Name: ID:

Math 317 Quiz 3 Solutions

Feb. 3, 2014

- The quiz has three problems. Total 10 + 1 points. It should be completed in 20 minutes.

Question 1. (5 pts) *Prove* ***by definition*** *that*

$$f_n(x) = \frac{n\,x^3 + 5}{3\,n^2 + 7} \tag{1}$$

converges uniformly to 0 *on* $[0, 1]$.

Solution. For any $\varepsilon > 0$, take $N > \max\{5, \varepsilon^{-1}\}$, then for any $n > N$,

$$\left| \frac{n\,x^3 + 5}{3\,n^2 + 7} \right| \leqslant \frac{n + 5}{3\,n^2 + 7} < \frac{2\,n}{2\,n^2} < \frac{1}{n} < \frac{1}{N} < \varepsilon. \tag{2}$$

Question 2. (5 pts)

a) *Find a positive convergent series* $\sum_{n=1}^{\infty} a_n$ *such that*

$$\limsup_{n \to \infty} \frac{a_{n+1}}{a_n} > 1. \tag{3}$$

b) *Find a positive series* $\sum_{n=1}^{\infty} a_n$ *such that* $\limsup_{n \to \infty} |a_n|^{1/n} > 1$ *but the following does not hold:*

$$\exists N \in \mathbb{N}, \quad \forall n > N, \qquad |a_n|^{1/n} \geqslant 1. \tag{4}$$

Solution.

a) Take $a_n = \frac{2 + (-1)^n}{n^2}$.

b) Take $a_n = [1 + (-1)^n]\, 2^n$.

Question 3. (1 bonus pt) *Prove that*

$$\sum_{n=1}^{\infty} x \sin\left(\frac{1}{n^4\, x^2} \right) \tag{5}$$

converges uniformly on $(0,\infty)$.

Solution. Take any $R>1$. For $0<x\leqslant\frac{1}{n^2}$, we have

$$\left|x\sin\left(\frac{1}{n^4x^2}\right)\right|\leqslant\frac{1}{n^2}. \tag{6}$$

For $\frac{1}{n^2}<x<\infty$, we have

$$\left|x\sin\left(\frac{1}{n^4x^2}\right)\right|<\frac{x}{n^4x^2}=\frac{1}{n^4x}<\frac{1}{n^2}. \tag{7}$$

Therefore we have

$$\forall x\in(0,\infty),\qquad\left|x\sin\left(\frac{1}{n^4x^2}\right)\right|\leqslant\frac{1}{n^2}. \tag{8}$$

Thus by Weierstrass' M-test the series converges uniformly on $(0,\infty)$.

Remark 4. Alternatively, define $u(x):=x\sin(1/x^2)$. Then

$$x\sin\left(\frac{1}{n^4x^2}\right)=n^{-2}u(n^2x). \tag{9}$$

We can prove that $u(x)$ is uniformly bounded on $(0,\infty)$ which again leads to

$$\forall x\in(0,\infty),\qquad\left|x\sin\left(\frac{1}{n^4x^2}\right)\right|\leqslant\frac{1}{n^2}. \tag{10}$$