NAME: ID:

Math 317 Quiz 6 Solutions

MAR. 31, 2014

- The quiz has three problems. Total $10+1$ points. It should be completed in 20 minutes.

Question 1. (5 pts) *Let* $\boldsymbol{f}(x,y,z) := \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ *and* S *be the portion of* $x^2+y^2=z$ *with* $1 \leqslant z \leqslant 4$*, oriented so that the normal points upward. Calculate* $\int_S \boldsymbol{f} \cdot \mathbf{d}\boldsymbol{S}$.

Solution. We parametrize S as

$$x=u, y=v, z=u^2+v^2, \qquad (u,v) \in D := \{1 \leqslant u^2+v^2 \leqslant 4\}. \tag{1}$$

Then

$$\boldsymbol{r}_u = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}, \boldsymbol{r}_v = \begin{pmatrix} 0 \\ 1 \\ 2v \end{pmatrix} \Longrightarrow \boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix}. \tag{2}$$

As the 3rd component is $1>0$, $\boldsymbol{r}_u \times \boldsymbol{r}_v$ points upward. Thus we have

$$\begin{aligned} \int_S \boldsymbol{f} \cdot \mathbf{d}\boldsymbol{S} &= \int_D \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix} \mathrm{d}(u,v) \\ &= \int_D [-6u-4v+1]\, \mathrm{d}(u,v) \\ \int [-6u-4v]=0 \text{ due to symmetry} &= \int_D \mathrm{d}(u,v) = 3\pi. \end{aligned} \tag{3}$$

Remark. Alternative solution.

Solution.

$$x = r\cos\theta, y = r\sin\theta, z = r^2. \tag{4}$$

Then

$$\boldsymbol{r}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2r \end{pmatrix}, \quad \boldsymbol{r}_\theta = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} \Longrightarrow \boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix}. \tag{5}$$

Thus we have

$$\begin{aligned}\int_S \boldsymbol{f}\cdot \mathbf{d}\boldsymbol{S} &= \int_{[1,2]\times[0,2\pi]} -6\,r^2\cos\theta - 4\,r^2\sin\theta + r\,\mathrm{d}(r,\theta)\\ &= \int_1^2 \left[\int_0^{2\pi} -6\,r^2\cos\theta - 4\,r^2\sin\theta + r\,\mathrm{d}\theta\right]\mathrm{d}r\\ &= \int_1^2 2\,\pi\,r\,\mathrm{d}r = 3\,\pi. \qquad (6)\end{aligned}$$

Question 2. (5 pts) *Let $f, g \in C^1$ and let $V \subset \mathbb{R}^3$ satisfy hypotheses of Gauss's Theorem. Prove that*

$$\int_V f\,\frac{\partial g}{\partial x}\,\mathrm{d}(x,y,z) = \int_{\partial V} f\,g\,n_x\,\mathrm{d}S - \int_V g\,\frac{\partial f}{\partial x}\,\mathrm{d}(x,y,z) \qquad (7)$$

where n_x is the x-component of the unit outer normal $\boldsymbol{n}$, that is $\boldsymbol{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$.

Solution. All we need to prove is

$$\int_V \frac{\partial (f\,g)}{\partial x}\,\mathrm{d}(x,y,z) = \int_{\partial V} f\,g\,n_x\,\mathrm{d}S. \qquad (8)$$

Set $\boldsymbol{F}(x,y,z) := \begin{pmatrix} f\,g \\ 0 \\ 0 \end{pmatrix}$. Application of Gauss's Theorem now gives

$$\int_{\partial V} f\,g\,n_x\,\mathrm{d}S = \int_{\partial V} \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_V \mathrm{div}\,(\boldsymbol{F})\,\mathrm{d}(x,y,z) = \int_V \frac{\partial (f\,g)}{\partial x}\,\mathrm{d}(x,y,z). \qquad (9)$$

Question 3. (1 bonus pt) *Prove Green's Theorem for $D = \{(x,y) |\, x, y \geqslant 0, x + y \leqslant 1\}$.*

Solution. Let f, g be arbitrary C^1 functions. Denote by S_1, S_2, S_3 the three parts of ∂D:

$$S_1\colon (0,0) \to (1,0);\ S_2\colon (1,0) \to (0,1);\ S_3\colon (0,1) \to (0,0). \qquad (10)$$

All three are straight line segments. We parametrize:

$$S_1\colon \begin{pmatrix} u \\ 0 \end{pmatrix};\quad S_2\colon \begin{pmatrix} 1-u \\ u \end{pmatrix};\quad S_3\colon \begin{pmatrix} 0 \\ 1-u \end{pmatrix} \qquad (11)$$

all three parametrizations are $u \in [0,1]$.

Now calculate

$$\begin{aligned}
\int_{\partial D} f\,\mathrm{d}x + g\,\mathrm{d}y &= \int_{S_1} f\,\mathrm{d}x + g\,\mathrm{d}y \\
&\quad + \int_{S_2} f\,\mathrm{d}x + g\,\mathrm{d}y \\
&\quad + \int_{S_3} f\,\mathrm{d}x + g\,\mathrm{d}y \\
&= \int_0^1 f(u,0)\,\mathrm{d}u + \int_0^1 [-f(1-u,u) + g(1-u,u)]\,\mathrm{d}u \\
&\quad - \int_0^1 g(0,1-u)\,\mathrm{d}u \\
\text{(Change of variable } 1-u=v\text{)} &= \int_0^1 f(u,0)\,\mathrm{d}u + \int_0^1 [-f(v,1-v)]\,\mathrm{d}v \\
&\quad + \int_0^1 g(1-u,u)\,\mathrm{d}u - \int_0^1 g(0,v)\,\mathrm{d}v \\
&= \int_0^1 [f(u,0) - f(u,1-u)]\,\mathrm{d}u \\
&\quad + \int_0^1 [g(u,1-u) - g(0,1-u)]\,\mathrm{d}u \\
&= -\int_0^1 \left[\int_0^{1-u} \frac{\partial f}{\partial y}(u,v)\,\mathrm{d}v\right]\mathrm{d}u \\
&\quad + \int_0^1 \left[\int_0^{1-u} \frac{\partial g}{\partial x}(u,v)\,\mathrm{d}v\right]\mathrm{d}u \\
&= \int_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right]\mathrm{d}(x,y). \qquad (12)
\end{aligned}$$