)u^h(y)dy \\ &\quad +\alpha(u^h)^4 - \nabla\cdot(\sigma(u^h)\phi^h\nabla\phi^h), \end{aligned}$$

l, h_l, $[v]_l$ and $\bar{v}_z$ are defined in Lemma 3.1.

Proof. From (2.12), (3.8) and (3.13) we obtain that for any function $\rho \in H_0^1(\Omega)$ and $t \in J$,

$$\begin{aligned} <\mathcal{G}_h,\rho> &= (u_t^h,\rho) + (k(u^h)\nabla u^h,\nabla\rho) + \eta(\int_\Omega G(x,y)u^h(y)dy,\rho) \\ &\quad +\alpha((u^h)^4,\rho) + (\sigma(u^h)\phi^h\nabla\phi^h,\nabla\rho) + (\beta_h,\rho) \\ &= \sum_{z\in\mathcal{C}_h}\Big((u_t^h,\rho\psi_z) + (k(u^h)\nabla u^h,\nabla(\rho\psi_z)) \\ &\quad +\eta(\int_\Omega G(x,y)u^h(y)dy,\rho\psi_z) + \alpha((u^h)^4,\rho\psi_z) \\ &\quad +(\sigma(u^h)\phi^h\nabla\phi^h,\nabla(\rho\psi_z)) + (\beta_{hz}\psi_z,\rho)\Big) \\ &\quad +\sum_{z\in\mathcal{N}_h\backslash\mathcal{C}_h}\Big((u_t^h,\rho\psi_z) + (k(u^h)\nabla u^h,\nabla(\rho\psi_z)) \\ &\quad +\eta(\int_\Omega G(x,y)u^h(y)dy,\rho\psi_z) + \alpha((u^h)^4,\rho\psi_z) \\ &\quad +(\sigma(u^h)\phi^h\nabla\phi^h,\nabla(\rho\psi_z)) + (\beta_{hz}\psi_z,\rho)\Big) \\ &= A_1 + A_2. \end{aligned} \tag{3.16}$$

Note that $u^h = 0$ on ω_z if $z \in \mathcal{C}_h$. Then it follows from (3.9) and (3.10) that

$$\begin{aligned} A_1 &= \sum_{z\in\mathcal{C}_h}\Big((u_t^h,\rho\psi_z) + (k(u^h)\nabla u^h,\nabla(\rho\psi_z)) \\ &\quad +\eta(\int_\Omega G(x,y)u^h(y)dy,\rho\psi_z) + \alpha((u^h)^4,\rho\psi_z) \\ &\quad +(\sigma(u^h)\phi^h\nabla\phi^h,\nabla(\rho\psi_z)) + (\beta_{hz}\psi_z,\rho)\Big) \\ &= \sum_{z\in\mathcal{C}_h}\Big(-(P_z,\rho\psi_z) + (P_z\psi_z,\rho)\Big) = 0. \end{aligned} \tag{3.17}$$

Let

$$\hat{\rho}_z = \frac{\int_{\omega_z}\rho\psi_z}{\int_{\omega_z}\psi_z}. \tag{3.18}$$

Note that $\psi_z = \varphi_z$ if $\bar{\omega}_z \cap \partial\Omega = \emptyset$. Then, it follows from (3.9) and (3.11) that

$$
\begin{aligned}
A_2 = & \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h} \Big((u_t^h, \rho\psi_z) + (k(u^h)\nabla u^h, \nabla(\rho\psi_z)) \\
& + \eta(\int_\Omega G(x,y)u^h(y)dy, \rho\psi_z) + \alpha((u^h)^4, \rho\psi_z) \\
& + (\sigma(u^h)\phi^h\nabla\phi^h, \nabla(\rho\psi_z)) + (\beta_{hz}\psi_z, \rho) \Big)
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, \bar{\omega}_z\cap\partial\Omega=\emptyset} \Big((u_t^h, (\rho-\hat{\rho}_z)\psi_z) + (k(u^h)\nabla u^h, \nabla((\rho-\hat{\rho}_z)\psi_z)) \\
& + \eta(\int_\Omega G(x,y)u^h(y)dy, (\rho-\hat{\rho}_z)\psi_z) + \alpha((u^h)^4, (\rho-\hat{\rho}_z)\psi_z) \\
& + (\sigma(u^h)\phi^h\nabla\phi^h, \nabla((\rho-\hat{\rho}_z)\psi_z)) \Big) \\
& + \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, \bar{\omega}_z\cap\partial\Omega\neq\emptyset} \Big((u_t^h, \rho\psi_z - \hat{\rho}_z\varphi_z) + (k(u^h)\nabla u^h, \nabla(\rho\psi_z - \hat{\rho}_z\varphi_z)) \\
& + \eta(\int_\Omega G(x,y)u^h(y)dy, \rho\psi_z - \hat{\rho}_z\varphi_z) + \alpha((u^h)^4, \rho\psi_z - \hat{\rho}_z\varphi_z) \\
& + (\sigma(u^h)\phi^h\nabla\phi^h, \nabla(\rho\psi_z - \hat{\rho}_z\varphi_z)) \Big) \\
= & \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, \bar{\omega}_z\cap\partial\Omega=\emptyset} (F, (\rho-\hat{\rho}_z)\psi_z) \\
& + \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, \bar{\omega}_z\cap\partial\Omega=\emptyset, l\subset\omega_z} \int_l [k(u^h)\frac{\partial u^h}{\partial n}](\rho-\hat{\rho}_z)\psi_z \\
& + \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, \bar{\omega}_z\cap\partial\Omega\neq\emptyset} (F, \rho\psi_z - \hat{\rho}_z\varphi_z) \\
& + \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, \bar{\omega}_z\cap\partial\Omega\neq\emptyset, l\subset\omega_z, l\cap\partial\Omega=\emptyset} \int_l [k(u^h)\frac{\partial u^h}{\partial n}](\rho\psi_z - \hat{\rho}_z\varphi_z) \\
& + \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, \bar{\omega}_z\cap\partial\Omega=\emptyset, l\subset\omega_z} \int_l [\sigma(u^h)\phi^h\frac{\partial \phi^h}{\partial n}](\rho-\hat{\rho}_z)\psi_z \\
& + \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, \bar{\omega}_z\cap\partial\Omega\neq\emptyset, l\subset\omega_z, l\cap\partial\Omega=\emptyset} \int_l [\sigma(u^h)\phi^h\frac{\partial \phi^h}{\partial n}](\rho\psi_z - \hat{\rho}_z\varphi_z),
\end{aligned} \tag{3.19}
$$

where

$$
\begin{aligned}
F = & \; u_t^h - \nabla\cdot(k(u^h)\nabla u^h) + \eta\int_\Omega G(x,y)u^h(y)dy + \alpha(u^h)^4 \\
& - \nabla\cdot(\sigma(u^h)\phi^h\nabla\phi^h).
\end{aligned} \tag{3.20}
$$

Note that (the proof is similar to Lemma 2.1)

$$\int_{\omega_z} \psi_z(\rho - \hat{\rho}_z) = 0,$$

$$\|\rho - \hat{\rho}_z\|_{0,\omega_z} \le Ch_z|\rho|_{1,\omega_z}, \quad |\rho - \hat{\rho}_z|_{1,\omega_z} \le C|\rho|_{1,\omega_z},$$

and for $\rho \in H_0^1(\Omega)$, $\bar{\omega}_z \cap \partial\Omega \neq \emptyset$,

$$\|\rho\|_{0,\omega_z} \leq Ch_z|\rho|_{1,\omega_z}, \quad \|\hat{\rho}_z\|_{0,\omega_z} \leq Ch_z|\rho|_{1,\omega_z}, \quad |\hat{\rho}_z|_{1,\omega_z} = 0.$$

Then it follows that

$$\begin{aligned} A_2 \;\leq\;& C\Big(\sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h,\bar{\omega}_z\cap\partial\Omega=\emptyset} \int_{\omega_z} h_z^2(F-\bar{F}_z)^2 + \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h,\bar{\omega}_z\cap\partial\Omega\neq\emptyset} h_z^2 \int_{\omega_z} F^2 \\ &+ \sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [k(u^h)\frac{\partial u^h}{\partial n}]^2 + \sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [\sigma(u^h)\phi^h\frac{\partial\phi^h}{\partial n}]^2 \Big)^{\frac{1}{2}} \\ &\times\Big(\sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h,\bar{\omega}_z\cap\partial\Omega=\emptyset} (h_z^{-2}\|\rho-\hat{\rho}_z\|_{0,\omega_z}^2 + |\rho-\hat{\rho}_z|_{1,\omega_z}^2) \\ &+ \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h,\bar{\omega}_z\cap\partial\Omega\neq\emptyset} (h_z^{-2}\|\rho\|_{0,\omega_z}^2 + h_z^{-2}\|\hat{\rho}_z\|_{0,\omega_z}^2 + |\rho|_{1,\omega_z}^2 + |\hat{\rho}_z|_{1,\omega_z}^2) \Big)^{\frac{1}{2}} \\ \leq\;& C(\eta_2^2+\eta_4^2+\eta_5^2)^{\frac{1}{2}}|\rho|_{1,\Omega}. \end{aligned} \tag{3.21}$$

Therefore, it follows from (3.16), (3.17) and (3.21) that

$$< \mathcal{G}_h, \rho > \leq \; C(\eta_2^2+\eta_4^2+\eta_5^2)^{\frac{1}{2}}\|\rho\|_{1,\Omega} \quad \forall \rho \in H_0^1(\Omega). \tag{3.22}$$

This proves (3.15). □

Lemma 3.3. *Let u and u^h be the solutions of the problem (2.1) and (2.7), respectively. Assume that $u \in W^{1,\infty}(\Omega)$, $\phi \in W^{1,p}(\Omega)$, $3 < p < 4$, and $\phi\nabla\phi \in (L^\infty(\Omega))^2$. Then for any $t \in (0,T)$,*

$$\|(u-u^h)(t)\|_{0,\Omega}^2 + \int_0^t \|u-u^h\|_{1,\Omega}^2 \leq C\Theta^2, \tag{3.23}$$

where

$$\Theta^2 = \eta_0^2 + \int_0^t (\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2+\eta_5^2+\eta_6^2)dt,$$

$$\eta_0^2 = \|(u-u^h)(0)\|_{0,\Omega}^2,$$

η_i, $i = 1,2,3$, are defined in Lemma 3.1, η_i, $i = 4,5$, are defined in Lemma 3.2,

$$\eta_6^2 = \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h} (-Q_z)\int_{\omega_z} u^h\psi_z,$$

where Q_z and ψ_z are defined by (3.11) and (2.11), respectively.

Proof. Let $e = u^h - u$. It follows from (3.14), Poincare's inequality and the assumptions on $k(\cdot)$ that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|e\|_{0,\Omega}^2 + c\|e\|_{1,\Omega}^2 \leq (e_t,e) + (k(u^h)\nabla u^h - k(u^h)\nabla u, \nabla e) \\ =\;& < \mathcal{G}_h, e > + ((k(u)-k(u^h))\nabla u, \nabla e) \\ & -\eta\int_\Omega\int_\Omega G(x,y)e(x)e(y)dxdy + \alpha(u^4-(u^h)^4, e) \\ & +(\sigma(u)\phi\nabla\phi - \sigma(u^h)\phi^h\nabla\phi^h, \nabla e) + < \beta-\beta_h, e > \\ =\;& A_1+A_2+A_3+A_4+A_5+A_6. \end{aligned} \tag{3.24}$$

It follows from Lemma 3.2 that

$$\begin{aligned} A_1 &= < \mathcal{G}_h, e > \leq C\|\mathcal{G}_h\|^2_{-1,\Omega} + C\delta\|e\|^2_{1,\Omega} \\ &\leq C(\eta_2^2 + \eta_4^2 + \eta_5^2) + C\delta\|e\|^2_{1,\Omega}. \end{aligned} \tag{3.25}$$

It is easy to see that

$$\begin{aligned} A_2 &= ((k(u) - k(u^h))\nabla u, \nabla e) \leq C\|u - u^h\|_{0,\Omega}|u|_{1,\infty,\Omega}|e|_{1,\Omega} \\ &\leq C\|e\|^2_{0,\Omega} + C\delta\|e\|^2_{1,\Omega}, \end{aligned} \tag{3.26}$$

$$A_3 = -\eta \int_\Omega \int_\Omega G(x,y)e(x)e(y)dxdy \leq C\|e\|^2_{0,\Omega}, \tag{3.27}$$

and

$$\begin{aligned} A_4 &= \alpha(u^4 - (u^h)^4, e) = \alpha((u^h - u)((u^h)^3 + (u^h)^2 u + u^h u^2 + u^3), e) \\ &\leq C\|e\|^2_{0,\Omega}\|(u^h)^3 + (u^h)^2 u + u^h u^2 + u^3\|_{0,\infty,\Omega} \leq C\|e\|^2_{0,\Omega}. \end{aligned} \tag{3.28}$$

It follows from Lemma 3.1 and Poincare' inequality that

$$\begin{aligned} A_5 &= (\sigma(u)\phi\nabla\phi - \sigma(u^h)\phi^h\nabla\phi^h, \nabla e) \\ &\leq \|\sigma(u)\phi\nabla\phi - \sigma(u^h)\phi^h\nabla\phi^h\|_{0,\Omega}|e|_{1,\Omega} \\ &\leq C\|u - u^h\|_{0,\Omega}|e|_{1,\Omega}\|\phi\nabla\phi\|^2_{0,\infty,\Omega} \\ &\quad + C|\phi^h|_{0,\infty,\Omega}|\phi - \phi^h|_{1,\Omega}|e|_{1,\Omega} \\ &\quad + C|\phi|_{1,p,\Omega}\|\phi - \phi^h\|_{0,\frac{2p}{p-2},\Omega}|e|_{1,\Omega} \\ &\leq C\|e\|^2_{0,\Omega} + C\|\phi - \phi^h\|^2_{1,\Omega} + C\delta|e|^2_{1,\Omega} \\ &\leq C\|e\|^2_{0,\Omega} + C|\phi - \phi^h|^2_{1,\Omega} + \|\phi - \phi^h\|^2_{0,\partial\Omega_D} + C\delta|e|^2_{1,\Omega} \\ &\leq C\|e\|^2_{0,\Omega} + C(\eta_1^2 + \eta_2^2 + \eta_3^2) + C\delta|e|^2_{1,\Omega}. \end{aligned} \tag{3.29}$$

Next, we have:

$$\begin{aligned} A_6 &= < \beta - \beta_h, e > = < \beta, u^h - u > + < \beta_h, u - u^h > \\ &= < \beta, u^h - u > + \sum_{z \in \mathcal{C}_h} (\beta_{hz}\psi_z, u - u^h) \\ &\quad + \sum_{z \in \mathcal{N}_h \setminus \mathcal{C}_h} (\beta_{hz}\psi_z, u) - \sum_{z \in \mathcal{N}_h \setminus \mathcal{C}_h} (\beta_{hz}\psi_z, u^h). \end{aligned} \tag{3.30}$$

Note that $u^h \in K$. It follows from (2.1) and (3.7) that

$$< \beta, u^h - u > \leq 0. \tag{3.31}$$

Observe that $\beta_{hz} \leq 0$, $\psi_z \geq 0$, $u \geq 0$, and $u^h = 0$ in ω_z if $z \in \mathcal{C}_h$. Hence,

$$\sum_{z \in \mathcal{C}_h} (\beta_{hz}\psi_z, u - u^h) \leq 0. \tag{3.32}$$

Similarly,

$$\sum_{z \in \mathcal{N}_h \setminus \mathcal{C}_h} (\beta_{hz}\psi_z, u) \leq 0. \tag{3.33}$$

Then, equations (3.30)-(3.33) yield

$$
\begin{aligned}
A_6 &\leq -\sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h}(\beta_{hz}\psi_z, u^h) \\
&= \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h}(-Q_z)\int_{\omega_z} u^h\psi_z = \eta_6^2. \qquad (3.34)
\end{aligned}
$$

Summing up, we obtain from equations (3.24)-(3.29) and (3.34) that

$$
\begin{aligned}
\frac{d}{dt}\|e\|_{0,\Omega}^2 + \|e\|_{1,\Omega}^2 &\leq C(\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2+\eta_5^2+\eta_6^2) \\
&\quad +C\|e\|_{0,\Omega}^2 + C\delta\|e\|_{1,\Omega}^2.
\end{aligned}
$$

That is

$$
\begin{aligned}
&\|e(t)\|_{0,\Omega}^2 + \int_0^t \|e\|_{1,\Omega}^2 dt \\
\leq\ & \|e(0)\|_{0,\Omega}^2 + C\int_0^t (\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2+\eta_5^2+\eta_6^2)dt \\
&+C\int_0^t \|e\|_{0,\Omega}^2 dt + C\delta\int_0^t \|e\|_{1,\Omega}^2 dt \\
\leq\ & C\Theta^2 + C\int_0^t \|e\|_{0,\Omega}^2 dt + C\delta\int_0^t \|e\|_{1,\Omega}^2 dt,
\end{aligned}
$$

and therefore,

$$
\|e(t)\|_{0,\Omega}^2 + \int_0^t \|e\|_{1,\Omega}^2 dt \leq C\Theta^2 + C\int_0^t \|e\|_{0,\Omega}^2 dt. \qquad (3.35)
$$

Then (3.23) follows from (3.35) and Gronwall's inequality. □

Lemma 3.4. *Let β and β_h be defined by (3.7) and (3.8), respectively. Assume that $u \in W^{1,\infty}(\Omega)$, $\phi \in W^{1,p}(\Omega)$, $3 < p < 4$, and $\phi\nabla\phi \in (L^\infty(\Omega))^2$. Then, for all $t \geq t_0 > 0$,*

$$
\|\beta - \beta_h\|_{-1,\Omega\times J_t} \leq C(\frac{1+t}{t})^{\frac{1}{2}}\Theta, \qquad (3.36)
$$

where $J_t = (0,t) \subset (0,T) = J$, and Θ is defined in Lemma 3.3.

Proof. For any $\rho \in H^1(0,t;H_0^1(\Omega))$, it follows from (3.14) and Lemma 3.1-3.3 that

$$
\begin{aligned}
&\int_0^t < \beta_h - \beta, \rho > dt \\
=\ & \int_0^t \Big(< \mathcal{G}_h, \rho > -(e_t,\rho) - (k(u^h)\nabla u^h - k(u)\nabla u, \nabla\rho) \\
&-\eta(\int_\Omega G(x,y)e(y)dy, \rho) - \alpha((u^h)^4 - u^4, \rho) \qquad (3.37) \\
&-(\sigma(u^h)\phi^h\nabla\phi^h - \sigma(u)\phi\nabla\phi, \nabla\rho)\Big)dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^t \Big(< \mathcal{G}_h, \rho > +(e,\rho_t) - (k(u^h)\nabla u^h - k(u)\nabla u, \nabla\rho) \\
&\quad -\eta(\int_\Omega G(x,y)e(y)dy, \rho) - \alpha((u^h)^4 - u^4, \rho) \\
&\quad -(\sigma(u^h)\phi^h\nabla\phi^h - \sigma(u)\phi\nabla\phi, \nabla\rho)\Big)dt + (e,\rho)(0) - (e,\rho)(t) \qquad (3.38)\\
&\le C\Big(\int_0^t (\|\mathcal{G}_h\|^2_{-1,\Omega} + \|e\|^2_{1,\Omega} + \|\phi-\phi^h\|^2_{1,\Omega}dt \\
&\quad +C(\|e(t)\|^2_{0,\Omega} + \|e(0)\|^2_{0,\Omega}\Big)^{\frac{1}{2}} \\
&\quad \times\Big(\int_0^t (\|\rho\|^2_{1,\Omega} + \|\rho_t\|^2_{0,\Omega})dt + \|\rho(t)\|^2_{0,\Omega} + \|\rho(0)\|^2_{0,\Omega}\Big)^{\frac{1}{2}} \\
&\le C(\frac{1+t}{t})^{\frac{1}{2}}\Big(\int_0^t (\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2+\eta_5^2+\eta_6^2)dt \\
&\quad +\|e(0)\|^2_{0,\Omega}\Big)^{\frac{1}{2}} \|\rho\|_{1,\Omega\times J_t} \\
&\le C(\frac{1+t}{t})^{\frac{1}{2}}\Theta\|\rho\|_{1,\Omega\times J_t},
\end{aligned}
$$

where we used the trace theorem: for all $t \ge t_0 > 0$,

$$
\|\rho(t)\|^2_{0,\Omega} \le C(\frac{1+t}{t})\|\rho\|^2_{1,\Omega\times J_t}, \quad \|\rho(0)\|^2_{0,\Omega} \le C(\frac{1+t}{t})\|\rho\|^2_{1,\Omega\times J_t}.
$$

Thus, (3.36) follows from the definition of the negative norm and (3.38). □

Using Lemma 3.1-3.4, we can state the following a posteriori upper bound for the error.

Theorem 3.1. *Let (u,ϕ) and (u^h,ϕ^h) be the solutions of (2.1)-(2.2) and (2.7)-(2.8), and β, β_h be defined by (3.7) and (3.8). Let*

$$
E^2 = \|(u-u^h)(0)\|^2_{0,\Omega} + \|(u-u^h)(t)\|^2_{0,\Omega} + \int_0^t (\|u-u^h\|^2_{1,\Omega} + \|\phi-\phi^h\|^2_{1,\Omega})dt.
$$

Assume that $u \in W^{1,\infty}(\Omega)$, $\phi \in W^{1,p}(\Omega)$, $3 < p < 4$, and $\phi\nabla\phi \in (L^\infty(\Omega))^2$. Then, for all $t \ge t_0 > 0$,

$$
E^2 + \frac{t}{1+t}\|\beta-\beta_h\|^2_{-1,\Omega\times J_t} \le C\Theta^2, \qquad (3.39)
$$

where Θ is defined in Lemma 3.3.

Proof. It follows from Lemma 3.1, 3.3 and Poincare's inequality that

$$
\begin{aligned}
\int_0^t \|\phi-\phi^h\|^2_{1,\Omega}dt &\le C\int_0^t |\phi-\phi^h|^2_{1,\Omega}dt + C\int_0^t \|\phi-\phi^h\|^2_{0,\partial\Omega_D}dt \\
&\le C\int_0^t (\eta_1^2+\eta_2^2+\eta_3^2)dt + C\int_0^t \|u-u^h\|^2_{1,\Omega}dt \le C\Theta^2. \qquad (3.40)
\end{aligned}
$$

Then, (3.39) follows from Lemma 3.3, Lemma 3.4 and (3.40). □

3.2. Lower bound. Let us consider the a posteriori lower bound of the error concerning the efficiency of the a posteriori error estimator.

Theorem 3.2. *Let E, Θ, β, β_h be as defined in Theorem 3.1. Assume that $u \in W^{1,\infty}(\Omega)$, $\phi \in W^{1,p}(\Omega)$, $3 < p < 4$. Then,*

$$\Theta^2 \leq CE^2 + C\int_0^t \|\beta - \beta_h\|^2_{-1,\Omega} + C\epsilon^2, \tag{3.41}$$

where

$$\epsilon^2 \quad = \quad \int_0^t (\eta_1^2 + \eta_3^2 + \eta_4^2)dt + h^2\int_0^t \|u_t - u_t^h\|^2_{0,\Omega}dt.$$

Proof. We only need to consider $\int_0^t \eta_i^2 dt$, $i = 2, 5, 6$.

Firstly, we introduce some notations (the idea is same as in [13]). Let element $\tau = \triangle z_\tau^1 z_\tau^2 z_\tau^3$, edge $l = \bar{\tau}_l^1 \cap \bar{\tau}_l^2 = \overline{z_l^1 z_l^2}$. Set $\gamma_\tau = \varphi_{z_\tau^1}\varphi_{z_\tau^2}\varphi_{z_\tau^3}$, $\gamma_l = \varphi_{z_l^1}\varphi_{z_l^2}$, where φ_z is the base function on the node z. For the element τ_l^j, $j = 1, 2$, choose $\alpha_{\tau_l^j, z^i_{\tau_l^j}}$, $i = 1, 2, 3$, such that

$$\sum_{i=1}^3 \alpha_{\tau_l^j, z^i_{\tau_l^j}} \int_{\tau_l^j} \gamma_{\tau_l^j}\varphi_{z^i_{\tau_l^j}}\psi_{z^k_{\tau_l^j}} = \int_{\tau_l^j}\gamma_l\psi_{z^k_{\tau_l^j}} \qquad k = 1, 2, 3, \tag{3.42}$$

where ψ_z is defined by (2.11), and set $\psi_{z^k_{\tau_l^j}} = 0$, $\alpha_{z^k_{\tau_l^j}} = 0$ if $z^k_{\tau_l^j} \cap \partial\Omega \neq \emptyset$. Set

$$w_l = \bar{J}_h|_l(\gamma_l - \sum_{j=1}^2\sum_{i=1}^3 \alpha_{\tau_l^j, z^i_{\tau_l^j}}\gamma_{\tau_l^j}\varphi_{z^i_{\tau_l^j}}),$$

where

$$\bar{J}_h|_l = \frac{\int_l J_h}{\int_l 1}, \quad J_h = [\sigma(u^h)\frac{\partial\phi^h}{\partial n}].$$

Then, it is easy to see that $supp\{w_l\} = \tau_l^1 \cup \tau_l^2$, because $supp\{\gamma_l\} = \tau_l^1 \cup \tau_l^2$ and $supp\{\gamma_{\tau_l^j}\} = \tau_l^j$, $j = 1, 2$. The definition of $\alpha_{\tau_l^j, z^i_{\tau_l^j}}$ with equation (3.42) implies that

$$\int_{\tau_l^1\cup\tau_l^2} w_l\psi_y = 0 \qquad \forall y \in \mathcal{N}_h.$$

Moreover, considering that $\sigma(s) \geq c > 0$, and $|\sigma(s) - \sigma(s') \leq C|s - s'|$, we have that if h is small enough,

$$\|J_h\|^2_{0,l} \leq C\|\bar{J}_h\|^2_{0,l} \leq C\sum_{y\in\mathcal{N}_h\cap l}\int_l \bar{J}_h w_l\psi_y \leq C\sum_{y\in\mathcal{N}_h\cap l}\int_l J_h w_l\psi_y.$$

It is easy to check that

$$\|w_l\|^2_{0,\tau_l^1\cup\tau_l^2} \leq Ch_l\|J_h\|^2_{0,l}, \quad |w_l|^2_{1,\tau_l^1\cup\tau_l^2} \leq Ch_l^{-1}\|J_h\|^2_{0,l}.$$

Then using the technique in, say, [9], [13], [21], it follows from (2.2) that

$$\begin{aligned}
& h_l\|J_h\|_{0,l}^2 \\
\leq\ & Ch_l\sum_{y\in\mathcal{N}_h\cap l}\int_l J_h w_l\psi_y = Ch_l\sum_{y\in\mathcal{N}_h\cap l}\int_l[\sigma(u^h)\frac{\partial\phi^h}{\partial n}-\sigma(u)\frac{\partial\phi}{\partial n}]w_l\psi_y \\
=\ & Ch_l\sum_{y\in\mathcal{N}_h\cap l}\int_{\tau_l^1\cup\tau_l^2}(\sigma(u^h)\nabla\phi^h-\sigma(u)\nabla\phi)\nabla(w_l\psi_y) \\
& +Ch_l\sum_{y\in\mathcal{N}_h\cap l}\int_{\tau_l^1\cup\tau_l^2}\nabla\cdot(\sigma(u^h)\nabla\phi^h)w_l\psi_y \\
=\ & Ch_l\sum_{y\in\mathcal{N}_h\cap l}\int_{\tau_l^1\cup\tau_l^2}(\sigma(u^h)\nabla\phi^h-\sigma(u)\nabla\phi)\nabla(w_l\psi_y) \\
& +Ch_l\sum_{y\in\mathcal{N}_h\cap l}\int_{\tau_l^1\cup\tau_l^2}\left(\nabla\cdot(\sigma(u^h)\nabla\phi^h)-(\overline{\nabla\cdot(\sigma(u^h)\nabla\phi^h)})_y\right)w_l\psi_y \\
\leq\ & C\|\sigma(u^h)\nabla\phi^h-\sigma(u)\nabla\phi\|_{0,\tau_l^1\cup\tau_l^2}^2+C\delta(h_l^2|w_l|_{1,\tau_l^1\cup\tau_l^2}^2+\|w_l\|_{0,\tau_l^1\cup\tau_l^2}^2) \\
& +Ch_l^2\sum_{y\in\mathcal{N}_h\cap l}\int_{\tau_l^1\cup\tau_l^2}\left(\nabla\cdot(\sigma(u^h)\nabla\phi^h)-(\overline{\nabla\cdot(\sigma(u^h)\nabla\phi^h)})_y\right)^2 \\
\leq\ & C(\|u-u^h\|_{1,\tau_l^1\cup\tau_l^2}^2+|\phi-\phi^h|_{1,\tau_l^1\cup\tau_l^2}^2)+C\delta h_l\|J_h\|_{0,l}^2 \\
& +Ch_l^2\sum_{y\in\mathcal{N}_h\cap l}\int_{\tau_l^1\cup\tau_l^2}\left(\nabla\cdot(\sigma(u^h)\nabla\phi^h)-(\overline{\nabla\cdot(\sigma(u^h)\nabla\phi^h)})_y\right)^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^t\eta_2^2dt &= \int_0^t\sum_{l\cap\partial\Omega=\emptyset}h_l\|J_h\|_{0,l}^2dt \\
&\leq C\int_0^t(\|u-u^h\|_{1,\Omega}^2+|\phi-\phi^h|_{1,\Omega}^2)dt \\
&\quad +C\sum_{z\in\mathcal{N}_h}h_z^2\int_{\omega_z}\left(\nabla\cdot(\sigma(u^h)\nabla\phi^h)-(\overline{\nabla\cdot(\sigma(u^h)\nabla\phi^h)})_z\right)^2 \\
&\leq CE^2+C\int_0^t\eta_1^2dt\leq CE^2+C\epsilon^2.
\end{aligned}\tag{3.43}$$

Note that $[k(u^h)\frac{\partial u^h}{\partial n}]_l=0$ if $l\subset\omega_z$, $z\in\mathcal{C}_h$, because of $u^h|_{\omega_z}=0$. Otherwise, let $J_h=[k(u^h)\frac{\partial u^h}{\partial n}]$, and set w_l as above. Moreover, note that $\int_{\tau_l^1\cup\tau_l^2}w_l\psi_y=0$, $\forall y\in\mathcal{N}_h\cap l$. We have that $\int_{\omega_y}w_l\psi_y=0$, and hence $(\hat{w}_l)_y=0$, for all $y\in\mathcal{N}_h\cap l$, where $(\hat{w}_l)_y$ is defined by (3.18). Furthermore, $w_l\psi_z=0$ if $z\cap\bar{l}=\emptyset$, because that $supp\{w_l\}=\tau_l^1\cup\tau_l^2$. Therefore, $(\hat{w}_l)_z=0$ for all $z\in\mathcal{N}_h$, and (3.16), (3.17) and

(3.19) imply that

$$
\begin{aligned}
& < \mathcal{G}_h, w_l > \\
= & \sum_{z\in\mathcal{N}_h\backslash\mathcal{C}_h,\bar{\omega}_z\cap\partial\Omega=\emptyset} (F,(w_l-(\hat{w}_l)_z)\psi_z) \\
& + \sum_{z\in\mathcal{N}_h\backslash\mathcal{C}_h,\bar{\omega}_z\cap\partial\Omega=\emptyset,l\subset\omega_z} \int_l [k(u^h)\frac{\partial u^h}{\partial n}](w_l-(\hat{w}_l)_z)\psi_z \\
& + \sum_{z\in\mathcal{N}_h\backslash\mathcal{C}_h,\bar{\omega}_z\cap\partial\Omega\neq\emptyset} (F,w_l\psi_z-(\hat{w}_l)_z\varphi_z) \\
& + \sum_{z\in\mathcal{N}_h\backslash\mathcal{C}_h,\bar{\omega}_z\cap\partial\Omega\neq\emptyset,l\subset\omega_z} \int_l [k(u^h)\frac{\partial u^h}{\partial n}](w_l\psi_z-(\hat{w}_l)_z\varphi_z) \\
= & \sum_{z\in\mathcal{N}_h\backslash\mathcal{C}_h} (F,w_l\psi_z) + \sum_{z\in\mathcal{N}_h\backslash\mathcal{C}_h} \int_l [k(u^h)\frac{\partial u^h}{\partial n}]w_l\psi_z \\
= & \sum_{y\in(\mathcal{N}_h\backslash\mathcal{C}_h)\cap l} ((F-\bar{F}_y),w_l\psi_y) + \sum_{y\in\mathcal{N}_h\cap l} \int_l [k(u^h)\frac{\partial u^h}{\partial n}]w_l\psi_y.
\end{aligned}
$$

Then, it follows from (3.14) that

$$
\begin{aligned}
& h_l\int_l [k(u^h)\frac{\partial u^h}{\partial n}]^2 \leq Ch_l \sum_{y\in\mathcal{N}_h\cap l} \int_l [k(u^h)\frac{\partial u^h}{\partial n}]w_l\psi_y \\
= & Ch_l < \mathcal{G}_h, w_l > -Ch_l \sum_{y\in(\mathcal{N}_h\backslash\mathcal{C}_h)\cap l} \int_{\omega_y} (F-\bar{F}_y)w_l\psi_y \\
= & Ch_l\Big((e_t,w_l)+(k(u^h)\nabla u^h-k(u)\nabla u,\nabla w_l) \\
& +\eta(\int_\Omega G(x,y)e(y)dy,w_l)+\alpha((u^h)^4-u^4,w_l) \\
& +(\sigma(u^h)\phi^h\nabla\phi^h-\sigma(u)\phi\nabla\phi,\nabla w_l)+ < \beta_h-\beta, w_l > \Big) \\
& -Ch_l \sum_{y\in(\mathcal{N}_h\backslash\mathcal{C}_h)\cap l} \int_{\omega_y} (F-\bar{F}_y)w_l\psi_y \\
\leq & C\Big(h_l^2\|e_t\|^2_{0,\tau_l^1\cup\tau_l^2} + \|k(u^h)\nabla u^h-k(u)\nabla u\|^2_{0,\tau_l^1\cup\tau_l^2} \\
& +h_l^2\|\int_\Omega G(x,y)e(y)dy\|^2_{0,\tau_l^1\cup\tau_l^2} + h_l^2\|(u^h)^4-u^4\|^2_{0,\tau_l^1\cup\tau_l^2} \\
& +\|\sigma(u^h)\phi^h\nabla\phi^h-\sigma(u)\phi\nabla\phi\|^2_{0,\tau_l^1\cup\tau_l^2} + \|\beta_h-\beta\|^2_{-1,\tau_l^1\cup\tau_l^2}
\end{aligned}
$$

$$
\begin{aligned}
&+h_l^2 \sum_{y\in(\mathcal{N}_h\setminus\mathcal{C}_h)\cap l} \int_{\omega_y} (F-\bar{F}_y)^2\Big) + C\delta(\|w_l\|^2_{0,\tau_l^1\cup\tau_l^2} + h_l^2|w_l|^2_{1,\tau_l^1\cup\tau_l^2}) \\
\le\ & C\Big(h_l^2\|e_t\|^2_{0,\tau_l^1\cup\tau_l^2} + \|e\|^2_{1,\tau_l^1\cup\tau_l^2} + h_l^2\|\int_\Omega G(x,y)e(y)dy\|^2_{0,\tau_l^1\cup\tau_l^2} \\
&+\|\phi-\phi^h\|^2_{1,\tau_l^1\cup\tau_l^2} + \|\beta_h-\beta\|^2_{-1,\tau_l^1\cup\tau_l^2} \\
&+h_l^2 \sum_{y\in(\mathcal{N}_h\setminus\mathcal{C}_h)\cap l} \int_{\omega_y} (F-\bar{F}_y)^2\Big) + C\delta h_l \int_l [k(u^h)\frac{\partial u^h}{\partial n}]^2,
\end{aligned}
$$

where F is defined by (3.20). Therefore,

$$
\begin{aligned}
\int_0^t \eta_5^2 dt &= \int_0^t \sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [k(u^h)\frac{\partial u^h}{\partial n}]^2 dt \\
&\le C\int_0^t \Big(h^2\|u_t-u_t^h\|^2_{0,\Omega} + \|u-u^h\|^2_{1,\Omega} + \|\phi-\phi^h\|^2_{1,\Omega} \\
&\quad +\|\beta_h-\beta\|^2_{-1,\Omega} + \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h} h_z^2\int_{\omega_z}(F-\bar{F}_z)^2\Big)dt \\
&\le CE^2 + Ch^2\int_0^t \|u_t-u_t^h\|^2_{0,\Omega}dt \\
&\quad +C\int_0^t\|\beta_h-\beta\|^2_{-1,\Omega}dt + \int_0^t \eta_4^2 dt \\
&\le CE^2 + C\epsilon^2 + C\int_0^t \|\beta_h-\beta\|^2_{-1,\Omega}dt.
\end{aligned} \tag{3.44}
$$

Last, let us consider $\int_0^t \eta_6 dt$. Note that

$$
\begin{aligned}
\int_0^t \eta_6^2 &= \int_0^t \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h} (-Q_z)\int_{\omega_z}\psi_z u_h \\
&= \int_0^t \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, u^h(z)>0} (-Q_z)\int_{\omega_z}\psi_z u_h \\
&\quad + \int_0^t \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, u^h|_{\omega_z}\equiv 0} (-Q_z)\int_{\omega_z}\psi_z u_h \\
&\quad + \int_0^t \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, u^h(z)=0, \exists x\in\omega_z, u^h(x)>0} (-Q_z)\int_{\omega_z}\psi_z u_h.
\end{aligned} \tag{3.45}
$$

Since $Q_z = 0$ if $u^h(z) > 0$, we have

$$
\int_0^t \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, u^h(z)>0} (-Q_z)\int_{\omega_z}\psi_z u_h = 0, \tag{3.46}
$$

and, clearly,

$$
\int_0^t \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, u^h|_{\omega_z}\equiv 0} (-Q_z)\int_{\omega_z}\psi_z u_h = 0. \tag{3.47}
$$

When $z \in \mathcal{N}_h \setminus \mathcal{C}_h$, $u^h(z) = 0$ and there exists a point $x \in \omega_z$ such that $u^h(x) > 0$, there exists an element $\tau_z \subset \omega_z$ such that $y, z \in \mathcal{N}_h \cap \bar{\tau}_z$, $u^h(y) > 0$ and hence

$Q_y = 0$. It follows that

$$\begin{aligned} -Q_z &= |Q_z| = |Q_z - Q_y| = |\beta_h(z) - \beta_h(y)| = |(y-z)\cdot(\nabla\beta_h)_{\tau_z}| \\ &\leq Ch_z|\beta_h|_{1,\infty,\tau_z} = Ch_z|\beta_h + \bar{F}_z|_{1,\infty,\tau_z} \leq Ch_z^{-1}\|\beta_h + \bar{F}_z\|_{0,\tau_z}, \end{aligned} \tag{3.48}$$

where $\bar{F}_z$ is the average of F on ω_z, and F is defined by (3.20). Let $w_\tau \in H_0^1(\tau_z)$ be a polynomial, such that

$$\|\beta_h + \bar{F}_z\|_{0,\tau_z}^2 \leq C\int_{\tau_z}(\beta_h + \bar{F}_z)w_\tau, \tag{3.49}$$

and

$$\|w_\tau\|_{0,\tau_z}^2 \leq C\|\beta_h + \bar{F}_z\|_{0,\tau_z}^2, \quad |w_\tau|_{1,\tau_z}^2 \leq Ch_z^{-2}\|\beta_h + \bar{F}_z\|_{0,\tau_z}^2. \tag{3.50}$$

Then, it follows from (3.13), (3.14) and (3.49), (3.50) that

$$\begin{aligned} &\|\beta_h + \bar{F}_z\|_{0,\tau_z}^2 \\ \leq\ & C\int_{\tau_z}(\beta_h + \bar{F}_z)w_\tau = C\int_{\tau_z}(\beta_h + F_z)w_\tau + C\int_{\tau_z}(\bar{F}_z - F_z)w_\tau \\ =\ & C < \mathcal{G}_h, w_\tau > + C\int_{\tau_z}(\bar{F}_z - F_z)w_\tau \\ =\ & C\Big((e_t, w_\tau) + (k(u^h)\nabla u^h - k(u)\nabla u, \nabla w_\tau) + \eta(\int_\Omega G(x,y)e(y)dy, w_\tau) \\ & + \alpha((u^h)^4 - u^4, w_\tau) + (\sigma(u^h)\phi^h\nabla\phi^h - \sigma(u)\phi\nabla\phi, \nabla w_\tau) \\ & + < \beta_h - \beta, w_\tau > \Big) + C\int_{\tau_z}(\bar{F}_z - F_z)w_\tau \\ \leq\ & C\Big(\|e_t\|_{0,\tau_z}^2 + h_z^{-2}\|k(u^h)\nabla u^h - k(u)\nabla u\|_{0,\tau_z}^2 + \|\int_\Omega G(x,y)e(y)dy\|_{0,\tau_z}^2 \\ & + \|(u^h)^4 - u^4\|_{0,\tau_z}^2 + h_z^{-2}\|\sigma(u^h)\phi^h\nabla\phi^h - \sigma(u)\phi\nabla\phi\|_{0,\tau_z}^2 \\ & + h_z^{-2}\|\beta_h - \beta\|_{-1,\tau_z}^2 + \int_{\tau_z}(\bar{F}_z - F_z)^2\Big) + C\delta(\|w_\tau\|_{0,\tau_z}^2 + h_z^2|w_\tau|_{1,\tau_z}^2) \\ \leq\ & C\Big(\|e_t\|_{0,\tau_z}^2 + h_z^{-2}\|u - u^h\|_{1,\tau_z}^2 + \|\int_\Omega G(x,y)e(y)dy\|_{0,\tau_z}^2 \\ & + h_z^{-2}\|\phi - \phi^h\|_{1,\tau_z}^2 + h_z^{-2}\|\beta_h - \beta\|_{-1,\tau_z}^2 + \int_{\tau_z}(\bar{F}_z - F_z)^2\Big) \\ & + C\delta\|\beta_h + \bar{F}_z\|_{0,\tau_z}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\beta_h + \bar{F}_z\|_{0,\tau_z}^2 \leq\ & C\Big(\|e_t\|_{0,\tau_z}^2 + h_z^{-2}\|u - u^h\|_{1,\tau_z}^2 + \|\int_\Omega G(x,y)e(y)dy\|_{0,\tau_z}^2 \\ & + h_z^{-2}\|\phi - \phi^h\|_{1,\tau_z}^2 + h_z^{-2}\|\beta_h - \beta\|_{-1,\tau_z}^2 + \int_{\tau_z}(\bar{F}_z - F_z)^2\Big). \end{aligned} \tag{3.51}$$

Note that $u^h \geq 0$, $0 \leq \psi_z \leq 1$ and $u^h(z) = 0$. Let B be the biggest ball with center z and radius r, $B \subset \omega_z$. Then

$$\int_{\omega_z}\psi_z u^h \leq \int_{\omega_z} u^h \leq C\int_B u^h = C\int_B (u^h(y) - u^h(z))dy. \tag{3.52}$$

Note that

$$
\begin{aligned}
\int_B (u^h(y) - u^h(z))dy &= \int_B (y-z)\nabla u^h dy = \frac{1}{2}\int_B \nabla(|y-z|^2)\nabla u^h dy \\
&\le C\int_B \nabla(|y-z|^2)k^*(u^h)\nabla u^h dy,
\end{aligned} \tag{3.53}
$$

where $k^*(u^h) = \min_{x\in B}\{k(u^h)\}$. Note that $k^*(u_h)$ is a constant on B, and u^h is piecewise constant. We have that $\nabla\cdot(k^*(u^h)\nabla u^h) = 0$ on every element. Therefore,

$$
\begin{aligned}
&\int_B \nabla(|y-z|^2)k^*(u^h)\nabla u^h dy \\
= &\sum_{l\cap B\neq\emptyset}\int_{l\cap B}[k^*(u^h)\frac{\partial u^h}{\partial n}]|y-z|^2 + \int_{\partial B} k^*(u^h)\frac{\partial u^h}{\partial n}|y-z|^2 \\
= &\sum_{l\cap B\neq\emptyset}\int_{l\cap B}[k^*(u^h)\frac{\partial u^h}{\partial n}]|y-z|^2 + r^2\int_{\partial B} k^*(u^h)\frac{\partial u^h}{\partial n} \\
= &\sum_{l\cap B\neq\emptyset}\int_{l\cap B}[k^*(u^h)\frac{\partial u^h}{\partial n}]|y-z|^2 - r^2\sum_{l\cap B\neq\emptyset}\int_{l\cap B}[k^*(u^h)\frac{\partial u^h}{\partial n}] \\
= &\sum_{l\cap B\neq\emptyset}\int_{l\cap B}[k^*(u^h)\frac{\partial u^h}{\partial n}](|y-z|^2 - r^2) \\
\le\ &Ch_z^2\sum_{l\cap B\neq\emptyset}\int_{l\cap B}|[k(u^h)\frac{\partial u^h}{\partial n}]| \\
\le\ &Ch_z^2\sum_{l\subset\omega_z}\int_l|[k(u^h)\frac{\partial u^h}{\partial n}]| \le Ch_z^{\frac{5}{2}}(\sum_{l\subset\omega_z}\int_l[k(u^h)\frac{\partial u^h}{\partial n}]^2)^{\frac{1}{2}}.
\end{aligned} \tag{3.54}
$$

It follows from (3.52)-(3.54) that

$$
\int_{\omega_z}\psi_z u^h \le Ch_z^{\frac{5}{2}}(\sum_{l\subset\omega_z}\int_l[k(u^h)\frac{\partial u^h}{\partial n}]^2)^{\frac{1}{2}}. \tag{3.55}
$$

Summing up, equations (3.44)-(3.48), (3.51) and (3.55) yield

$$
\begin{aligned}
\int_0^t \eta_6^2 &= \int_0^t \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, u^h(z)=0, \exists x\in\omega_z, u^h(x)>0} (-Q_z)\int_{\omega_z}\psi_z u_h \\
&\le C\int_0^t \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h, u^h(z)=0, \exists x\in\omega_z, u^h(x)>0} \Big(h_z^{-1}\|\beta_h+\bar{F}_z\|_{0,\tau_z} h_z^{\frac{5}{2}} \\
&\quad \times (\sum_{l\subset\omega_z}\int_l [k(u^h)\frac{\partial u^h}{\partial n}]^2)^{\frac{1}{2}}\Big) \qquad (3.56)\\
&\le C\int_0^t \sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h u^h(z)=0, \exists x\in\omega_z, u^h(x)>0} \Big(h_z^2\|\beta_h+\bar{F}_z\|_{0,\tau_z}^2\Big) \\
&\quad + C\int_0^t \sum_{l\cap\partial\Omega=\emptyset}\Big(h_l\int_l [k(u^h)\frac{\partial u^h}{\partial n}]^2\Big) \\
&\le C\int_0^t \Big(h^2\|u_t-u_t^h\|_{0,\Omega}^2+\|u-u^h\|_{1,\Omega}^2+\|\phi-\phi^h\|_{1,\Omega}^2 \\
&\quad +\|\beta_h-\beta\|_{-1,\Omega}^2+\sum_{z\in\mathcal{N}_h\setminus\mathcal{C}_h} h_z^2\int_{\omega_z}(\bar{F}_z-F_z)^2\Big)dt + C\int_0^t \eta_5^2 \\
&\le CE^2+C\epsilon^2+C\int_0^t \|\beta_h-\beta\|_{-1,\Omega}^2 dt.
\end{aligned}
$$

Therefore, (3.41) follows from (3.43), (3.44) and (3.56). □

Remark 3.1. *Note that*

$$
\frac{t}{1+t}\|\beta-\beta_h\|_{-1,\Omega\times J_t}^2 \ge Ct\ \|\beta-\beta_h\|_{-1,\Omega\times J_t}^2 = C\int_0^t \|\beta-\beta_h\|_{-1,\Omega\times J_t}^2 d\theta.
$$

It follows from theorems 3.1 and 3.2 that for all $t\ge t_0>0$,

$$
E^2+\int_0^t \|\beta-\beta_h\|_{-1,\Omega\times J_t}^2 d\theta \le C\Theta^2,
$$

and

$$
\Theta^2 \le CE^2 + C\int_0^t \|(\beta-\beta_h)(\theta)\|_{-1,\Omega}^2 d\theta + C\epsilon^2.
$$

Note that E *describes the error between the exact solution of the problem and its finite element approximation, and* $\beta-\beta_h$ *describes the approximation of the contact set. Generally,* $\epsilon^2=o(h^2)$ *is a higher order term. Therefore, it can be concluded that* Θ *is an efficient and reliable a posteriori estimator.*

4. Discussion

We have established the existence of the a posteriori error estimator Θ. Future work involves numerical experiments validating the use of Θ, and the establishment of similar results for more general systems and/or boundary conditions.

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Department of Mathematical and Statistics Science, University of Alberta, Edmonton AB, Canada T6G 2G1

Institute of System Sciences, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing(100080), China