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EVOLUTIONARY STABILITY OF IDEAL FREE DISPERSAL STRATEGIES: A NONLOCAL DISPERSAL MODEL

Dedicated to Herb Freedman on the occasion of his 70th birthday

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ABSTRACT. An important question in the study of the evolution of dispersal is what kind of dispersal strategies are evolutionary stable. This work is motivated by recent work of Cosner *et al.* [**9**], in which they introduced a class of ideal free dispersal kernels and found conditions suggesting that they determine evolutionarily stable dispersal strategies. The goals of this paper are to introduce a more general class of ideal free dispersal kernels and further to show that such ideal free dispersal strategies are indeed evolutionary stable. Our work also extends some recent work on the evolutionary stability of ideal free dispersal for reaction-diffusion equations and patch models to nonlocal dispersal models.

1 Nonlocal dispersal models Spatially explicit population models in continuous time traditionally have been formulated in terms of reaction-diffusion equations [**4**] or analogous discrete-diffusion systems [**5**]. In recent years there has been considerable interest in models where the dispersal process is described by a nonlocal integral operator; see for example [**2, 9, 12, 18, 19, 21, 23, 24, 29, 30**]. A problem of particular interest in spatial ecology is that of determining which dispersal strategies are likely to evolve and persist; see [**1, 5, 6, 7, 9, 17, 20, 21, 23, 25, 26**]. A class of strategies that have been found to be evolutionarily stable in various contexts are those that lead to an ideal free distribution of the population; see [**1, 5, 6, 7, 9**]. (An ideal free

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distribution of a population is the distribution that would arise if all individuals were able to locate themselves so as to optimize their fitness. In mathematical models, that situation is characterized by the population having a stable equilibrium distribution where fitness is equal in all locations and where there is no net movement at equilibrium; see for example [**5, 6, 9**].) The present work is motivated by recent work of Cosner *et al.* [**9**], in which they introduced a class of ideal free dispersal kernels and found conditions suggesting that they determine evolutionarily stable dispersal strategies. The goals of this paper are to introduce a more general class of ideal free dispersal kernels and further to show that such ideal free dispersal strategies are indeed evolutionary stable.

A dispersal strategy is evolutionarily stable with respect to a second strategy if a resident species using the first strategy can resist invasion by a small population of mutants using the second strategy. Mathematically, this situation can be studied by formulating a competition model for the two populations and determining the stability of the single-species equilibrium where only the resident species is present. Suppose that some resident species adopts an ideal free dispersal strategy and assume that a mutant, which adopts a non-ideal free dispersal strategy, tries to invade when rare. Linearized stability analysis reveals that the dominant eigenvalue of the corresponding linear eigenvalue problem is zero; i.e., one has neutral stability here and cannot even conclude from the local stability analysis whether the mutant can or cannot invade. In this paper we will construct some Lyapunov functionals which help establish the global asymptotic stability of the single-species equilibrium for a resident species using an ideal free strategy in competition with an invading population using any non-ideal free strategy.

Our work in this paper extends some recent work on the evolutionary stability of ideal free dispersal for reaction-diffusion equations [**1, 6**] and patch models [**7**] to nonlocal dispersal models. A new difficulty for nonlocal dispersal models is the loss of compactness of solution trajectories and we apply some new ideas from [**18**] to overcome such difficulty.

1.1 Single species model Cosner *et al.* [**9**] proposed the single species model

$$u_t = \int_\Omega k^*(x,y)u(y,t)\,dy - u(x,t)\int_\Omega k^*(y,x)\,dy + u[m(x)-u], \quad x\in\Omega,\ t>0, \tag{1.1}$$

where we assume that $m > 0$ in Ω and is non-constant. We also assume that $\Omega \subset \mathbb{R}^N$ is bounded, that $m \in C(\overline{\Omega})$, and that k^* is non-negative, $k^* \in C(\overline{\Omega} \times \overline{\Omega})$ and that for some $\delta > 0$, $k^*(x, y) > 0$ for $|x - y| < \delta$.

Definition. We say that $k^*(x, y)$ is an ideal free dispersal strategy if

$$\int_\Omega k^*(x,y)m(y)\,dy = m(x)\int_\Omega k^*(y,x)\,dy, \quad x \in \Omega. \tag{1.2}$$

Note that $m(x)$ is a steady state of (1.1) if and only if k^* satisfies (1.2). In particular, if $m(x)$ is a steady state of (1.1), the population distribution of the single species at equilibrium is ideal free; i.e., the fitness of the species, which is given by $m - u$, is equal to zero across the whole habitat Ω. This explains why it makes sense to refer to k^* as an ideal free dispersal strategy.

1.2 Competing species Following Cosner *et al.* [**9**], we propose to investigate the two-species competition model

$$\begin{cases} u_t = \displaystyle\int_\Omega k^*(x,y)u(y,t)\,dy \\ \qquad\qquad - u(x,t)\displaystyle\int_\Omega k^*(y,x)\,dy + u[m(x) - u - v], \\ v_t = \displaystyle\int_\Omega k(x,y)v(y,t)\,dy \\ \qquad\qquad - v(x,t)\displaystyle\int_\Omega k(y,x)\,dy + v[m(x) - u - v], \end{cases} \tag{1.3}$$

where we assume that k and k^* are non-negative, $k, k^* \in C(\overline{\Omega} \times \overline{\Omega})$ and that for some $\delta > 0$, $k(x, y) > 0$ and $k^*(x, y) > 0$ for $|x - y| < \delta$.

Our first main result is

Theorem 1. *Suppose that both k^* and k are continuous and positive in $\bar{\Omega} \times \bar{\Omega}$, k^* is an ideal free dispersal strategy and k is not an ideal dispersal strategy. Then, the steady state $(m(x), 0)$ of (1.3) is globally asymptotically stable in the $C(\overline{\Omega}) \times C(\overline{\Omega})$ norm relative to all positive and continuous initial data.*

Remark 1.1. If k is also an ideal free dispersal strategy, system (1.3) has a continuum of positive steady states in the form of the 1-parameter family $\{(u, v) = (sm, (1 - s)m) : 0 < s < 1\}$, as both species are using ideal free dispersal strategies. When this occurs, the steady states $\{(u,v) = (sm,(1-s)m) : 0 \le s \le 1\}$ are not locally asymptotically stable among positive continuous initial data.

Our second main result is the following integral inequality, which not only plays a crucial role in the proof of Theorem 1 but is of independent interest.

Theorem 2. *Let $h : \overline{\Omega} \times \overline{\Omega} \to [0,\infty)$ be a continuous non-negative function. Then the following two statements are equivalent:*

(i) $\displaystyle\int_\Omega h(x,y)\,dy = \int_\Omega h(y,x)\,dy$ *for all* $x \in \Omega$.

(ii) $\displaystyle\int_\Omega \int_\Omega h(x,y)\frac{f(x)}{f(y)}\,dx\,dy \ge \int_\Omega \int_\Omega h(x,y)\,dx\,dy$ *for all* $f \in C(\overline{\Omega})$ *with* $f(x) > 0$ *on* $\overline{\Omega}$.

If we further assume that $h(x,y)+h(y,x)$ does not vanish on any open set of Ω and $\int_\Omega h(x,y)dy = \int_\Omega h(y,x)dy$ for all $x \in \Omega$, then

$$\int_\Omega \int_\Omega h(x,y)\frac{f(x)}{f(y)}\,dx\,dy = \int_\Omega \int_\Omega h(x,y)\,dx\,dy$$

holds for $f \in C(\overline{\Omega})$ and $f(x) > 0$ in $\overline{\Omega}$ if and only f is a constant.

This paper is organized as follows: We first present some technical preliminaries in Section 2. In Section 3, we construct a Lyapunov functional for system (1.3). Theorems 1 and 2 are established in Sections 4 and 5, respectively.

2 Technical preliminaries Since operators of the form

$$(Ku)(x) = \int_\Omega k^*(x,y)u(y)\,dy \tag{2.1}$$

are bounded on $C(\overline{\Omega})$, the models (1.1) and (1.3) can be viewed as ordinary differential equations on the Banach spaces $C(\overline{\Omega})$ and $C(\overline{\Omega}) \times C(\overline{\Omega})$ respectively, so the existence of a unique local solution for any given initial data in those spaces follows from the contraction mapping principle. To establish global existence all that is needed is to show that on any finite time interval solutions remain bounded, because as long as they remain bounded they can be continued forward in time. We are interested only in nonnegative solutions. To see that solutions with nonnegative initial data remain nonnegative and bounded we can use results on monotonicity together with super-solutions and sub-solutions. Suppose that $u(x,t)$ is continuous on $[0,T)\times\overline{\Omega}$ and differentiable with respect to t. We say that u is a super-solution (sub-solution) of (1.1) if

$$u_t(x,t) \geq (\leq) \int_\Omega k^*(x,y)u(y,t)\,dy - u(x,t)\int_\Omega k^*(y,x)\,dy + u[m(x)-u] \tag{2.2}$$

on $[0,T)\times\overline{\Omega}$. We have

Lemma 2.1. *If u_1 and u_2 are respectively a super-solution and a sub-solution of* (1.1) on $[0,T)\times\overline{\Omega}$ with $u_1(x,0)\geq u_2(x,0)$ then $u_1(x,t)\geq u_2(x,t)$ on $[0,T)\times\overline{\Omega}$. If $u_1(x_0,0)>u_2(x_0,0)$ for some $x_0\in\overline{\Omega}$ then $u_1(x,t)>u_2(x,t)$ on $(0,T)\times\overline{\Omega}$.

Lemma 2.1 follows from the arguments used to prove Theorems 2.1 and 3.1 of [**23**], which in turn are partly based on the arguments used to prove Proposition 2.4 of [**22**]. See also [**29**], Proposition 2.1. The detailed assumptions on the structure of $k^*(x,y)$ in those results are slightly different from ours but the proofs are still valid in our case. Related results have been obtained by various other authors. Once this comparison principle for a single inequality is available, a corresponding result for the competition system (1.3) follows by using arguments analogous to those used in the case of reaction-diffusion systems to obtain a comparison principle for competitive systems from the one for a single equation. In the context of competition, an appropriate ordering for comparing (u_1,v_1) and (u_2,v_2) is given by

$$(u_1,v_1)\geq(u_2,v_2) \iff u_1\geq u_2 \text{ and } v_1\leq v_2. \tag{2.3}$$

For the system (1.3) we say that (u,v) is a super-solution (sub-solution) on $[0,T)\times\overline{\Omega}$ if

$$\begin{cases} u_t \geq (\leq) \displaystyle\int_\Omega k^*(x,y)u(y,t)\,dy - u(x,t)\int_\Omega k^*(y,x)\,dy + u[m(x)-u-v], \\ v_t \leq (\geq) \displaystyle\int_\Omega k(x,y)v(y,t)\,dy - v(x,t)\int_\Omega k(y,x)\,dy + v[m(x)-u-v] \end{cases} \tag{2.4}$$

on $[0,T)\times\overline{\Omega}$. We have

Lemma 2.2. *If* (u_1,v_1) *and* (u_2,v_2) *are respectively a super-solution and a sub-solution of* (1.3) *on* $[0,T)\times\overline{\Omega}$ *with* $(u_1(x,0),v_1(x,0))\geq(u_2(x,0),v_2(x,0))$ *in the sense of* (2.3), *then* $(u_1(x,t),v_1(x,t))\geq(u_2(x,t),v_2(x,t))$ *in the sense of* (2.3) *on* $[0,T)\times\overline{\Omega}$. *If* $u_1(x_0,0)>u_2(x_0,0)$ *for some* $x_0\in\overline{\Omega}$ *then* $u_1(x,t)>u_2(x,t)$ *on* $(0,T)\times\overline{\Omega}$. *If* $v_1(x_0,0)<v_2(x_0,0)$ *for some* $x_0\in\overline{\Omega}$ *then* $v_1(x,t)<v_2(x,t)$ *on* $(0,T)\times\overline{\Omega}$.

A result that includes a version of Lemma 2.2 is given in [**18**], Proposition 3.1. The proof is based on the single-equation comparison principle and is very similar to the proof of the corresponding result for reaction-diffusion models for competing species so we omit it.

The comparison principles have various implications. Global existence for solutions of (1.1) follows from the fact that 0 is a sub-solution and any sufficiently large constant is a super-solution on $[0,T)\times\overline{\Omega}$ for any $T>0$. Similarly, global existence for solutions of (1.3) follows from the fact that for any sufficiently large constant C the pairs $(C,0)$ and $(0,C)$ are respectively a super-solution and a sub-solution on $[0,T)\times\overline{\Omega}$ for any $T>0$.

A function $\bar{u}=\bar{u}(x)\in C(\overline{\Omega})$ that is independent of t but satisfies (2.2) is a super- (sub-) solution of the equilibrium problem for (1.1); similarly a pair $(\bar{u},\bar{v})\in C(\overline{\Omega})\times C(\overline{\Omega})$ that is independent of t but satisfies (2.4) is a super- (sub-) solution of the equilibrium problem for (1.3). A solution of (1.1) or (1.3) that is initially equal to a super-solution (sub-solution) of the corresponding equilibrium problem for (1.1) or (1.3) will decrease (increase) relative to the appropriate ordering. This follows from the comparison principles exactly as in the reaction-diffusion case. For example, if $\bar{u}$ is a super-solution to the equilibrium problem for (1.1) then by Lemma 2.1 the solution u of (1.1) with $u(x,0)=\bar{u}$ satisfies $u(x,t)\leq\bar{u}$, with strict inequality unless $\bar{u}$ is an equilibrium of (1.1). Note that for any $\tau>0$, we have that $u(x,t+\tau)$ is a solution of (1.1) with initial value $u(x,\tau)\leq\bar{u}(x)=u(x,0)$, so by Lemma 2.1 we have $u(x,t+\tau)\leq u(x,t)$ with strict inequality if $\bar{u}$ is not an equilibrium. Hence, $u(x,t)$ is decreasing, as claimed. The analogous results for other cases follow in the same way.

If it is possible to obtain a positive sub-solution for (1.1) then the existence, uniqueness, and global stability of a positive equilibrium u^* follow from the arguments used to prove Theorem 3.2 of [**23**]. (It follows that $(u^*,0)$ is a semi-trivial equilibrium of (1.3).) Related results on the existence, uniqueness, and stability of equilibria in single-species models are given in [**2, 12, 21**].

In the reaction-diffusion case it is often possible to construct super- or sub-solutions from the eigenfunctions associated with the principal eigenvalues of related linear problems. However, there are some delicate issues relative to the existence of eigenfunctions in the nonlocal case. Consider the eigenvalue problem

$$\int_\Omega k^*(x,y)\varphi(y)\,dy - \varphi(x)\int_\Omega k^*(y,x)\,dy + a(x)\varphi(x) = \lambda\varphi(x). \tag{2.5}$$

Let

$$b(x) = \int_\Omega k^*(y,x)\,dy.$$

For $\Omega \subset \mathbb{R}^N$, the eigenvalue problem (2.5) is guaranteed to have a principal eigenvalue with a positive eigenfunction only if $c(x) = -b(x) + a(x)$ has a global maximum at some point $x_0 \in \Omega$, and $1/(c(x_0) - c(x)) \notin L^1(\Omega)$. That will be true if $c(x) \in C^N(\overline{\Omega})$ when $N = 1, 2$ but requires the additional condition that all derivatives of $c(x)$ of order $N-1$ or less vanish at x_0 if $N \geq 3$. See [**10**], Theorems 1.1 and 1.2, [**18**] Theorem 2.6, and the counter-example in Section 5 of [**10**]. However, it turns out that we can construct an arbitrarily small positive sub-solution for the equilibrium problem for (1.1) under the original assumptions stated just after (1.1). By the hypotheses on k^* we have $b(x) > 0$ on $\overline{\Omega}$. It follows from the Krein-Rutman theorem that the operator

$$Lu = \frac{1}{b(x)}\int_\Omega k^*(x,y)u(y)\,dy$$

on $C(\overline{\Omega})$ has a principal eigenvalue $\mu > 0$ with eigenfunction $\psi > 0$. The eigenvalue problem for L is equivalent to

$$\int_\Omega k^*(x,y)\psi(y)\,dy - \mu\psi(x)\int_\Omega k^*(y,x)\,dy = 0. \tag{2.6}$$

Integrating (2.6) over Ω shows that $\mu = 1$. Assume that ψ is normalized by $\|\psi\|_\infty = 1$ and let $\underline{u} = \epsilon\psi$. It follows that

$$\int_\Omega k^*(x,y)\underline{u}(y,t)\,dy - \underline{u}(x,t)\int_\Omega k^*(y,x)\,dy + \underline{u}[m(x) - \underline{u}] = \epsilon\psi[m - \epsilon\psi],$$

so $\underline{u}$ is a sub-solution to the equilibrium problem for (1.1) for any sufficiently small $\epsilon > 0$. It then follows from the arguments used to prove Theorem 3.2 of [**23**] that (1.1) and the corresponding model with k^* replaced by k will each have a unique globally attracting positive equilibrium.

3 A Lyapunov functional For any $u, v \in C(\overline{\Omega})$ with $u > 0$ in $\overline{\Omega}$ and $v \geq 0$ in $\overline{\Omega}$, define

$$E(u,v) := \int_\Omega \left[u(x) - m(x) - m(x) \ln \frac{u(x)}{m(x)}\right] + \int_\Omega v(x)\, dx.$$

It is easy to check that if $u > 0$ in $\overline{\Omega}$, then $E(u,v) > 0$ for any $(u,v) \not\equiv (m,0)$ and $E(m,0) = 0$. That is, E attains its global (unique) minimum at $(u,v) = (m,0)$ in $C(\overline{\Omega}) \times C(\overline{\Omega})$.

Lemma 3.1. *Let* $(u(x,t), v(x,t))$ *be a positive solution of* (1.3)*. Set* $V(t) = E(u(x,t), v(x,t))$*, i.e.,*

$$V(t) := \int_\Omega \left[u(x,t) - m(x) + v(x,t) - m(x) \ln \frac{u(x,t)}{m(x)}\right] dx.$$

Then

$$\begin{aligned} \frac{dV}{dt} = &- \int_\Omega [m(x) - u(x,t) - v(x,t)]^2\, dx \\ &- \left(\int_\Omega \int_\Omega h(x,y) \frac{m(x)/u(x,t)}{m(y)/u(y,t)}\, dx\, dy \right. \\ &\left. - \int_\Omega \int_\Omega h(x,y)\, dx\, dy \right) \end{aligned} \tag{3.1}$$

for every $t \geq 0$*, where* $h(x,y) := k^*(x,y)m(y)$*.*

Proof. By the equation of u,

$$\begin{aligned} \frac{d}{dt} \int_\Omega u(x,t)\, dx &= \int_\Omega u_t(x,t)\, dx \\ &= \int_\Omega \int_\Omega k^*(x,y) u(y,t)\, dx\, dy \\ &\qquad - \int_\Omega \int_\Omega k^*(y,x) u(x,t)\, dx\, dy \\ &\qquad + \int_\Omega u[m - u - v]\, dx \\ &= \int_\Omega u[m - u - v]\, dx. \end{aligned} \tag{3.2}$$

Similarly, we have

$$\frac{d}{dt}\int_\Omega v(x,t)\,dx = \int_\Omega v[m-u-v]\,dx. \tag{3.3}$$

Finally, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega & m(x)\ln u(x,t)\,dx \\ &= \int_\Omega m(x)\frac{u_t}{u}(x,t)\,dx \\ &= \int_\Omega\int_\Omega k^*(x,y)u(y,t)\frac{m(x)}{u(x,t)} \\ &\quad - \int_\Omega\int_\Omega m(x)k^*(y,x)\,dx\,dy + \int_\Omega m[m-u-v].\end{aligned}$$

Since $k^*(x,y) = h(x,y)/m(y)$,

$$\begin{aligned}\frac{d}{dt}\int_\Omega & m(x)\ln u(x,t)\,dx \\ &= \int_\Omega\int_\Omega h(x,y)\frac{m(x)/u(x,t)}{m(y)/u(y,t)}\,dx\,dy \\ &\quad - \int_\Omega\int_\Omega h(x,y)\,dx\,dy + \int_\Omega m[m-u-v].\end{aligned} \tag{3.4}$$

Adding up equations (3.2) and (3.3) and subtracting (3.4), we see that (3.1) holds. □

Lemma 3.2. *Suppose that $k^*(x,y)$ is an ideal free dispersal strategy, continuous in $\overline{\Omega}\times\overline{\Omega}$, and $k^*(x,y)+k^*(y,x)$ does not vanish on any open set of $\Omega\times\Omega$. Then $dV/dt \le 0$ for $t \ge 0$. Moreover, $dV/dt(t_0) = 0$ for some t_0 if and only if $u(x,t_0) \equiv s_0 m(x)$ and $v(x,t_0) \equiv (1-s_0)m(x)$ in $\overline{\Omega}$ for some $s_0 \in (0,1]$.*

Proof. Let $h(x,y) = k^*(x,y)m(y)$. It follows from Theorem 2 that

$$\int_\Omega\int_\Omega h(x,y)\frac{m(x)/u(x,t)}{m(y)/u(y,t)}\,dx\,dy - \int_\Omega\int_\Omega h(x,y)\,dx\,dy \ge 0 \tag{3.5}$$

for any $t \ge 0$. Hence, it follows from (3.1) and (3.5) that $dV/dt \le 0$. If $dV/dt(t_0) = 0$ for some t_0, then from (3.1), (3.5) and Theorem 2 that $m(x) - u(x,t_0) - v(x,t_0) \equiv 0$ in Ω and $m(x)/u(x,t_0)$ is a positive constant. Therefore, $u(x,t_0) = s_0 m(x)$ for some $s_0 > 0$ and $v(x,t_0) = (1-s_0)m(x)$. Note that $s_0 \le 1$ since $v \ge 0$. □

4 Global convergence The goal of this section is to establish the global convergence result Theorem 1. The proof of Theorem 1 is divided into several steps. Throughout this section we assume that all assumptions in Theorem 1 hold.

We first collect a few results on qualitative properties of solutions to scalar equations. Let v_* denote the unique positive solution of the following nonlocal equation:

$$\int_\Omega k(x,y)v_*(y)\,dy - v_*(x)\int_\Omega k(y,x)\,dy + v_*[m(x)-v_*] = 0 \quad \text{in } \Omega. \tag{4.1}$$

Lemma 4.1. *$v_* \not\equiv m$ and*

$$\int_\Omega v_*(m - v_*) = 0. \tag{4.2}$$

Proof. If $v_* \equiv m$, by (4.1) we see that

$$\int_\Omega k(x,y)m(y)\,dy - m(x)\int_\Omega k(y,x)\,dy = 0 \quad \text{in } \Omega,$$

which contradicts our assumption that k is not an ideal free dispersal strategy. To establish (4.2), we integrate the equation of v_* in Ω to obtain

$$\begin{aligned} &\int_\Omega v_*(v_* - m) \\ &\quad = \int_\Omega\int_\Omega k(x,y)v_*(y)\,dxdy - \int_\Omega\int_\Omega k(y,x)v_*(x)\,dx\,dy = 0. \end{aligned} \tag{4.3}$$

□

Lemma 4.2. *Let $u(x,t)$ be a positive solution of*

$$u_t = \int_\Omega k^*(x,y)u(y,t)\,dy - u(x,t)\int_\Omega k^*(y,x)\,dy + u[m(x)-u] \tag{4.4}$$

with $u(x,0) \le (1+\eta)m(x)$ for all $x \in \overline{\Omega}$, where η is some non-negative constant. Then $u(x,t) \le (1+\eta)m(x)$ for any $t > 0$ and $x \in \overline{\Omega}$. If further assume that $u(x,0) \not\equiv (1+\eta)m(x)$, then $u(x,t) < (1+\eta)m(x)$ for any $t > 0$ and $x \in \overline{\Omega}$.

Proof. Note that $u_\eta(x) := (1+\eta)m(x)$ is a super-solution of (4.4) as

$$\int_\Omega k^*(x,y)u_\eta(y)\,dy - u_\eta(x)\int_\Omega k^*(y,x)\,dy + u_\eta[m(x)-u_\eta]$$
$$= -\eta(1+\eta)m(x) \le 0$$

in Ω, where we used the identity (1.2). By the comparison principle for single equations we see that for any x, $u(x,t)$ is monotone decreasing in t. In particular, $u(x,t) \le u(x,0) = (1+\eta)m(x)$. If further assume that $u(x,0) \not\equiv (1+\eta)m(x)$, then $u(x,t)$ is strictly monotone decreasing in t, which implies that $u(x,t) < (1+\eta)m(x)$ for any $t > 0$ and $x \in \overline{\Omega}$. □

The stability of the semi-trivial steady state $(0, v_*)$ is determined by the principal eigenvalue, if it exists, of the linear eigenvalue problem

$$\int_\Omega k^*(x,y)\varphi(y)\,dy - \varphi(x)\int_\Omega k^*(y,x)\,dy + (m-v_*)\varphi(x) = \lambda\varphi(x). \tag{4.5}$$

However, it is unclear whether this linear eigenvalue problem has a principal eigenvalue. To overcome this difficulty, we adopt an idea of Hetzer *et al.* [**18**] and first perturb the potential function $-\int_\Omega k^*(y,x)\,dy + m - v_*$ as follows: For any $\epsilon > 0$, we can always find some function $l_\epsilon^* \in C^N(\overline{\Omega})$ such that

$$-\int_\Omega k^*(y,x)\,dy + m - v_* \le l_\epsilon^* \le -\int_\Omega k^*(y,x)\,dy + m - v_* + \epsilon \quad \text{in } \Omega, \tag{4.6}$$

and there exists some $x_0 \in \Omega$ such that $l_\epsilon^*(x_0) = \max_{\overline{\Omega}} l_\epsilon^*$ and the partial derivatives of l_ϵ^* at x_0 up to $N-1$ order are zero. By Proposition 2.6 of Hetzer *et al.* [**18**], the (perturbed) linear eigenvalue problem

$$\int_\Omega k^*(x,y)\varphi(y)\,dy + l_\epsilon^*(x)\varphi(x) = \lambda\varphi(x) \tag{4.7}$$

in Ω has a principal eigenvalue, denoted by λ_ϵ, such that the corresponding eigenfunction (denoted as φ_ϵ) can chosen to be positive in $\overline{\Omega}$ and is uniquely determined by $\max_{\overline{\Omega}} \varphi_\epsilon = 1$. If we let

$$l_\epsilon(x) = l_\epsilon^*(x) + \int_\Omega k^*(y,x)\,dy,$$

then

$$m - v_* \le l_\epsilon \le m - v^* + \epsilon \quad \text{in } \Omega, \tag{4.8}$$

and (4.7) is equivalent to

$$\int_\Omega k^*(x,y)\varphi(y)\,dy - \varphi(x)\int_\Omega k^*(y,x)\,dy + l_\epsilon\varphi(x) = \lambda\varphi(x). \tag{4.9}$$

The following lower bound of λ_ϵ, which is independent of ϵ, plays an important role in later analysis:

Lemma 4.3. *The eigenvalue λ_ϵ satisfies*

$$\lambda_\epsilon \geq \frac{\int_\Omega (m-v_*)^2}{\int_\Omega m} > 0.$$

Proof. To establish Lemma 4.3, we divide the equation of φ_ϵ by $\varphi_\epsilon(x)/m(x)$ and integrating in Ω with respect to x, we have

$$\lambda_\epsilon \int_\Omega m = \int_\Omega\int_\Omega k^*(x,y)m(y)\frac{\varphi_\epsilon(y)/m(y)}{\varphi_\epsilon(x)/m(x)}\,dy - \int_\Omega\int_\Omega k^*(y,x)m(x)\,dx\,dy + \int_\Omega l_\epsilon m. \tag{4.10}$$

By the assumption on k^* and Theorem 2,

$$\int_\Omega\int_\Omega k^*(x,y)m(y)\frac{\varphi_\epsilon(y)/m(y)}{\varphi_\epsilon(x)/m(x)}\,dy \geq \int_\Omega\int_\Omega k^*(y,x)m(x)\,dx\,dy. \tag{4.11}$$

Hence, by (4.2), (4.8), (4.10) and (4.11),

$$\begin{aligned}\lambda_\epsilon \int_\Omega m &\geq \int_\Omega l_\epsilon m = \int_\Omega (m-v_*)^2 + \int_\Omega m[l_\epsilon - (m-v_*)] \\ &\geq \int_\Omega (m-v^*)^2 > 0,\end{aligned}$$

where the last inequality follows from $v_* \not\equiv m$. □

Let δ be any positive constant satisfying $\delta \leq \epsilon$. Set $(\tilde{u}(x), \tilde{v}(x)) = (\delta\varphi_\epsilon(x), (1+\epsilon)v_*(x))$.

Lemma 4.4. *If ϵ satisfies*

$$0 < \epsilon \leq \frac{\int_\Omega (m-v_*)^2}{\int_\Omega m}\frac{1}{2+\|v_*\|_\infty}, \tag{4.12}$$

then $(\tilde{u}, \tilde{v})$ *satisfies*

$$
\begin{aligned}
0 &\le \int_\Omega k^*(x,y)\tilde{u}(y)\,dy - \tilde{u}(x)\int_\Omega k^*(y,x)\,dy + \tilde{u}[m(x) - \tilde{u} - \tilde{v}],\\
0 &\ge \int_\Omega k(x,y)\tilde{v}(y)\,dy - \tilde{v}(x)\int_\Omega k(y,x)\,dy + \tilde{v}[m(x) - \tilde{u} - \tilde{v}]
\end{aligned} \tag{4.13}
$$

for any $x \in \Omega$.

Proof. The second inequality of (4.13) follows immediately from the equation of v_* and the positivity of $\tilde{u}$. For the first inequality of (4.13), it follows from the definition of λ_ϵ that

$$
\begin{aligned}
&\int_\Omega k^*(x,y)\tilde{u}(y)\,dy - \tilde{u}(x)\int_\Omega k^*(y,x)\,dy + \tilde{u}[m(x) - \tilde{u} - \tilde{v}]\\
&\qquad = \delta\varphi_\epsilon\left([m - v_* - l_\epsilon] + \lambda_\epsilon - \delta\varphi_\epsilon - \epsilon v_*\right).
\end{aligned} \tag{4.14}
$$

Since $m - v_* - l_\epsilon \ge -\epsilon$, $\delta \le \epsilon$ and $\|\varphi_\epsilon\|_\infty = 1$, by Lemma 4.3, which is applicable due to equation (1.2), we have

$$
\begin{aligned}
&\int_\Omega k^*(x,y)\tilde{u}(y)\,dy - \tilde{u}(x)\int_\Omega k^*(y,x)\,dy + \tilde{u}[m(x) - \tilde{u} - \tilde{v}]\\
&\qquad \ge \delta\varphi_\epsilon\left(\frac{\int_\Omega (m - v_*)^2}{\int_\Omega m} - 2\epsilon - \epsilon\|v_*\|_\infty\right)\\
&\qquad \ge 0,
\end{aligned} \tag{4.15}
$$

where the last inequality follows from (4.12). □

Let $(\underline{u}(x,t), \overline{v}(x,t))$ be the solution of

$$
\begin{cases}
\underline{u}_t = \displaystyle\int_\Omega k^*(x,y)\underline{u}(y,t)\,dy\\
\qquad\qquad - \underline{u}(x,t)\displaystyle\int_\Omega k^*(y,x)\,dy + \underline{u}[m(x) - \underline{u} - \overline{v}],\\
\overline{v}_t = \displaystyle\int_\Omega k(x,y)\overline{v}(y,t)\,dy\\
\qquad\qquad - \overline{v}(x,t)\displaystyle\int_\Omega k(y,x)\,dy + \overline{v}[m(x) - \underline{u} - \overline{v}],\\
\underline{u}(x,t_0) = \delta\varphi_\epsilon(x), \quad \overline{v}(x,t_0) = (1+\epsilon)v_*(x),
\end{cases} \tag{4.16}
$$

where $t_0 \ge 0$ is a constant which will be chosen later.

Lemma 4.5. *For $x \in \Omega$ and $t \geq t_0$, $\underline{u}(x,t)$ is monotone increasing in t and $\overline{v}(x,t)$ is monotone decreasing in t. Moreover, as $t \to \infty$, $\underline{u}(x,t) \to m(x)$ and $\overline{v}(x,t) \to 0$ in $C(\overline{\Omega})$ norm.*

Proof. By Lemma 4.4 and the comparison principle for system (1.3), we see that $\underline{u}(x,t)$ is monotone increasing in t and $\overline{v}(x,t)$ is monotone decreasing in t. Since $\underline{u}$ is uniformly bounded from above and $\overline{v} \geq 0$, we see that as $t \to \infty$, $\underline{u} \to u^*$ and $\overline{v} \to v^*$ for some bounded measurable functions u^* and v^*, and $u^* \geq \delta\varphi_\epsilon > 0$ in $\overline{\Omega}$ and $v^* \geq 0$ in Ω.

For $t \geq t_0$, define

$$V(t) = \int_\Omega \left[\underline{u}(x,t) - m(x) + \overline{v}(x,t) - m(x) \ln \frac{\underline{u}(x,t)}{m(x)}\right] dx.$$

Since $\underline{u} \to u^*$ and $\overline{v} \to v^*$, by (3.1) we see that

$$\begin{aligned} \lim_{t\to\infty} \frac{dV}{dt} = &- \int_\Omega [m(x) - u^* - v^*]^2 \, dx \\ &- \left(\int_\Omega \int_\Omega h(x,y) \frac{m(x)/u^*(x)}{m(y)/u^*(y)} \, dx \, dy \right. \\ &\left. - \int_\Omega \int_\Omega h(x,y) \, dx \, dy \right), \end{aligned} \tag{4.17}$$

where $h(x,y) := k^*(x,y)m(y)$. By Theorem 2,

$$\int_\Omega \int_\Omega h(x,y) \frac{m(x)/u^*(x)}{m(y)/u^*(y)} \, dx \, dy - \int_\Omega \int_\Omega h(x,y) \, dx \, dy \geq 0. \tag{4.18}$$

Therefore, the right hand side of (4.17) must be non-positive. We claim that the right hand side of (4.17) must be equal to zero: if not, there exist some positive constants δ and T such that $dV/dt \leq -\delta$ for any $t \geq T$. This, however, contradicts $V(t) \geq 0$ for all $t \geq T$. Hence, we have

$$\int_\Omega [m(x) - u^* - v^*]^2 = 0 \tag{4.19}$$

and

$$\int_\Omega \int_\Omega h(x,y) \frac{m(x)/u^*(x)}{m(y)/u^*(y)} \, dx \, dy - \int_\Omega \int_\Omega h(x,y) \, dx \, dy = 0. \tag{4.20}$$

By Theorem 2, (4.20) holds if and only if $u^*(x)/m(x)$ is a constant function in Ω. By (4.19), $u^* + v^* = m$ a.e. in Ω. Therefore, $u^* = \tau m$ and $v^* = (1-\tau)m$ for some constant τ. Since $u^* > 0$ and $v^* \geq 0$, we see that $0 < \tau \leq 1$. Note that v^* satisfies

$$\int_\Omega k(x,y)v^*(y)\,dy - v^*(x)\int_\Omega k(y,x)\,dy + v^*[m(x) - u^* - v^*] = 0 \quad \text{in } \Omega.$$

By $u^* + v^* = m$ and $v^* = (1-\tau)m$, we have

$$(1-\tau)\left[\int_\Omega k(x,y)m(y)\,dy - m(x)\int_\Omega k(y,x)\,dy\right] = 0 \quad \text{in } \Omega.$$

Since $k(x,y)$ is not an ideal free dispersal strategy, i.e.,

$$\int_\Omega k(x,y)m(y)\,dy - m(x)\int_\Omega k(y,x)\,dy \not\equiv 0 \quad \text{in } \Omega,$$

we see that $\tau = 1$. That is, $(u^*, v^*) = (m, 0)$. Therefore, $\underline{u}(x,t) \to m$ and $\overline{v}(x,t) \to 0$ pointwise in x as $t \to \infty$. As $m(x)$ is continuous, we see that these monotone convergences are uniform for $x \in \Omega$. □

Let $u(x,t)$ and $v(x,t)$ denote a solution of (1.3) with continuous positive initial data.

Lemma 4.6. *For any $\epsilon > 0$, there exists some $T_\epsilon \gg 1$ such that $v(x,t) \leq (1+\epsilon)v_*$ for any $t \geq T_\epsilon$, where v_* is the unique positive solution of* (4.1).

Proof. Since $u(x,t) \geq 0$ for all t, from (1.3) we see that $v(x,t)$ satisfies

$$v_t \leq \int_\Omega k(x,y)v(y,t)\,dy - v(x,t)\int_\Omega k(y,x)\,dy + v[m(x) - v]. \tag{4.21}$$

Let $w(x,t)$ denote the unique solution of

$$w_t = \int_\Omega k(x,y)w(y,t)\,dy - w(x,t)\int_\Omega k(y,x)\,dy + w[m(x) - w] \tag{4.22}$$

with the initial condition $w(x,0) = v(x,0)$. By the comparison principle, $v(x,t) \leq w(x,t)$ for all $x \in \Omega$ and $t \geq 0$. Since $w(x,t) \to v_*$ in $C(\overline{\Omega})$ as $t \to \infty$, we see that for any $\epsilon > 0$, there exists T_ϵ such that $w(x,t) \leq (1+\epsilon)v_*$ for any $t \geq T_\epsilon$, which implies that $v(x,t) \leq (1+\epsilon)v_*$ for any $t \geq T_\epsilon$. □

Proof of Theorem 1. We first establish the global convergence of $u(x,t)$, and $v(x,t)$ as $t \to \infty$. Let $(\underline{u}(x,t), \overline{v}(x,t))$ be the solution of (4.16) with $t_0 = T_\epsilon$, where T_ϵ is given in Lemma 4.6. Let δ be any positive constant satisfying $\delta \le \min\{\epsilon, \inf_{x\in\Omega} u(x,T_\epsilon)\}$. By the definition of δ and φ_ϵ,

$$\underline{u}(x,T_\epsilon) = \delta\varphi_\epsilon \le \delta \le u(x,T_\epsilon)$$

for any $x \in \overline{\Omega}$. By Lemma 4.6, we have

$$\overline{v}(x,T_\epsilon) = (1+\epsilon)v_*(x) \ge v(x,T_\epsilon)$$

for every $x \in \Omega$. By the comparison principle for system (1.3), we have $u(x,t) \ge \underline{u}(x,t)$ and $v(x,t) \le \overline{v}(x,t)$ for every $x \in \Omega$ and $t \ge T_\epsilon$. By Lemma 4.5, as $t \to \infty$, $\underline{u}(x,t) \to m(x)$ and $\overline{v}(x,t) \to 0$ in $C(\overline{\Omega})$ norm. Hence, $v(x,t) \to 0$ in $C(\overline{\Omega})$ norm, which implies that $u(x,t) \to m(x)$ in $C(\overline{\Omega})$ norm.

Next, we show that $(m(x),0)$ is locally stable in $C(\overline{\Omega})\times C(\overline{\Omega})$ topology, i.e., for $\epsilon > 0$, there exists $\gamma = \gamma(\epsilon) < \epsilon$ such that if $|u(x,0) - m(x)| < \gamma$ and $0 \le v(x,0) < \gamma$, then $|u(x,t) - m(x)| < \epsilon$ and $v(x,t) < \epsilon$ for all $t \ge 0$. To this end, let $(\underline{u}(x,t), \overline{v}(x,t))$ be the solution of (4.16) with $t_0 = 0$. By Lemma 4.5, there exists some $T^* = T^*(\epsilon) > 0$ such that

$$\underline{u}(x,t) \ge m(x) - \epsilon, \quad \overline{v}(x,t) < \epsilon, \quad \forall\, x \in \overline{\Omega},\ t \ge T^*. \tag{4.23}$$

By Lemma 4.2, $\underline{u}(x,t) < m(x)$ for all $x \in \Omega$ and $t \ge 0$ as $\underline{u}(x,0) \le \epsilon < m(x)$ in $\overline{\Omega}$, provided that ϵ is positive and sufficiently small. Set

$$\gamma = \min\left\{ \min_{x\in\overline{\Omega}} [m(x) - \underline{u}(x,T^*)], \quad \min_{x\in\overline{\Omega}} \overline{v}(x,T^*), \quad \epsilon \cdot \frac{\min_{\overline{\Omega}} m}{\max_{\overline{\Omega}} m} \right\}.$$

By the choice of γ, if $|u(x,0) - m(x)| < \gamma$ and $v(x,0) < \gamma$, then $u(x,0) > m(x) - \gamma \ge \underline{u}(x,T^*)$ and $v(x,0) < \gamma \le \overline{v}(x,T^*)$. By the comparison principle for system (1.3), $u(x,t) \ge \underline{u}(x,t+T^*)$ and $v(x,t) \le \overline{v}(x,t+T^*)$ for any x and $t \ge 0$. Since $\underline{u}(x,t)$ is monotone increasing in t and $\overline{v}(x,t)$ is monotone decreasing in t, $u(x,t) \ge \underline{u}(x,T^*)$ and $v(x,t) \le \overline{v}(x,T^*)$. This together with (4.23) implies that for any $t \ge 0$ and $x \in \overline{\Omega}$, $u(x,t) \ge \underline{u}(x,T^*) \ge m(x) - \epsilon$ and $v(x,t) \le \overline{v}(x,T^*) \le \epsilon$.

It remains to show that $u(x,t) \le m(x) + \epsilon$ for $t \ge 0$ and $x \in \Omega$. By the choice of γ, if $|u(x,0) - m(x)| < \gamma$, then $u(x,0) < (1 + \epsilon/\max_{\overline{\Omega}} m)m(x)$ for $x \in \overline{\Omega}$. Let $\hat{u}(x,t)$ be the solution of

$$\begin{cases} \hat{u}_t = \displaystyle\int_\Omega k^*(x,y)\hat{u}(y,t)\,dy - \hat{u}(x,t)\int_\Omega k^*(y,x)\,dy \\ \qquad\qquad + \hat{u}[m(x) - \hat{u}], \quad x \in \Omega,\ t > 0, \\ \hat{u}(x,0) = (1 + \epsilon/\max_{\overline{\Omega}} m)m(x). \end{cases} \tag{4.24}$$

By Lemma 4.2,

$$\hat{u}(x,t) \leq (1+\epsilon/\max_{\overline{\Omega}} m)m(x), \quad t \geq 0. \tag{4.25}$$

As $u(x,t)$ satisfies

$$\begin{aligned} u_t = & \int_\Omega k^*(x,y)u(y,t)\,dy \\ & - u(x,t)\int_\Omega k^*(y,x)\,dy + u[m(x)-u-v] \\ \leq & \int_\Omega k^*(x,y)u(y,t)\,dy \\ & - u(x,t)\int_\Omega k^*(y,x)\,dy + u[m(x)-u] \end{aligned} \tag{4.26}$$

for $t \geq 0$, by the comparison principle for single equations we have $u(x,t) \leq \hat{u}(x,t)$ for $x \in \overline{\Omega}$ and $t \geq 0$. Hence, by (4.25) we have $u(x,t) \leq \hat{u}(x,t) \leq (1+\epsilon/\max_{\overline{\Omega}} m)m(x) \leq m(x)+\epsilon$. □

5 A new integral inequality We first introduce the definition of line-symmetric matrix, which is a natural generalization of symmetric matrix. A $n \times n$ matrix A is called line-symmetric if for every $1 \leq i \leq n$, the sum of the elements in the i-th row of A equals the sum of the elements in the i-th column of A. The following result, which gives a classification of line-symmetric matrix, can be found in Corollary 3 of [**14**].

Theorem 3. *Let A be an $n \times n$ nonnegative matrix. Then A is line-sum-symmetric if and only*

$$\sum_{i,j=1}^{n} a_{ij}\frac{x_i}{x_j} \geq \sum_{i,j=1}^{n} a_{ij} \tag{5.1}$$

for all $x_i > 0$, $1 \leq i \leq n$. Moreover, if A is irreducible and line-sum-symmetric, equality in (5.1) *holds if and only if all the coordinates of $x = (x_1, ..., x_n)$ coincide, i.e., $x_i = x_j$ for any $1 \leq i, j \leq n$.*

A continuous version of Theorem 3 in terms of integrals can be stated as follows.

Theorem 4. *Let $h : \Omega \times \Omega \to [0, \infty]$ be a continuous Riemann integrable function. Then the following are equivalent*

(i) $\int_\Omega h(x,y)\,dy = \int_\Omega h(y,x)\,dy$ *for all* $x \in \Omega$.

(ii) $\int_\Omega \int_\Omega h(x,y)\frac{f(x)}{f(y)}\,dx\,dy \geq \int_\Omega \int_\Omega h(x,y)\,dx\,dy$ *for all* $f \in C(\overline{\Omega})$ *with* $f(x) > 0$ *on* $\overline{\Omega}$.

Proof. The proof of (i) implies (ii) will make use of a matrix inequality for line-sum symmetric matrices. Suppose h satisfies (i) and $f \in C(\overline{\Omega})$ with $f(x) > 0$ on $\overline{\Omega}$. By choosing specific sample points in a partition of $\Omega \times \Omega$, the Riemann sum approximation of $\int_\Omega \int_\Omega h(x,y)\,dx\,dy$ can be made into a line-sum symmetric matrix and Theorem 3 can be applied to the approximations of the double integrals in (ii). Let $\{\Omega_i\}_{i=1}^n$ be a partition of Ω and $\{(x_{ij}, y_{ij}), \Omega_i \times \Omega_j\}_{i,j=1}^n$ be a partition of $\Omega \times \Omega$ with sample points $(x_{ij}, y_{ij}) \in \Omega_i \times \Omega_j$ chosen so that

$$h(x_{ij}, y_{ij}) = \frac{1}{|\Omega_i \times \Omega_j|} \int_{\Omega_j} \int_{\Omega_i} h(x,y)\,dx\,dy. \tag{5.2}$$

Such sample points (x_{ij}, y_{ij}) exist for all $i, j = 1, \ldots, n$ because h is continuous. Define

$$a_{ij} = h(x_{ij}, y_{ij})|\Omega_i \times \Omega_j|. \tag{5.3}$$

Then (i) $\implies A = (a_{ij})$ is a line-sum symmetric matrix because

$$\begin{aligned} \sum_{i=1}^n a_{ij} &= \sum_{i=1}^n \int_{\Omega_j} \int_{\Omega_i} h(x,y)\,dx\,dy = \int_{\Omega_j} \int_{\Omega} h(x,y)\,dx\,dy \\ &= \int_{\Omega_j} \int_{\Omega} h(y,x)\,dx\,dy \\ &= \sum_{i=1}^n \int_{\Omega_j} \int_{\Omega_i} h(y,x)\,dx\,dy = \sum_{i=1}^n a_{ji}. \end{aligned} \tag{5.4}$$

The Riemann sum corresponding to the first double integral in (ii) can be manipulated to make use of Theorem 3

$$\begin{aligned}\sum_{i,j=1}^{n} a_{ij}\frac{f(x_{ij})}{f(y_{ij})} &= \sum_{i,j=1}^{n} a_{ij}\left(\frac{f(x_{i1})}{f(x_{j1})}+\left(\frac{f(x_{ij})}{f(y_{ij})}-\frac{f(x_{i1})}{f(x_{j1})}\right)\right)\\ &\geq \sum_{i,j=1}^{n} a_{ij}\left(1+\frac{f(x_{ij})}{f(y_{ij})}-\frac{f(x_{i1})}{f(x_{j1})}\right).\end{aligned} \tag{5.5}$$

A lower bound on the coefficient of a_{ij} in right hand side of (5.5) follows from

$$\begin{aligned}&\left|\frac{f(x_{ij})}{f(y_{ij})}-\frac{f(x_{i1})}{f(x_{j1})}\right|\\ &\qquad=\left|\frac{f(x_{ij})f(x_{j1})-f(x_{i1})f(y_{ij})}{f(y_{ij})f(x_{j1})}\right|\\ &\qquad\leq\frac{f(x_{ij})\left|f(x_{j1})-f(y_{ij})\right|+f(y_{ij})\left|f(x_{ij})-f(x_{i1})\right|}{f(y_{ij})f(x_{j1})}.\end{aligned} \tag{5.6}$$

Recall that $(x_{ij}, y_{ij}) \in \Omega_i \times \Omega_j$ so, x_{ij}, $x_{i1} \in \Omega_i$ and y_{ij}, $x_{j1} \in \Omega_j$. Because f is uniformly continuous on $\overline{\Omega}$, and hence bounded above and away from zero below, this quantity can be made arbitrarily small by choosing a sufficiently fine partition. More precisely, for all $k > 1$, there exists a h_k such that

$$\sum_{i,j=1}^{n} a_{ij}\frac{f(x_{ij})}{f(y_{ij})} \geq (1-2^{-k})\sum_{i,j=1}^{n} a_{ij} \tag{5.7}$$

for any partition of Ω with $|\Omega_i| < h_k$ for all i. Taking the limit of (5.7) as the size of the partition of Ω goes to zero yields the integral inequality in (ii).

Now, assume h satisfies (ii). To prove (ii) $\Longrightarrow$ (i) it is useful to work with the function

$$G(\varepsilon) = \int_\Omega\int_\Omega h(x,y)\frac{1+\varepsilon g(x)}{1+\varepsilon g(y)}\,dx\,dy \tag{5.8}$$

which for any $g \in C(\overline{\Omega})$ is differentiable in a neighborhood of 0. Because

$$G(0) = \int_\Omega\int_\Omega h(x,y)\,dx\,dy \tag{5.9}$$

we have $G(\varepsilon) \geq G(0)$ for all ε in a neighborhood of 0. Therefore, a local minimum of G occurs at 0 and hence, $G'(0) = 0$. We can differentiate (5.8) to get

$$G'(0) = \int_\Omega \int_\Omega h(x,y)\,(g(x) - g(y))\; dx\,dy\,. \tag{5.10}$$

Therefore,

$$\int_\Omega \int_\Omega g(x)h(x,y)\,dx\,dy = \int_\Omega \int_\Omega g(y)h(x,y)\,dx\,dy\,. \tag{5.11}$$

We can change the order of integration on the left hand side of (5.11) and relabel the right hand side of (5.11) so that the role of x and y are changed to get

$$\int_\Omega g(x)\left(\int_\Omega h(x,y)\,dy\right)\,dx = \int_\Omega g(x)\left(\int_\Omega h(y,x)\,dy\right)\,dx\,. \tag{5.12}$$

Since (5.12) holds for any $g \in C(\overline{\Omega})$, we can conclude that h satisfies (i). ☐

Theorem 5. *Let $h : \Omega\times\Omega \to [0,\infty]$ be a continuous Riemann integrable function such that $h(x,y) + h(y,x)$ does not vanish on any open set of Ω. Let $f \in C(\overline{\Omega})$, $f(x) > 0$ on $\overline{\Omega}$. Furthermore, assume*

$$\int_\Omega h(x,y)dy = \int_\Omega h(y,x)dy \text{ for all } x \in \Omega\,.$$

Then

$$\int_\Omega \int_\Omega h(x,y)\frac{f(x)}{f(y)}\,dx\,dy = \int_\Omega \int_\Omega h(x,y)\,dx\,dy \iff f \text{ is a constant.}$$

Proof. It is clear that when f is a constant equality will hold. Suppose f is not a constant but the double integrals are equal. Then define

$$G(\varepsilon) = \int_\Omega \int_\Omega h(x,y)\frac{f(x) + \varepsilon g(x)}{f(y) + \varepsilon g(y)}\,dx\,dy \tag{5.13}$$

which, for any $g \in C(\overline{\Omega})$, is a differentiable function in a neighborhood of 0 that has a local minimum at 0. Hence, $G'(0) = 0$, where

$$G'(0) = \int_\Omega \int_\Omega h(x,y)\frac{g(x)f(y) - g(y)f(x)}{f(y)^2}\,dx\,dy. \tag{5.14}$$

Choosing $g(x) = f(x)^2$ in (5.14) yields

$$\int_\Omega \int_\Omega h(x,y) \frac{f(x)^2 - f(x)f(y)}{f(y)} \, dx \, dy = 0. \tag{5.15}$$

Because

$$\int_\Omega h(x,y) \, dy = \int_\Omega h(y,x) \, dy \text{ for all } x \in \Omega,$$

we can show

$$\begin{aligned} \int_\Omega \int_\Omega h(x,y) f(x) \, dx \, dy &= \int_\Omega f(x) \left(\int_\Omega h(x,y) \, dy \right) dx \\ &= \int_\Omega f(x) \left(\int_\Omega h(y,x) \, dy \right) dx \\ &= \int_\Omega \int_\Omega h(y,x) f(x) \, dy \, dx \\ &= \int_\Omega \int_\Omega h(x,y) f(y) \, dx \, dy. \end{aligned} \tag{5.16}$$

Therefore,

$$\begin{aligned} \int_\Omega \int_\Omega h(x,y) \frac{f(y)^2 - f(x)f(y)}{f(y)} \, dx \, dy \\ = \int_\Omega \int_\Omega h(x,y) \left(f(y) - f(x) \right) \, dx \, dy = 0. \end{aligned} \tag{5.17}$$

Adding (5.15) and (5.17) yields

$$\int_\Omega \int_\Omega h(x,y) \frac{\left(f(y) - f(x) \right)^2}{f(y)} \, dx \, dy = 0, \tag{5.18}$$

and interchanging x and y in (5.18) gives

$$\int_\Omega \int_\Omega h(y,x) \frac{\left(f(x) - f(y) \right)^2}{f(x)} \, dx \, dy = 0. \tag{5.19}$$

If f is not constant, then there must be an open set $U \subset \Omega \times \Omega$ where $\left(f(x) - f(y) \right)^2 > 0$. However, we know that $h(x,y) + h(y,x)$ does not vanish on any open set so either $h(x,y)$, or $h(y,x)$ must be positive on some open subset of U. Therefore (5.18) and (5.19) cannot both be 0. This is a contradiction. □

Finally, we give another application of Theorem 4. We say that $\lambda(K)$ is a principal eigenvalue of the problem

$$\int_{\Omega} K(x,y)\varphi(y)\,dy = \lambda(K)\varphi(x) \tag{5.20}$$

if (5.20) has a solution $\varphi(x)$ which is continuous and positive in $\overline{\Omega}$.

Let $K(x,y)$ be a positive continuous function in $\overline{\Omega}\times\overline{\Omega}$ and $\lambda(K)$ be a principal eigenvalue of (5.20). If $\int_\Omega K(x,y)\,dy = \int_\Omega K(y,x)\,dy$ for every $x\in\Omega$, then

$$\lambda(K) \geq \frac{1}{|\Omega|}\int_\Omega\int_\Omega K(x,y)\,dx\,dy.$$

If $\int_\Omega K(x,y)\,dy$ is non-constant, then

$$\lambda(K) > \frac{1}{|\Omega|}\int_\Omega\int_\Omega K(x,y)\,dx\,dy.$$

Proof. Dividing (5.20) by $\varphi(x)$ and integrating in Ω, we have

$$\lambda|\Omega| = \int_\Omega\int_\Omega K(x,y)\frac{\varphi(y)}{\varphi(x)}\,dx\,dy \geq \int_\Omega K(x,y)\,dx\,dy,$$

which the last inequality follows from Theorem 4. If $K(x,y)+K(y,x)$ does not vanish on any open set of Ω, we see that

$$\lambda(K) = \int_\Omega\int_\Omega K(x,y)\,dx\,dy.$$

if and only if φ is a constant function, which together with (5.20) would imply that $\int_\Omega K(x,y)\,dy$ is also a constant function. This contradicts our assumption. ☐

Remark 5.1. Corollary 5 for the symmetric case $K(x,y)=K(y,x)$ was established in Appendix A of [**16**]. We thank Mark Lewis for bringing this reference to our attention.

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