Math 436 Fall 2012 Homework 5

Due Nov. 22 in Class

Note. All problem numbers refer to "Updated" version of lecture note.

- **Ex. 3.11.** Write the following equations into S-L form and discuss whether they are regular or singular. Determine what is the orthogonality relation their eigenfunctions should satisfy.

  a) Legendre's equation:

  $$(1-x^2)\,y''-2\,x\,y'+\lambda\,y=0, \qquad -1<x<1 \tag{1}$$

  b) Chebyshev's equation

  $$(1-x^2)\,y''-x\,y'+\lambda\,y=0, \qquad -1<x<1 \tag{2}$$

  c) Laguerre's equation

  $$x\,y''+(1-x)\,y'+\lambda\,y=0, \qquad 0<x<\infty \tag{3}$$

  d) Hermite's equation

  $$y''-2\,x\,y'+\lambda\,y=0, \qquad -\infty<x<\infty \tag{4}$$

  e) Bessel's equation of order n

  $$x^2\,y''+x\,y'+(\lambda\,x^2-n^2)\,y=0, \qquad 0<x<1. \tag{5}$$

- **Ex. 3.12.** Give any second order equation

  $$a(x)\,y''+b(x)\,y'+c\,(x)\,y=0. \tag{6}$$

  Prove that there exists a multiplier $h(x)$ such that

  $$h(x)\,[a(x)\,y''+b(x)\,y'+c\,(x)\,y]=(p(x)\,y')'+q(x)\,y. \tag{7}$$

  Note that the term of first order derivative disappears.

- **Ex. 3.13.** Consider the S-L problem

  $$(p\,y')'+q\,y+\lambda\,y=0, \qquad a<x<b, \qquad y(a)=0, y(b)=0. \tag{8}$$

  Show that if $p(x)\geqslant 0, q(x)\leqslant M$, then any eigenvalue $\lambda\geqslant -M$.

- **Ex. 3.17.** Prove that the Green's function $G(x,\ \xi)$ as defined in the notes is symmetric: $G(x,\xi)=G(\xi,x)$. (Hint: Show that $p\,[y_1'\,y_2-y_1\,y_2']$ is constant).

- **Ex. 3.19.** Let K be defined as

$$K f := \int_a^b k(x;\xi)\, f(\xi)\, \mathrm{d}\xi \tag{9}$$

where

$$k(x;\xi) = r(x)^{1/2}\, G(x;\xi)\, r(\xi)^{1/2} \tag{10}$$

with $G(x;\xi)$ the Green's function for the operator

$$-(p\, y')' + q\, y \text{ with boundary condition } y(a) = y(b) = 0 \tag{11}$$

in the sense that the solution to

$$-(p\, y')' + q\, y = f, \qquad y(a) = y(b) = 0 \tag{12}$$

is given by

$$y(x) = \int_a^b G(x;\xi)\, f(\xi)\, \mathrm{d}\xi. \tag{13}$$

Assume $p,\ q,\ r > 0$. Show that K is a non-negative operator, that is $(K z,\ z) := \int_a^b [K z]\, z\, \mathrm{d}x \geqslant 0$ for all continuous functions z.