MATH 436 FALL 2012 HOMEWORK 1 SOLUTIONS

DUE SEPT. 27 IN CLASS

Note. All problem numbers refer to "Updated" version of lecture note.

- Ex. 1.1. Derive in detail the equation for random walk in the following general case: The probability of the particle at (x,t) moving left, right, staying, and disappearing are $p_L(x,t)$, $p_R(x,t)$, $p_S(x,t)$ and $p_D(x,t)$. Assume the sum of these probabilities is 1. Note that as $h,\tau \to 0$, these probabilities need to satisfy certain relations (similar to $p_R - p_L \sim h$).

  Solution. We have[1]

  $$u(x,t+\tau) = p_L(x+h,t)\,u(x+h,t) + p_R(x-h,t)\,u(x-h,t) + p_S(x,t)\,u(x,t) \tag{1}$$

  Taylor expansion gives: (we omit (x,t))

  $$\begin{aligned} u + u_t\,\tau + o(\tau) &= p_L\,u + (p_L\,u)_x\,h + \frac{1}{2}\,(p_L\,u)_{xx}\,h^2 + o(h^2) \\ &\quad + p_R\,u - (p_R\,u)_x\,h + \frac{1}{2}\,(p_R\,u)_{xx}\,h^2 + o(h^2) \\ &\quad + p_S\,u \end{aligned} \tag{2}$$

  Divide by τ, setting $h^2/\tau = 2\,D$ and letting $\tau, h \longrightarrow 0$, we reach

  $$u_t = \left(-\lim \frac{p_D}{\tau}\right) u - \left(\lim \frac{p_R - p_L}{\tau}\,h\,u\right)_x + D\,[(p_L + p_R)\,u]_{xx}. \tag{3}$$

  From here we see that we need to further set $\frac{p_D}{\tau} = c$ and $\frac{p_R - p_L}{\tau}\,h = b$. Then we reach

  $$u_t + (b\,u)_x + c\,u = D\,[(p_L + p_R)\,u]_{xx}. \tag{4}$$

  Note that here $b = b(x,t), c = c(x,t)$ and $p_L + p_R$ is also function of (x,t).

- Ex. 1.3. Design an underlying random walk rule which gives the equation

  $$u_t = D\,u_{xx} + f(x,t), \qquad u(x,0) = u_0(x). \tag{5}$$

  Then use it to explain the following Duhamel's principle:

  Duhamel's principle. The solution to the above problem is given by

  $$u(x,t) = U(x,t) + \int_0^t v(x,t;s)\,\mathrm{d}s \tag{6}$$

  where $U(x,t)$ is the solution to

  $$u_t = D u_{xx}, \qquad u(x,0) = u_0(x) \tag{7}$$

  and $v(x,t;s)$ (with $t \geqslant s$) solves

  $$v_t = D v_{xx}, \qquad v(x,s;s) = f(x,s). \tag{8}$$

  Solution. The model is as follows: Consider a large amount of particles moving with equal probability to left and right with spatial step size h and time step size τ, with $h^2/\tau = 2\,D$, while $f(x,t)$ is the "rate" of particles added at location x at time t. Then we have

  $$u(x,t+\tau) = \frac{1}{2}\,u(x-h,t) + \frac{1}{2}\,u(x+h,t) + \tau\,f(x,t) \tag{9}$$

  which leads to the desired equation.

1. Here a possible explanation seems to be, the particles arrive "just before" $t+\tau$ while they disappear "just after" $t+\tau$, therefore p_D does not explicitly appear in the difference equation.

From this point of view, Duhamel's principle can be understood as follows. $U(x,t)$ represents the density at time t of those particles that are present at time $t=0$, $v(x,t;s)$ represents the density at time t of those particles that are added (into action) at time s. So if we consider $s=n\,\tau$, the density of all particles together is

$$u(x,t)=U(x,t)+\sum_{n=1}^{t/\tau}\tau\,v(x,t;n\,\tau)\longrightarrow U(x,t)+\int_0^t v(x,t;s)\,\mathrm{d}s \tag{10}$$

as $\tau\longrightarrow 0$ since that second term is a Riemann sum.

- Ex. 1.5. Design a random walk rule which leads to the equation

$$u_{tt}-\gamma^2\,u_{xx}+c\,u_x+2\,\lambda\,u_t=0. \tag{11}$$

Solution. Consider the case $p^{\pm}=1-\lambda^{\pm}\,\tau$, $q^{\pm}=\lambda^{\pm}\,\tau$ where $+$ corresponds to right moving particles ($\alpha(x,t)$) and $-$ left moving particles ($\beta(x,t)$). Then we have

$$\begin{aligned}\alpha(x,t+\tau) &= p^+\,\alpha(x-h,t)+q^-\,\beta(x-h,t) \quad (12)\\ \beta(x,t+\tau) &= p^-\,\beta(x+h,t)+q^+\,\alpha(x+h,t). \quad (13)\end{aligned}$$

Taylor expand:

$$\begin{aligned}\alpha+\alpha_t\,\tau+o(\tau) &= p^+\,[\alpha-\alpha_x\,h+o(h)]+q^-\,[\beta-\beta_x\,h+o(h)] \quad (14)\\ \beta+\beta_t\,\tau+o(\tau) &= p^-\,[\beta+\beta_x\,h+o(h)]+q^+\,[\alpha+\alpha_x\,h+o(h)]. \quad (15)\end{aligned}$$

Setting $h/\tau=\gamma$ and substituting $p^{\pm}=1-\lambda^{\pm}\,\tau$, $q^{\pm}=\lambda^{\pm}\,\tau$, we obtain

$$\begin{aligned}\alpha_t\,\tau+o(\tau) &= (-\lambda^+\,\tau)\,[\alpha-\alpha_x\gamma\,\tau]-\alpha_x\,\gamma\,\tau+\lambda^-\tau\,[\beta-\beta_x\,\gamma\,\tau] \quad (16)\\ \beta_t\,\tau+o(\tau) &= (-\lambda^-\,\tau)\,[\beta+\beta_x\gamma\,\tau]+\beta_x\,\gamma\,\tau+\lambda^+\tau\,[\alpha+\alpha_x\gamma\,\tau]. \quad (17)\end{aligned}$$

Dividing both sides by τ and let $\tau\longrightarrow 0$, we obtain

$$\begin{aligned}\alpha_t &= -\lambda^+\,\alpha+\lambda^-\,\beta-\gamma\,\alpha_x \quad (18)\\ \beta_t &= -\lambda^-\,\beta+\lambda^+\,\alpha+\gamma\,\beta_x \quad (19)\end{aligned}$$

Setting $u=\alpha+\beta, v=\alpha-\beta$, we have

$$\begin{aligned}u_t+\gamma\,v_x &= 0 \quad (20)\\ v_t+\gamma\,u_x &= -2\,\lambda^+\,\alpha+2\,\lambda^-\,\beta=-(\lambda^++\lambda^-)\,v-(\lambda^+-\lambda^-)u\,. \quad (21)\end{aligned}$$

Denote $2\,\lambda=\lambda^++\lambda^-$ and $\gamma\,(\lambda^+-\lambda^-)=-c$, we have

$$u_{tt}=-\gamma\,v_{xt}=\gamma^2\,u_{xx}-2\,\lambda\,u_t-c\,u_x. \tag{22}$$

- Ex. 1.8. Consider particle moving with probabilities (p_1, q_1 right/left; p_2/q_2 up/down)

$$p_i(x,y)=\frac{1}{4}\,[a_i(x,y)+b_i(x,y)\,h],\qquad q_i(x,y)=\frac{1}{4}\,[a_i(x,y)-b_i(x,y)\,h]. \tag{23}$$

Show that for Problem No.1 we get

$$a_1\,u_{xx}+a_2\,u_{yy}+2\,b_1\,u_x+2\,b_2\,u_y=0. \tag{24}$$

For Problem No. 2 we get

$$(a_1\,w)_{xx}+(a_2\,w)_{yy}-2\,(b_1\,w)_x-2\,(b_2\,w)_y=-\delta(x-\xi)\,\delta(y-\eta). \tag{25}$$

Solution.

  - For problem 1, we have

    $$u(x,y)=p_1(x,y)\,u(x+h,y)+q_1(x,y)\,u(x-h,y)+p_2(x,y)\,u(x,y+h)+q_2(x,y)\,u(x,y-h). \tag{26}$$

Taylor expansion leads to the desired equation.

- For problem 2, we have[2]

$$u(x,y)=\begin{cases} 1+p_1(x+h,y)\,u(x+h,y)+\cdots & (x,y)=(\xi,\eta) \\ p_1(x+h,y)\,u(x+h,y)+\cdots & \text{elsewhere} \end{cases}. \tag{27}$$

Taylor expansion leads[3] to the desired equations.

2. This seems to mean that we have to interpret $w(x,y)$ as something like the number of paths reaching (ξ,η).
3. Expand $p_1\,u$ together as $(p_1\,u)(x+h,y)=p_1\,u(x,y)+(p_1\,u)_x\,h.....$