MATH 436 FALL 2012 HOMEWORK 6

DUE DEC. 4 IN CLASS

Note. All problem numbers refer to "Updated" version of lecture note.

- **Ex. 3.22.** Construct a sequence $f_n(x) \longrightarrow 0$ for every $x \in \mathbb{R}$, but $\|f_n\| = \left(\int_{\mathbb{R}} f(x)^2 \,\mathrm{d}x\right)^{1/2} = 1$ for all n.

- **Ex. 3.23.** Let V be a linear vector space. A norm $\|\cdot\|$ is a mapping $V \mapsto \mathbb{R}$ satisfying

  a) For any $v \in V$, $\|v\| \geqslant 0$, and $\|v\| = 0 \Longleftrightarrow v = 0$.

  b) For any $v \in V$ and $a \in \mathbb{R}$, $\|a\, v\| = |a|\, \|v\|$.

  c) For any $u, v \in V$, $\|u + v\| \leqslant \|u\| + \|v\|$.

  Prove that

  $$\|v\| := \sup_{x \in [a,b]} |f(x)| \tag{1}$$

  is a norm on $V = \{f(x) \colon [a, b] \mapsto \mathbb{R} |\ f(x) \text{ is bounded}\}$. Then show that this norm does not come from an inner product, that is there can be no inner product that $\|v\|^2 = (v, v)$. (Hint: Show that if $(\cdot, \cdot)$ is an inner product, then $(u + v, u + v) + (u - v, u - v) = 2\,(u, u) + 2\,(v, v)$.)

- **Ex. 4.4.** Consider the linear ODE system

  $$\dot{\boldsymbol{x}} = A(t)\, \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{2}$$

  Explain why in general

  $$\exp\left[\int_0^t A(s)\,\mathrm{d}s\right] \boldsymbol{x}_0 \tag{3}$$

  is not a solution.

- **Ex. 4.5.** Analyze the well/ill-posedness of

  $$u_{xx} + u_{tt} + u_t = 0 \tag{4}$$

  using normal modes analysis.