MATH 436 FALL 2012 HOMEWORK 4 SOLUTIONS

DUE NOV. 8 IN CLASS

Note. All problem numbers refer to "Updated" version of lecture note.

- **Exercise 3.1.** Consider the Telegrapher's equation

$$u_{xx} = u_{tt} + \lambda\, u_t \tag{1}$$

(recall that $\lambda > 0$) over the interval $x \in [0, L]$ subject to conditions

$$u(0,t) = u(L,t) = 0; \qquad u(x,0) = f(x), \quad u_t(x,0) = h(x). \tag{2}$$

Use the method of separation of variables to study the limiting behavior of u as $t \longrightarrow \infty$.

Solution. We separate the variables: Substitute $X(x)\,T(t)$ into the equation:

$$T(t)\,X''(x) = T''(t)\,X(x) + \lambda\,T'(t)\,X(x). \tag{3}$$

Divide both sides by $X(x)\,T(t)$ we get

$$\frac{X''}{X} = \frac{T''}{T} + \lambda\,\frac{T'}{T} = c^2\,K. \tag{4}$$

Thus we get the eigenvalue problem:

$$X'' - K\,X = 0, \qquad X(0) = X(L) = 0. \tag{5}$$

and the equation for T:

$$T'' + \lambda\,T' - K\,T = 0. \tag{6}$$

We solve the eigenvalue problem to obtain

$$K_n = -\left(\frac{n\,\pi}{L}\right)^2, \qquad X_n = \sin\frac{n\,\pi\,x}{L}, \qquad n = 1, 2, 3, \ldots \tag{7}$$

Now we expand

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\frac{n\,\pi\,x}{L}, \qquad h(x) = \sum_{n=1}^{\infty} h_n \sin\frac{n\,\pi\,x}{L} \tag{8}$$

where

$$f_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x, \qquad h_n = \frac{2}{L}\int_0^L h(x)\sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x. \tag{9}$$

Next we solve the T_n equation:

$$T_n'' + \lambda\,T_n' + \left(\frac{n\,\pi}{L}\right)^2 T_n = 0. \tag{10}$$

Its auxiliary equation is

$$r^2 + \lambda\,r + \left(\frac{n\,\pi}{L}\right)^2 = 0 \Longrightarrow r_{1,2} = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\left(\frac{n\,\pi}{L}\right)^2}}{2}. \tag{11}$$

Now it is clear that $\Re r_{1,2} < 0$ for all possible values of λ. As a consequence the solution would read

$$u(x,t) = \sum e^{-\alpha_n t}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \tag{12}$$

with all $\alpha_n > 0$. Therefore $u(x,t) \longrightarrow 0$ as $t \longrightarrow \infty$.

(To Cody: The following is not required)

More specifically, we have

- If $\lambda < \frac{2\pi}{L}$, then $\lambda^2 - 4\left(\frac{n\pi}{L}\right)^2 < 0$ for all $n \in \mathbb{N}$. Consequently the real part of all α_n is $-\frac{\lambda}{2}$. Thus $u(x,t) \longrightarrow 0$ like $e^{-\lambda t/2}$.
- If $\lambda = \frac{2\pi}{L}$, $u(x,t) \longrightarrow 0$ like $e^{-\lambda t/2}\, t$ (since now we have a repeated root).
- If $\lambda > \frac{2\pi}{L}$, then

  $$\Re \alpha_n \geqslant \frac{-\lambda + \sqrt{\lambda^2 - 4\left(\frac{\pi}{L}\right)^2}}{2} \tag{13}$$

  which means $u(x,t) \longrightarrow 0$ like $\exp\left[\frac{-\lambda + \sqrt{\lambda^2 - 4\left(\frac{\pi}{L}\right)^2}}{2}\, t\right]$.

- **Exercise 3.5.** Consider the boundary value problem for $u(x,y)$ in the annular region:

$$u_{xx} + u_{yy} = 0 \quad \rho^2 < x^2 + y^2 < 1; \qquad u(x,y) = \begin{cases} f & x^2 + y^2 = \rho^2 \\ g & x^2 + y^2 = 1 \end{cases}. \tag{14}$$

Obtain the formula for the solution using separation of variables.

Solution. The solution process is identical to that of Laplace equation in the disc untile we need to solve the R equation:

$$r^2 R'' + r R' - n^2 R = 0, \qquad R(\rho_1) = f_{ni}, \qquad R(\rho_2) = g_{ni}. \tag{15}$$

Here f_{ni}, g_{ni} comes from expansions

$$f(\theta) = \sum_{n=0}^{\infty} [f_{n1} \cos n\theta + f_{n2} \sin n\theta], \qquad g(\theta) = \sum_{n=0}^{\infty} [g_{n1} \cos n\theta + g_{n2} \sin n\theta]. \tag{16}$$

When $n = 0$, the general solution is

$$R(r) = C_1 + C_2 \log r \tag{17}$$

which leads to

$$C_1 + C_2 \log \rho = f_{0i}, \qquad C_1 = g_{0i}. \tag{18}$$

Consequently

$$C_2 = (f_{0i} - g_{0i})/(\log \rho), \qquad C_1 = g_{0i}. \tag{19}$$

When $n > 0$ the general solution is

$$R(r) = C_1 r^n + C_2 r^{-n}. \tag{20}$$

Then the boundary conditions lead to

$$C_1\,\rho^n + C_2\,\rho^{-n} = f_{ni}, \qquad C_1 + C_2 = g_{ni}. \tag{21}$$

The solution is

$$C_1 = \frac{f_{ni}\,\rho^n - g_{ni}}{\rho^{2n}-1}, \qquad C_2 = \rho^n\,\frac{g_{ni}\,\rho^n - f_{ni}}{\rho^{2n}-1}. \tag{22}$$

Therefore the solution to the problem is

$$u(r,\theta) = \left[g_{01} + \frac{f_{01}-g_{01}}{\log\rho}\right] + \sum_{n=1}^{\infty}\left[\left(\frac{f_{n1}\,\rho^n - g_{n1}}{\rho^{2n}-1}\,r^n + \left(\frac{\rho}{r}\right)^n\frac{g_{n1}\,\rho^n - f_{n1}}{\rho^{2n}-1}\right)\cos(n\,\theta) + \left(\frac{f_{n2}\,\rho^n - g_{n2}}{\rho^{2n}-1}\,r^n + \left(\frac{\rho}{r}\right)^n\frac{g_{n2}\,\rho^n - f_{n2}}{\rho^{2n}-1}\right)\sin(n\,\theta)\right]. \tag{23}$$

- **Exercise 3.8.** Consider the equation

$$r^2\,R'' + r\,R' + (\lambda\,r^2 - n^2)\,R = 0, \qquad R(0), R'(0) \text{ bounded}, \; R(1) = 0. \tag{24}$$

  a) Prove that there are no negative eigenvalues (along the way you will see why the boundedness of $R'(0)$ is needed.)

  b) Prove that $\lambda = 0$ is not an eigenvalue.

  c) Let $\lambda_k \neq \lambda_l$ be eigenvalues, prove that the eigenfunctions $R_k(r), R_l(r)$ satisfy

$$\int_0^1 R_k(r)\,R_l(r)\,r\,\mathrm{d}r = 0. \tag{25}$$

(Hint: For a) and c), write the equation as

$$\frac{1}{r}\left[(r\,R')' - \frac{n^2}{r}\,R\right] + \lambda\,R = 0 \tag{26}$$

first.)

Proof. We follow the hint and write the equation as

$$\frac{1}{r}\left[(r\,R')' - \frac{n^2}{r}\,R\right] + \lambda\,R = 0 \tag{27}$$

- $\lambda < 0$ cannot be eigenvalue.

  Let λ be an eigenvalue with corresponding eigenfunction R. Multiply both sides of the equation by $r\,R$ and integrate from 0 to 1 we have

$$\begin{aligned} 0 &= \int_0^1 \left\{\frac{1}{r}\left[(r\,R')' - \frac{n^2}{r}\,R\right] + \lambda\,R\right\} R\,r\,\mathrm{d}r \\ &= \int_0^1 \left[(r\,R')' - \frac{n^2}{r}\,R\right] R\,\mathrm{d}r + \int_0^1 \lambda\,R^2\,\mathrm{d}r \\ &= \int_0^1 (r\,R')'\,R\,\mathrm{d}r - \int_0^1 \left(\frac{n^2}{r} - \lambda\right) R^2\,\mathrm{d}r \\ &= r\,R'\,R|_0^1 - \int_0^1 r\,(R')^2 - \int_0^1 \left(\frac{n^2}{r} - \lambda\right) R^2\,\mathrm{d}r. \end{aligned} \tag{28}$$

As $R(1)=0, R'(0)\,R(0)$ bounded, the boundary terms vanish. Consequently

$$\int_0^1 \left(\frac{n^2}{r}-\lambda\right) R^2\,\mathrm{d}r \leqslant 0 \tag{29}$$

which is not possible for nonzero R if $\lambda<0$.

Remark 1. Note that in fact the above also shows $\lambda=0$ is not an eigenvalue.

- $\lambda=0$ is not an eigenvalue.

  When $\lambda=0$ the equation becomes

$$r^2 R''+r\,R'-n^2 R=0 \tag{30}$$

which general solution is

$$R(r)=\begin{cases} C_1+C_2\log r & n=0 \\ C_1 r^n+C_2 r^{-n} & n\neq 0 \end{cases}. \tag{31}$$

The boundary condition at 0 leads to $C_2=0$. Then the other boundary condition $R(1)=0$ requires $C_1=0$.

- If

$$\frac{1}{r}\left[(r\,R_k')'-\frac{n^2}{r}R_k\right]+\lambda_k R_k=0, \qquad \frac{1}{r}\left[(r\,R_l')'-\frac{n^2}{r}R_l\right]+\lambda_l R_l=0 \tag{32}$$

then we can multiply the first equation by $r\,R_l$ and integrate from 0 to 1:

$$\begin{aligned} 0 &= \int_0^1 \left[(r\,R_k')'-\frac{n^2}{r}R_k\right]R_l+\lambda_k\int_0^1 R_k R_l\,r\,\mathrm{d}r \\ &= \int_0^1 (r\,R_k')'\,R_l-\int_0^1 \frac{n^2}{r}R_k R_l+\lambda_k\int_0^1 R_k R_l\,r\,\mathrm{d}r \\ &= [r\,R_k' R_l]|_0^1-\int_0^1 r\,R_k' R_l'-\int_0^1 \frac{n^2}{r}R_k R_l+\lambda_k\int_0^1 R_k R_l\,r\,\mathrm{d}r \\ &= 0-[r\,R_k R_l']|_0^1+\int_0^1 R_k\,(r\,R_l')'-\int_0^1 \frac{n^2}{r}R_k R_l+\lambda_k\int_0^1 R_k R_l\,r\,\mathrm{d}r \\ &= \int_0^1 \left[(r\,R_l')'-\frac{n^2}{r}R_l\right]R_k\,\mathrm{d}r+\lambda_k\int_0^1 R_k R_l\,r\,\mathrm{d}r \\ &= (\lambda_k-\lambda_l)\int_0^1 R_k R_l\,r\,\mathrm{d}r. \end{aligned} \tag{33}$$

Since $\lambda_k-\lambda_l\neq 0$ we conclude

$$\int_0^1 R_k R_l\,r\,\mathrm{d}r=0 \tag{34}$$

which is the desired orthogonality relation. □