Math 436 Fall 2012 Homework 2

Due Oct. 11 in Class

Note. All problem numbers refer to "Updated" version of lecture note.

- **Ex. 2.2.** Find the solution of the following Cauchy problems.

  a) $x\,u_x + y\,u_y = 2\,x\,y$, with $u = 2$ on $y = x^2$.

  b) $u\,u_x - u\,u_y = u^2 + (x+y)^2$ with $u = 1$ on $y = 0$.

  Solution.

  a) We have

  $$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}u}{2\,x\,y}. \tag{1}$$

  The first equality gives $x\,\mathrm{d}y = y\,\mathrm{d}x \Longrightarrow \mathrm{d}(y/x) = 0$ so $\Phi = y/x$. The second inequality gives $2\,x\,\mathrm{d}y = \mathrm{d}u$ which together with $x\,\mathrm{d}y = y\,\mathrm{d}x$ gives $\mathrm{d}u = \mathrm{d}(x\,y)$. Therefore $\Psi = u - x\,y$. Finally the general solution is given by

  $$F(y/x, u - x\,y) = 0 \Longrightarrow u = x\,y + f(y/x). \tag{2}$$

  Applying the initial condition, we have

  $$2 = x\,x^2 + f(x) \Longrightarrow f(x) = 2 - x^3. \tag{3}$$

  Therefore the solution is

  $$u = x\,y + 2 - \frac{y^3}{x^3}. \tag{4}$$

  b) We have

  $$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{-u} = \frac{\mathrm{d}u}{u^2 + (x+y)^2}. \tag{5}$$

  The first $=$ gives $\mathrm{d}(x+y) = 0$. Taking advantiage of this we have

  $$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}u}{u^2 + (x+y)^2} \Rightarrow 2\,\mathrm{d}x = \frac{\mathrm{d}(u^2)}{u^2 + (x+y)^2} \Rightarrow \mathrm{d}(2\,x - \ln[u^2 + (x+y)^2]) = 0 \tag{6}$$

  Therefore the general solution is

  $$F(x+y, 2\,x - \ln[u^2 + (x+y)^2]) = 0. \tag{7}$$

  This simplifies to

  $$\ln[u^2 + (x+y)^2] = 2\,x - f(x+y). \tag{8}$$

  Applying the initial condition we have

  $$\ln[1 + x^2] = 2\,x - f(x) \Longrightarrow f(x) = 2\,x - \ln[1 + x^2]. \tag{9}$$

  So finally the solution is

  $$\ln[u^2 + (x+y)^2] = \ln[1 + (x+y)^2] - 2\,y \tag{10}$$

which simplifies to

$$u^2 + (x+y)^2 = e^{-2y}\,[1 + (x+y)^2]. \tag{11}$$

- **Ex. 2.4.** Consider a quasi-linear equation

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \tag{12}$$

(without specifying any initial conditions). Let $u_1(x,\, y)$, $u_2(x,\, y)$ be two solutions. Assume that the surfaces $u - u_1(x,y) = 0$ and $u - u_2(x,y) = 0$ intersects along a curve Γ in the xyu space. Show that Γ must be a characteristic curve.

Proof. Let Γ be parametrized by s, that is $\Gamma = \begin{pmatrix} x_0(s) \\ y_0(s) \\ u_0(s) \end{pmatrix}$. Since u_1, u_2 are different solutions, we have $\begin{pmatrix} u_{1x} \\ u_{1y} \\ -1 \end{pmatrix} \neq \begin{pmatrix} u_{2x} \\ u_{2y} \\ -1 \end{pmatrix}$ except may be at isolated points along Γ. This implies

$$\begin{pmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{u}_0 \end{pmatrix} // \left[\begin{pmatrix} u_{1x} \\ u_{1y} \\ -1 \end{pmatrix} \times \begin{pmatrix} u_{2x} \\ u_{2y} \\ -1 \end{pmatrix} \right] // \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{13}$$

Therefore Γ must be a characteristic curve. □

- **Ex. 2.7.** Show that the initial value problem

$$u_t + u_x = 0, \qquad u = x \text{ on } x^2 + t^2 = 1 \tag{14}$$

has no solution. However, if the initial data are given only over the semicircle that lies in the half-plane $x + t \leqslant 0$, the solution exists but is not differentiable along the characteristic base curves that issue from the two end points of the semicircle.

Proof. The characteristic curves are $x - t = c$ with $u =$ constant along each curve. Therefore the problem does not have classical solution. On the other hand, if the initial condition is $u = x$ on $x^2 + t^2 = 1$, $x + t \leqslant 0$, then we can use $\tau = x - t$ to parametrize the initial curve and obtain

$$u_0(\tau) = \frac{\sqrt{2 - \tau^2} + \tau}{2}. \tag{15}$$

If we take $s = x + t$, the solution is $u(\tau,\, s) = \frac{\sqrt{2-\tau^2} + \tau}{2}$ which gives $u(x,\, t) = \frac{1}{2}\,[(2 - (x-t))^2 + (x-t)]$. Observe that

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\,\frac{1}{\sqrt{2-\tau^2} + \tau}\left[\frac{-\tau}{\sqrt{2-\tau^2}} + 1\right] \tag{16}$$

which becomes ∞ at $\tau = \pm\sqrt{2}$. It is clear that u_x, u_t becomes ∞ at the end points of the semicircle.

Intuitively the reason is clear: Approaching the end points, the ratio between the distance between two chacteristics and the distance along the semicircle of the two intersection points become larger and larger, approaching infinity. □

- **Ex. 2.12.** Consider the wave equation

$$u_{tt} - u_{xx} = 0, \qquad u(x,0) = g(x), \quad u_t(x,0) = h(x). \tag{17}$$

Show that

a) If we set $v(x,t) = u_t - u_x$, then v satisfies

$$v_t + v_x = 0, \qquad v(x,0) = h(x) - g'(x). \tag{18}$$

b) Use method of characteristics to solve the v equation and then the u equation. Show that the solution is given by the d'Alembert's formula

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(s)\,\mathrm{d}s. \tag{19}$$

Solution. a) is obvious. For b), we first solve the v equation:

$$v_t + v_x = 0, \qquad v(x,0) = h(x) - g'(x) \tag{20}$$

to obtain

$$v(x,t) = h(x-t) - g'(x-t). \tag{21}$$

Now we solve

$$u_t - u_x = v = h(x-t) - g'(x-t), \qquad u(x,0) = g(x). \tag{22}$$

The characteristics are $x + t = c$. So we introduce new variables $\tau = x + t$, $s = x - t$. The equation becomes

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{1}{2}[g'(s) - h(s)], \qquad u_0(\tau) = g(\tau) \text{ along } x_0(\tau) = \tau, t_0(\tau) = 0. \tag{23}$$

which in the new variables becomes

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{1}{2}[g'(s) - h(s)], \qquad u_0(\tau,\tau) = g(\tau). \tag{24}$$

The solution is

$$u(s,\tau) = g(\tau) - \frac{1}{2}\left[\int_\tau^s h(\xi)\,\mathrm{d}\xi + g(s) - g(\tau)\right] = \frac{g(\tau) + g(s)}{2} - \frac{1}{2}\int_\tau^s h(\xi)\,\mathrm{d}\xi\,. \tag{25}$$

This leads to

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(s)\,\mathrm{d}s. \tag{26}$$

- **Ex. 2.18.** Solve (that is construct entropy solution for all t)

$$u_t + \left(\frac{u^4}{4}\right)_x = 0, \qquad u(0,x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}. \tag{27}$$

Solution. The speed of the shock is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1/4 - 0}{1 - 0} = \frac{1}{4} \tag{28}$$

therefore the solution is

$$u(x,t) = \begin{cases} 1 & x < t/4 \\ 0 & x > t/4 \end{cases}. \tag{29}$$

- **Ex. 2.19.** Compute explicitly the unique entropy solution of

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \qquad u(0,x) = g \tag{30}$$

for

$$g(x) = \begin{cases} 1 & x < -1 \\ 0 & -1 < x < 0 \\ 2 & 0 < x < 1 \\ 0 & x > 1 \end{cases}. \tag{31}$$

Draw a picture of your answer. Be sure to illustrate what happens for all times $t > 0$.
Solution. It is clear that initially we have two shocks and one rarefaction wave. The two shocks are

1. Starting from $(-1,0)$ with slope 2,
2. Starting from $(1,0)$ with slope $1/2$.

Note that after passing $(0,2)$ and $(2,1)$ both shocks are not straight anymore. Denote them by $x_1(t)$ and $x_2(t)$. First consider $x_1(t)$.

When $t < 2$ we have $x_1(t) = \frac{1}{2}t - 1$. For $t \geqslant 2$ we have

$$\dot{x}_1(t) = \frac{1}{2}\left(\frac{x_1}{t} + 1\right), \qquad x_1(2) = 0. \tag{32}$$

Now let $y(t) = x_1(t) - t$. We have

$$\dot{y}(t) = \dot{x}_1(t) - 1 = \frac{x_1}{2t} - \frac{1}{2} = \frac{y}{2t} \implies y = C t^{1/2}. \tag{33}$$

Now as $y(2) = x_1(2) - 2 = -2$, we have $C = -\sqrt{2}$. Thus

$$x_1(t) = \begin{cases} \frac{1}{2}t - 1 & t \leqslant 2 \\ t - \sqrt{2}\, t^{1/2} & t > 2 \end{cases}. \tag{34}$$

For $x_2(t)$ we have

$$\dot{x}_2(t) = \frac{1}{2}\left(\frac{x_2}{t} + 0\right), \qquad x_2(1) = 2. \tag{35}$$

Solving the equation we have

$$\ln x = \frac{1}{2}\ln t + C \implies x = C t^{1/2}. \tag{36}$$

Using $x_2(1) = 2$ we have

$$C = 2. \tag{37}$$

Thus the right shock is

$$x_2(t) = \begin{cases} 2t & t \leqslant 1 \\ 2t^{1/2} & t > 1 \end{cases}. \tag{38}$$

Setting $x_1(t) = x_2(t)$ we see that the two shocks meet at the point $\left(2 + \sqrt{2}, 6 + 4\sqrt{2}\right)$.

Finally, after $t = 6 + 4\sqrt{2}$, there is only one shock with speed $1/2$. To the left $u = 1$ and to the right $u = 0$.

- **Ex. 2.22.** Prove that

$$u(t,x) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & x > t \end{cases} \tag{39}$$

is a weak solution to the problem

$$u_t + u\, u_x = 0, \qquad u(0,x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}. \tag{40}$$

Proof. It suffices to prove the following: Consider finitely many regions Ω_i such that $\Omega_i \cap \Omega_j = \varnothing$ and $\cup \Omega_i = \mathbb{R} \times \{t > 0\}$. Let $u(x,t)$ be such that $u_t + f(u)_x = 0$ in each Ω_i, continuous across every $\Gamma_{ij} := \bar{\Omega}_i \cap \bar{\Omega}_j$, and furthermore $u(x,t) \longrightarrow u_0(x)$ at every x when $t \longrightarrow 0$, then u is a weak solution.

Take any $\phi \in C_0^1(\mathbb{R}^2)$. We have, using integration by parts,

$$\begin{aligned} \int\!\!\int u\,\phi_t + f(u)\,\phi_x \,\mathrm{d}x\,\mathrm{d}t + \int u_0(x)\,\phi(x,0)\,\mathrm{d}x &= \sum_i \int\!\!\int_{\Omega_i} u\,\phi_t + f(u)\,\phi_x\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int u_0(x)\,\phi(x,0)\,\mathrm{d}x \\ &= -\sum_i \int\!\!\int_{\Omega_i} (u_t + f(u)_x)\,\phi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \sum_{i,j} \int_{\Gamma_{ij}} n_t\,[u] + n_x\,[f(u)]\,\mathrm{d}s \\ &\quad - \sum_i \int_{\partial\Omega_i \cap \{t=0\}} u\,\phi\,\mathrm{d}x \\ &\quad + \int u_0(x)\,\phi(x,0)\,\mathrm{d}x \\ &= 0 \end{aligned} \tag{41}$$

thanks to the fact that u is continuous across Γ_{ij}, which means $[u] = 0$. ☐