Math 317 Week 12: Lebesgue Measure: A Brief Introduction

April 1, 2014

Table of contents

References.

- (BEAR) Bear, H. S., *A Primer of Lebesgue Integration*, 2nd ed., Academic Press, 2002. Chapters 3 & 4.
- (CIESIELSKI) Ciesielski, Krzysztof, *How Good is Lebesgue Measure?*, The Mathematical Intelligencer, 11(2), 54 – 58.
- (GOWERS) Gowers, Timothy, ed., *The Princeton Companion to Mathematics*, IV.22 Set Theory.
- (HAWKINS BOOK) Hawkins, Thomas, *Lebesgue's Theory of Integration: Its Origins and Development*, The University of Wisconsin Press, 1970.
- (HAWKINS CHAPTER) Hawkins, Thomas, *The Origins of Modern Theories of Integration*, Chapter 4 in *From the Calculus to Set Theory 1630 - 1910: An Introductory History*, edited by I. Grattan-Guinness, Princeton University Press, 1980.
- (KURATOWSKI) Kuratowski, K. and Mostowski, A., *Set Theory: With an Introduction to Descriptive Set Theory*, North-Holland 1976, Chapter XII.
- (LORENTZ) Lorentz, G. G., *Who discovered analytic sets?*, Mathematical Intelligencer, 23 (4), 2001, pp. 29 – 32. Online at http://www.math.nsc.ru/LBRT/g2/english/ssk/lorentz_a-sets.pdf.
- (SOLOMON) Solomon, Isaac, *Measure Theory and Lebesgue Integration: An Introductory Course*. Online note.[1]
- (WIKI) Wikipedia: Borel sets; Borel hierarchy; Descriptive set theory.

1. Looks like this: http://www.math.brown.edu/~ysolomon/index.html should be the author. However the article is not on the website.

1. LEBESGUE OUTER MEASURE

NOTATION. *In this and the following sections, we use* μ, μ_{in}, μ_{out} *to denote Jordan measure, Jordan inner measure, Jordan outer measure; And use* m, m_*, m^* *to denote Lebesgue measure, Lebesgue inner measuer, Lebesgue outer measure.*

1.1. Lebesgue outer measure

The inspiration comes from the following idea proposed by Èmile Borel: Let $(a_i, b_i), i = 1, 2, 3, \ldots$ be disjoint finite open intervals. Let $A := \cup_{i=1}^{\infty}(a_i, b_i)$. Then its measure should be

$$\sum_{i=1}^{\infty} (b_i - a_i). \tag{1}$$

Exercise 1. Prove that this infinite sum always exists (may be infinity). [2]

DEFINITION 1. *Let* $A \subseteq \mathbb{R}$*. Then we define its Lebesgue outer measure through*

$$m^*(A) := \inf \left[\sum_{i=1}^{\infty} (b_i - a_i) \right] \tag{2}$$

where the infimum is taken over all countable sequences of open intervals $\{(a_i, b_i)\}$ *such that*

$$A \subseteq \cup_{i=1}^{\infty}(a_i, b_i), \tag{3}$$

that is over all countable coverings of A *by open intervals.*

Exercise 2. Show that the (a_i, b_i)'s can be taken disjoint in Definition 1. (Hint:[3])

Exercise 3. Show that the "open intervals" in Definition 1 can be replaced by "closed intervals", "half-open, half-closed intervals", or simply "intervals".

Exercise 4. Show that the requirement that the covering is countable can be dropped. (Hint:[4])

Remark 2. Note that we cannot drop both "countable" and "open", since otherwise we have $A \subseteq \cup_{a \in A}[a, a]$ for any set A, and clearly $\sum_{a \in A} |a - a| = \sum_{a \in A} 0 = 0$ in any reasonable definition.

Remark 3. Just like Jordan outer measure, Lebesgue outer measure is defined for every set.

Exercise 5. Why is it defined for every set? (Hint:[5])

Exercise 6. Let μ_{out} denote the Jordan outer measure. Prove that for every $A \subseteq \mathbb{R}$,

$$m^*(A) \leqslant \mu_{\text{out}}(A). \tag{4}$$

2. The RHS is a positive series so either converges to a finite value or to $+\infty$.

3. Any open set in $\mathbb{R}$ can be written as a countable union of disjoint open intervals.

4. Prove that any covering of a set A by open intervals $A \subseteq \cup_{I \in W} I$ has a countable sub-covering. To see this, let $A_n := A \cap [\cup_{I \in W_n} I]$, where $W_n := \{I \in W \mid |I| \geqslant 1/n\}$. Prove that the covering $A_n \subseteq \cup_{I \in W_n} I$ has a countable sub-covering through considering rational points.

5. Because $\mathbb{R}$ has the least upper bound property which guarantees the existence of infimum.

From finite to countable

We see that the only difference between Jordan outer measure and Lebesgue outer measure is that, the latter allows countably many intervals in the covering.

This idea, that is replacing finite covering in the definition of Jordan outer measure by countable covering, though feels very natural to us today, was not so when it was first proposed by Borel. The concern was that it immediately leads to the conclusion that the set of rational numbers has zero outer measure and is thus "negligible". This was anything but natural to most mathematicians in the late 19th century due to the fact that rational numbers are dense. See (HAWKINS BOOK) for a detailed account of the historical origin and development of Lebesgue's theory.

Example 4. Let $A := \cup_{i=1}^{\infty}(a_i, b_i)$ where (a_i, b_i) are disjoint from each other. Then

$$m^*(A) := \sum_{i=1}^{\infty} (b_i - a_i) \tag{5}$$

Proof. Clearly $A \subseteq \cup_{i=1}^{\infty}(a_i, b_i)$ therefore $m^*(A) \leqslant \sum_{i=1}^{\infty} (b_i - a_i)$.

For the other direction, let

$$A \subseteq \cup_{i=1}^{\infty}(c_i, d_i) \tag{6}$$

where (c_i, d_i) may or may not be disjoint.

Let $\epsilon > 0$ be arbitrary. Take any $n \in \mathbb{N}$ and define $A_n := \cup_{i=1}^{n}\left[a_i + \frac{\varepsilon}{2n}, b_i - \frac{\varepsilon}{2n}\right]$ ($[a, b] = \varnothing$ if $b < a$). Then we have A_n compact and

$$A_n \subseteq \cup_{i=1}^{\infty}(c_i, d_i). \tag{7}$$

Consequently we have a finite subcover, say

$$A_n \subseteq \cup_{i=1}^{k}(c_i, d_i). \tag{8}$$

Both sides are Jordan measurable. Thus we easily see that

$$\sum_{i=1}^{n} (b_i - a_i) - \varepsilon \leqslant \mu(A_n) \leqslant \mu\big(\cup_{i=1}^{k}(c_i, d_i)\big) \leqslant \sum_{i=1}^{k} (d_i - c_i) \leqslant \sum_{i=1}^{\infty} (d_i - c_i). \tag{9}$$

As n is arbitrary, this gives

$$\sum_{i=1}^{\infty} (b_i - a_i) - \varepsilon \leqslant \sum_{i=1}^{\infty} (d_i - c_i). \tag{10}$$

This in turn gives

$$\sum_{i=1}^{\infty} (b_i - a_i) \leqslant \sum_{i=1}^{\infty} (d_i - c_i) \tag{11}$$

thanks to the arbitrariness of $\varepsilon > 0$.

Taking infimum over all $\{(c_i, d_i)\}$ we reach $\sum_{i=1}^{\infty} (b_i - a_i) \leqslant m^*(A)$. □

Exercise 7. Prove that $m^*([0,1]) = 1$. (Hint:[6])

Exercise 8. Prove that $m^*(A) = 0$ for any finite set A.

PROPOSITION 5. *The Lebesgue outer measure has the following properties.*

a) $A \subseteq B \Longrightarrow m^*(A) \leqslant m^*(B)$;

b) Let $A_1, A_2, \ldots \subseteq \mathbb{R}$ *be a countable sequence of sets. Then*

$$m^*(\cup_{n=1}^{\infty} A_n) \leqslant \sum_{n=1}^{\infty} m^*(A_n). \tag{12}$$

Proof. The proof for a) is left as exercise. For b), let $\varepsilon > 0$ be arbitrary. Let $A_n \subseteq \cup_{k=1}^{\infty} (a_{n,k}, b_{n,k})$ such that

$$\sum_{k=1}^{\infty} (b_{n,k} - a_{n,k}) \leqslant m^*(A_n) + \frac{\varepsilon}{2^n}. \tag{13}$$

Then we have

$$\cup A_n \subseteq \cup_{n,k=1}^{\infty} (a_{n,k}, b_{n,k}) \tag{14}$$

which is still a countable convering and furthermore

$$\sum_{n,k=1}^{\infty} (b_{n,k} - a_{n,k}) = \sum_{n=1}^{\infty} \left[\sum_{k=1}^{\infty} (b_{n,k} - a_{n,k}) \right] \leqslant \varepsilon + \sum_{n=1}^{\infty} m^*(A_n). \tag{15}$$

The conclusion now follows from the arbitrariness of ε. □

Remark 6. Naturally[7] one would like to construct $\{A_n\}$ such that the inequality in (12) becomes strict. However any attempt to explicitly construct such sets is doomed to fail due to the fact that their construction is only possible if we accept Axiom of Choice. More on this later when we discuss measurability.

Exercise 9. Prove (15). (Hint:[8])

Exercise 10. Let $A_1, A_2, \ldots$ be a sequence of disjoint compact sets. Then $m^*(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m^*(A_n)$.(Hint:[9])

Exercise 11. Let $A = \cup_{i=1}^{\infty} [a_i, b_i]$ where (a_i, b_i) are disjoint – note that $a_i = b_j$ could happen. Prove that

$$m^*(A) = \sum_{i=1}^{\infty} (b_i - a_i). \tag{16}$$

6. Let $[0,1] \subseteq \cup_{i=1}^{\infty} (a_i, b_i)$. Since $[0,1]$ is compact, there is a finite sb-covering.

7. If this thought didn't pop up when you are reading this, please slow down to allow yourself some thinking time.

8. The convergence of non-negative series is not affected by re-arrangement.

9. Prove that for any $N \in \mathbb{N}$, $\sum_{n=1}^{N} m^*(A_n) \leqslant m^*(\cup A_n)$. To see this, consider an open covering $\cup (a_i, b_i)$ of $\cup A_n$. Then there is a finite subcovering of $\cup_{n=1}^{N} A_n$. Now construct appropriate coverings based on these (a_i, b_i) – break up some (a_i, b_i) into two or more intervals if necessary – of each A_n.

(Hint:[10])

1.2. Zero measure sets

1.2.1. Definition and properties

DEFINITION 7. (ZERO MEASURE SET) $A \subseteq \mathbb{R}$ *is said to have zero Lebesgue measure if and only if* $m^*(A) = 0$.[11] *In other words if and only if the following holds:*

Given any $\varepsilon > 0$*, there is a countable sequence of open intervals* $\{(a_i, b_i)\}$ *such that* $A \subseteq \cup_{i=1}^{\infty} (a_i, b_i)$ *and* $\sum_{i=1}^{\infty} (b_i - a_i) < \varepsilon$.

PROPOSITION 8. *Let* $A \subset \mathbb{R}$ *be countable. Then* $m(A) = 0$.

Proof. This follows directly from Proposition 5. But a direct proof is also easy.

As A is countable, it can be listed $A = \{a_1, a_2, \ldots\}$. Now for any $\varepsilon > 0$, we have

$$A \subseteq \cup_{i=1}^{\infty} \Big(a_i - \frac{\varepsilon}{2^{i+2}}, a_i + \frac{\varepsilon}{2^{i+2}} \Big). \quad (17)$$

Note that

$$\sum_{i=1}^{\infty} \Big[\Big(a_i + \frac{\varepsilon}{2^{i+2}} \Big) - \Big(a_i - \frac{\varepsilon}{2^{i+2}} \Big) \Big] = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2} < \varepsilon. \quad (18)$$

Thus $m(A) = 0$ by definition. □

Exercise 12. Prove that $[0, 1]$ is uncountable. (Hint:[12])

Remark 9. Note that $m(A) = 0$ does not imply that A is countable, as the Cantor set shows.

THEOREM 10. *Let* $A_n, n \in \mathbb{N}$ *be such that* $m(A_n) = 0$ *for all* n*. Then* $m(\cup A_n) = 0$.

Proof. Exercise. (Hint:[13]) □

Almost everywhere

Let $P(x)$ be a statement regarding real numbers. If there is $A \subset \mathbb{R}$ such that $P(x)$ is true for all $x \in \mathbb{R} - A$, we say $P(x)$ holds "almost everywhere", denoted "$P(x) \quad a.e.$". For example, let $f(x) = \begin{cases} 0 & x \in \mathbb{R} - \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}$, then $f(x) = 0 \quad a.e.$.

10. Note that $A \supseteq \cup (a_i, b_i)$.

11. Also denoted $m(A) = 0$ since any such set is in fact Lebesgue measurable.

12. If it is countable then it should have zero outer measure.

13. For A_n, find a covering of open intervals whose length add up to less than $\frac{\varepsilon}{2^{n+1}}$. Or apply Proposition 5.

Exercise 13. Show that the Cantor function (devil's staircase) – check wiki if you don't know what it is – is constant (and thus differentiable with derivative 0) almost everywhere.

1.2.2. Lebesgue's theorem on Riemann integrability

LEMMA 11. *Let $A \subseteq \mathbb{R}$ be compact. Then $\mu_{\text{out}}(A) = m^*(A)$.*

Proof. All we need to prove is $\mu_{\text{out}}(A) \leqslant m^*(A)$. Let $\varepsilon > 0$ be arbitrary. Take any countable covering $\cup(a_i, b_i) \supseteq A$ such that $\sum_{i=1}^{\infty} (b_i - a_i) \leqslant m^*(A) + \varepsilon$. Then there is a finite subcovering (a_{i_k}, b_{i_k}), $k = 1, 2, \ldots, K$. Now we have

$$\mu_{\text{out}}(A) \leqslant \sum_{k=1}^{K} (b_{i_k} - a_{i_k}) \leqslant \sum_{i=1}^{\infty} (b_i - a_i) \leqslant m^*(A) + \varepsilon. \tag{19}$$

The conclusion now follows from the arbitrariness of ε. □

Exercise 14. Find a compact set $A \subseteq [0, 1]$ that is not Jordan measurable. (Hint:[14])

THEOREM 12. (LEBESGUE) *Let $f: [a, b] \mapsto \mathbb{R}$ be bounded. Let $A := \{x \in [a, b] | \, f$ is not continuous at $x\}$. Then f is Riemann integrable on $[a, b]$ if and only if $m^*(A) = 0$. In other words, f is Riemann integrable on $[a, b]$ if and only if f is continuous almost everywhere.*

Proof. As the bulk of the proof is in fact several exercises of Math 217, we only sketch the new ideas here.

The key is the following "oscillation function", originally proposed by Riemann.

$$\omega_f(x) := \lim_{\delta \to 0+} \left[\sup_{|y-x|<\delta} f(y) - \inf_{|y-x|<\delta} f(y) \right]. \tag{20}$$

It can be shown that f is Riemann integrable if and only if for every $\varepsilon > 0$,

$$\mu_{\text{out}}(\{x \in [a, b] | \, \omega_f(x) > \varepsilon\}) = 0. \tag{21}$$

Since $\mu_{\text{out}} \geqslant m^*$, we have $m^*\{\omega_f(x) > \varepsilon\} = 0$ and $m^*(A) = 0$ follows from Proposition 5 and $A = \cup_{n \in \mathbb{N}} \{x | \, \omega_f(x) > 1/n\}$.

For the other direction, the key is the prove that for every $\varepsilon > 0$, the set $\{x \in [a, b] | \, \omega_f(x) > \varepsilon\}$ is bounded and closed, and is thus compact thanks to Heine-Borel. □

Remark 13. Recall that a set A is Jordan measurable if and only if $\mu_{\text{out}}(\partial A) = 0$. By Theorem 12 we see that A is Jordan measurable if and only if $m^*(\partial A) = 0$. Thanks to Lemma 11, there is no inconsistency here, as the boundary set ∂A is always compact.

14. Construct $B \subseteq [0, 1]$ open and not Jordan measurable through "fattening" the rationals. Then take $A = [0, 1] - B$.

2. Lebesgue Measure on $[0,1]$

For simplicity of presentation we restrict our consideration in this section to subsets of $[0,1]$. Only a small part of the proofs needs to be changed – be warned: there are indeed things that need to be changed! – when considering the general case.

From now on, we will denote $A^c := [0,1] - A$.

2.1. Lebesgue measurability

Definition 14. (Inner Measure) *Let $A \subseteq [0,1]$. We define its Lebesgue inner measure through*

$$m_*(A) := 1 - m^*(A^c). \tag{22}$$

Exercise 15. Let $A \subseteq [0,1]$. Then $\mu_{\text{in}}(A) \leqslant m_*(A)$.

Proposition 15. *Let $A \subseteq [0,1]$. Then $m_*(A) \leqslant m^*(A)$.*

Proof. We have

$$m^*(A) + m^*(A^c) \geqslant m^*(A \cup A^c) = m^*([0,1]) = 1. \tag{23}$$

The conclusion follows. □

Definition 16. (Measurable sets) *$A \subseteq [0,1]$ is Lebesgue measurable if and only if $m_*(A) = m^*(A)$ (that is $m^*(A) + m^*(A^c) = 1$). The common value is called the Lebesgue measure of A, denoted $m(A)$.*

From now on, "measurable" means Lebesgue measurable.

Exercise 16. Prove: $A \subseteq [0,1]$ is measurable if and only if $m^*(A) + m^*(A^c) \leqslant 1$.

Theorem 17. *If $A \subseteq [0,1]$ is Jordan measurable then it is measurable. In this case furthermore $\mu(A) = m(A)$.*

Proof. Exercise. (Hint:[15]) □

Exercise 17. Prove: $A \subseteq [0,1]$ is Jordan measurable if and only if $\mu_{\text{out}}(A) + \mu_{\text{out}}(A^c) = \mu_{\text{out}}([0,1]) = 1$. (Hint:[16])

Proposition 18. *Let $A \subseteq [0,1]$ be such that $m^*(A) = 0$. Then A is measurable, and $m(A) = 0$.*

Proof. All we need to prove is $m^*(A^c) = 1$. We have on one hand $m^*(A^c) \leqslant m^*([0,1]) = 1$ while on the other $m^*(A^c) \geqslant m^*([0,1]) - m^*(A) = 1$. Therefore $m^*([0,1] - A) = 1$. □

Exercise 18. Prove that open, closed, half-open/half-closed intervals are all measurable.

15. $\mu_{\text{in}} \leqslant m_* \leqslant m^* \leqslant \mu_{\text{out}}$.

16. Take any simple graph $B \supseteq A^c$. Then $B^c := [0,1] - B$ is measurable and furthermore $[0,1] - B \subseteq A$ which implies $\mu_{\text{in}}(A) = \mu_{\text{in}}(B^c) = \mu(B^c) = \mu([0,1]) - \mu(B) = 1 - \mu(B)$. This gives $\mu_{\text{in}}(A) \geqslant 1 - \mu_{\text{out}}(A^c) = \mu_{\text{out}}(A)$.

It turns out that $m^*(A) + m^*(A^c) = 1$, that is $m^*(A \cap [0, 1]) + m^*(A^c \cap [0, 1]) = m^*([0, 1])$, is equivalent to the following condition which looks like a much stronger condition.

THEOREM 19. (CARATHEODORY'S CRITERION) $A \subseteq [0,1]$ *is Lebesgue measurable if and only if for every* $E \subseteq [0,1]$*, measurable or not, there holds*

$$m^*(E) = m^*(A \cap E) + m^*(A^c \cap E). \tag{24}$$

Proof. See §4.1. □

Remark 20. When we need to prove the measurability of a set, we only need to check Definition 16; But once we know a set is measurable, we can immediately employ the power of (24).

Are you surprised?

Instead of proving the theorem, we provide some explanation and rough idea here.

- At the level of intuition, we note that, just like Jordan measurability, (Lebesgue) measurability of a set A can also be understood as a requirement that ∂A is "not too irregular". Thus if $m^*(A \cap [0, 1]) + m^*(A^c \cap [0, 1]) = m^*([0, 1])$, then the whole ∂A must be "not too irregular". On the other hand, when we consider $m^*(A \cap E) + m^*(A^c \cap E)$, only part of ∂A is involved here, so the mess it creates is just part of that created by the whole ∂A. As the latter is negligible, the former must also be.
- The strategy of proof would be as follows.
  - Prove (24) when E is an open interval;
  - Prove (24) when E is a finite union of disjoint open intervals;
  - Prove (24) for general E using the fact that the outer measure of any set is defined through approximation by open sets.

2.2. Properties of Lebesgue measure

THEOREM 21. *The following hold.*

a) *A is measurable $\Longleftrightarrow$ A^c is measurable;*

b) *A, B are measurable then so are $A \cap B$, $A \cup B$; If furthermore A, B are disjoint, then $m(A \cup B) = m(A) + m(B)$.*

c) *$A_1, A_2, \ldots$ are measurable and disjoint. Then $\cup_{n=1}^{\infty} A_n$ is measurable and furthermore*

$$m(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n). \tag{25}$$

Proof.

a) Exercise. (Hint:[17])

17. $(A^c)^c = A$.

b) As $A\cup B=(A^c\cap B^c)^c$, it suffices to prove the measurability of $A\cap B$ for any measurable A, B. We apply Theorem 19:

$$m^*(A\cap B)+m^*(A^c\cap B)=m^*(B); \tag{26}$$

$$m^*(A\cap B^c)+m^*(A^c\cap B^c)=m^*(B^c). \tag{27}$$

Now adding the two equalities up, we have

$$m^*(A\cap B)+m^*(A^c\cap B)+m^*(A\cap B^c)+m^*(A^c\cap B^c)=1. \tag{28}$$

But

$$(A^c\cap B)\cup(A^c\cap B^c)\cup(A\cap B^c)=(A\cap B)^c \tag{29}$$

which gives, thanks to Proposition 5,

$$m^*(A^c\cap B)+m^*(A\cap B^c)+m^*(A^c\cap B^c)\geqslant m^*((A\cap B)^c). \tag{30}$$

Thus we have

$$m^*(A\cap B)+m^*((A\cap B)^c)\leqslant 1. \tag{31}$$

Applying Proposition 5 one more time we obtain $m^*(A\cap B)+m^*((A\cap B)^c)=1$ and measurability follows.

Now assume $A\cap B=\varnothing$. In this case (28) becomes (note that the $*$ vanishes due to measurability of all sets involved)

$$m(B)+m(A)+m(A^c\cap B^c)=1. \tag{32}$$

But as $A^c\cap B^c=(A\cup B)^c$ is measurable, we have

$$1-m(A^c\cap B^c)=m(A\cup B) \tag{33}$$

and the conclusion follows.

c) Denote $A:=\cup_{n=1}^{\infty}A_n$. Then we have

$$A^c\subseteq(A_1\cup\cdots\cup A_N)^c\Longrightarrow m^*(A^c)\leqslant m^*((A_1\cup\cdots\cup A_N)^c)=m((A_1\cup\cdots\cup A_N)^c). \tag{34}$$

Thus

$$m^*(A^c)\leqslant 1-\sum_{n=1}^{N}m(A_n) \tag{35}$$

for any $N\in\mathbb{N}$. This leads to

$$m^*(A^c)\leqslant 1-\sum_{n=1}^{\infty}m(A_n)\leqslant 1-m^*(A)\leqslant m^*(A^c). \tag{36}$$

Thus all the relations must be "=" and both measurability and $m(A)=\sum_{n=1}^{\infty}m(A_n)$ follow. □

Exercise 19. Let A, B be measurable. Prove that $A-B$ is measurable. (Hint:[18])

18. $A-B=A\cap B^c$.

Exercise 20. Let $A_1, A_2, \ldots$ be measurable. Prove that $\cup_{n=1}^{\infty} A_n$ is also measurable. Note that the A_n's may not be disjoint. (Hint:[19])

Exercise 21. Let $A_1, A_2, \ldots.$ be measurable. Prove that $\cap_{n=1}^{\infty} A_n$ is also measurable. (Hint:[20])

Exercise 22. Prove that there doesn't exist a sequence of disjoint Jordan measurable sets $A_1, A_2, \ldots$ such that $\cup_{n=1}^{\infty} A_n$ is also Jordan measurable but $\mu(\cup_{n=1}^{\infty} A_n) \neq \sum_{n=1}^{\infty} \mu(A_n)$. (Hint:[21])

Exercise 23. Prove that there doesn't exist a sequence of non-negative functions $u_n(x)$ such that each $u_n(x)$ is Riemann integrable on $[a, b]$ and $f(x) = \sum_{n=1}^{\infty} u_n(x)$ is also Riemann integrable on $[a, b]$, but

$$\int_a^b f(x)\,\mathrm{d}x \neq \sum_{n=1}^{\infty} \int_a^b u_n(x)\,\mathrm{d}x. \tag{37}$$

(Hint:[22])

PROPOSITION 22. *Open sets are measurable; Closed sets are measurable.*

Proof. Exercise. (Hint:[23]) □

2.3. Non-measurable sets

PROPOSITION 23. (GUISEPPE VITALI, 1905) *There is $A \subset [0, 1)$ that is not measurable.*

Proof. We divide $[0, 1)$ into a countable union of disjoint sets as follows.

First divide $[0, 1)$ into disjoint equivalence classes: x, y are equivalent, denoted $x \sim y$, if and only if $x - y \in \mathbb{Q}$. Now by Axiom of Choice, there is a subset $A \subset [0, 1)$ such that its intersection with each equivalence class consists of exactly one number.

Now denote $\mathbb{Q} \cap (0, 1)$ by $\{r_1, r_2, \ldots\}$. Define the set

$$A_i := \{a + r_i \bmod 1 \mid a \in A\}. \tag{38}$$

Denote $A_0 := A$.

We claim i) $[0, 1) = \cup_{i=0}^{\infty} A_i$; ii) $i \neq j \Longrightarrow A_i \cap A_j = \varnothing$.

- Proof of i). Assume the contrary, that is there is $x \in [0, 1)$ such that $x \notin A_i$ for every i. By definitio of A there is $y \sim x$ such that $y \in A$. But then $y - x \in \mathbb{Q}$ which means $x = y + r_i$ mod 1 for some i which means $x \in A_i$. Contradiction.

- Proof of ii). Assume the contrary, there are i, j such that $A_i \cap A_j \neq \varnothing$. Take $x \in A_i \cap A_j$. By definition there are $y, z \in A$ such that

  $$x = y + r_i = z + r_j \bmod 1 \tag{39}$$

  which gives

  $$y - z \in \mathbb{Q}. \tag{40}$$

  This contradicts the definition of A.

19. Set $B_1 = A_1$; $B_2 = A_2 - A_1$; $B_3 = A_3 - (A_1 \cup A_2)$; and so on.
20. $\cap_{n=1}^{\infty} A_n = [\cup_{n=1}^{\infty} A_n^c]^c$.
21. Jordan measurable $\Longrightarrow$ measurable.
22. $\int_a^b u_n(x)\,\mathrm{d}x = \mu(\{(x, y) \mid x \in [a, b], 0 \leqslant y < u_n(x)\})$.
23. Any open set is the union of countably many open intervals.

Now assume A is measurable. Then we have on one hand

$$\forall i \qquad m(A) = m(A_i) \tag{41}$$

while on the other

$$\sum_{i=0}^{\infty} m(A_i) = m(\cup_{i=0}^{\infty} A_i) = m([0,1)) = 1. \tag{42}$$

It then follows $m(A_i) = 0$ for all i. But then $\sum_{i=0}^{\infty} m(A_i) = 0$. Contradiction. $\square$

Exercise 24. Prove that A is measurable $\Longrightarrow$ every A_i is measurable and furthermore $m(A_i) = m(A)$ for every i.

Exercise 25. The proof can be simplified a bit: Define $A_i := A + r_i$ for every $r_i \in \mathbb{Q} \cap [-1, 1]$. Prove that $[0,1] \subseteq \cup_{i=1}^{\infty} A_i \subseteq [-1,2]$ and obtain contradiction.

Remark 24. In 1970, Robert M. Solovay (1938 –)[24] proved that if we replace Axiom of Choice by Axiom of Dependent Choice[25] and assume the existence of certain large cardinal[26], then the statement "Every $A \subseteq \mathbb{R}$ is measurable" is consistent with the Zermelo-Fraenkel set theory. Thus we see that there is no hope giving an explicit construction – even a inductive one – of a set A that is not measurable. See (CIESIELSKI) for further discussions on this topic.

In particular, regarding (12), we see that if all of A_i's are measurable, then necessarily $\cup A_i$ is also measurable and $m^*(\cup A_i) = \sum m^*(A_i)$ must hold. For the inequality to be strict, at least one of the A_i's must be non-measurable.

Exercise 26. Let $A \subset [0,1]$ be a non-measurable set. Then

$$m^*(A) + m^*(A^c) > m^*(A \cup A^c) = m^*[0,1] = 1. \tag{43}$$

(Hint:[27])

Exercise 27. Is it possible to find a sequence of disjoint sets $\{A_n\}$ such that $m^*(\cup_{n=1}^{\infty} A_n) < \sum_{n=1}^{\infty} m^*(A_n)$ but for every $N \in \mathbb{N}$ we have $m^*(\cup_{n=1}^{N} A_n) = \sum_{n=1}^{N} m^*(A_n)$? (Hint:[28])

Remark 25. (THE MEASURE PROBLEM) It is clear that Vitali's construction forbids any countably additive and translation-invariant measure to have the whole $\mathcal{P}(\mathbb{R})$ as its measurable sets.

The "Measure Problem" asks the following question: What if we drop the requirement of translation invariance? Does there now exist a measure that can measure all subsets of $\mathbb{R}$?

It turns out that, if such a measure exists, then the cardinality of $\mathbb{R}$ must be extremely large: $\mathfrak{c} > \aleph_n$ for any $n \in \mathbb{N}$.

Stanislav Ulam (1919 – 1984) proved in 1930 that if the answer is negative, then there is a uncountable cardinal κ such that there exists a nontrivial $\{0,1\}$ valued κ-additive measure. Such a κ is called "measurable". Existence of measurable cardinals cannot be proved or disproved in ZFC.

24. *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math., 92 (1970), 1 – 56.

25. Which enables us to define infinitely many objects inductively, such as "take x_1, then take x_2, and so on ..."

26. In 1984 Saharon Shelah (1945 –) showed (*Can you take Solovay's inaccessible away?*, Israel J. Math. 48(1), 1984) that this assumption cannot be dropped, in the sense that ZF + Axiom of Dependent Choice + All sets are measurable implies ZF + existence of such large cardinals.

27. Definition of measurability.

28. If $m^*(\cup_{n=1}^{N} A_n) = \sum_{n=1}^{N} m^*(A_n)$, then necessarily $m^*(\cup_{n=1}^{\infty} A_n) \geqslant \sum_{n=1}^{\infty} m^*(A_n)$.

3. STRUCTURE OF MEASURABLE SETS

NOTATION. *"measurable" means "Lebesgue measurable" in this section.*

3.1. G_δ and F_σ

Now we try to get some concrete idea of what measurable sets look like. First recall that for Jordan measurable sets we have the following result on their structures:

THEOREM 26. *A set $A \subseteq [0,1]$ is Jordan measurable, then $A = B \cup C$ where B is open and C has Jordan measure 0; Also $A = D - E$ where D is closed and E has Jordan measure 0.*

Proof. Exercise. (Hint:[29]) □

Thus a Jordan measurable set is almost open or closed. For measurable sets we have the following understanding:

> $A \subseteq [0,1]$ is measurable if and only if A is almost the result of countably many set operations (intersection, union, complement) on open (or closed) sets.

DEFINITION 27. (G_δ AND F_σ SETS) *A set is called G_δ if it is the intersection of countably many open sets; A set is called F_σ if it is the union of countably many closed sets.*

Exercise 28. Prove that every closed set is G_δ and every open set is F_σ. (Hint:[30])

Exercise 29. Prove that A is G_δ if and only if A^c is F_σ.

Exercise 30. Here δ comes from the German word "Durchschnitt", meaning intersection; σ comes from the French word "somme", meaning sum.

Now explain: Why don't we consider G_σ and F_δ sets? (Hint:[31])

Exercise 31. Explain why we didn't encounter G_δ and F_σ sets when discussing Jordan measure. (Hint:[32]).

PROPOSITION 28. *G_δ and F_σ sets are measurable.*

Proof. Exercise. (Hint:[33]) □

THEOREM 29. *Let $A \subseteq [0,1]$ be measurable. Then there are a G_δ set $B \supseteq A$ and a F_σ set $C \subseteq A$ such that $m^*(B - A) = m^*(A - C) = 0$.*

Proof. By definition of outer measure there is a countable sequence of open sets $B_n \supseteq A$ such that $m^*(B_n) \searrow m^*(A)$. Take $B = \cap_{n=1}^{\infty} B_n \supseteq A$. Then $m^*(B) \leqslant m^*(A) \Longrightarrow m^*(B) = m^*(A)$. Since A is measurable, $m^*(A) + m^*(B - A) = m^*(A \cap B) + m^*(A^c \cap B) = m^*(B) = m^*(A)$ which gives $m^*(B - A) = 0$. The proof of the other half is left as exercise. □

29. $A = A^o \cup (A - A^o)$. Note that $A - A^o \subseteq \partial A$.
30. Let A be closed. Set $A_n := \{x | \operatorname{dist}(x, A) < 1/n\}$.
31. Union of open sets is open; Intersection of closed sets is closed.
32. Finite intersection of open sets is still open.
33. Theorem 21.

Remark 30. From the above proof we see that for any set A, not necessarily measurable, there are G_δ and F_σ sets $B \supseteq A, C \subseteq A$ such that $m^*(B) = m^*(A), m_*(C) = m_*(A)$. However only when A is measurable can we conclude from this that $m^*(B - A) = m^*(A - C) = 0$.

3.2. Borel sets

3.2.1. Borel sets

Exercise 32. Let $f: \mathbb{R} \mapsto \mathbb{R}$. Prove that the set of points where f is continuous is G_δ. (Hint:[34])

Borel sets are generalizations of G_δ and F_σ sets. Put it simply, Borel sets are the results of application of countably many times union, intersection, complement on open sets or closed sets.

DEFINITION 31. (σ-ALGEBRA) *A σ-algebra on $[0,1]$ is a subset $\mathcal{X}$ of $\mathcal{P}([0,1])$ satisfying:*

i. $A \in \mathcal{X} \Longrightarrow A^c \in \mathcal{X}$;

ii. $A_1, A_2, \ldots \in \mathcal{X} \Longrightarrow \cup_{n=1}^\infty A_n, \cap_{n=1}^\infty A_n \in \mathcal{X}$.

Exercise 33. Show that we can drop either $\cup_{n=1}^\infty A_n$ or $\cap_{n=1}^\infty A_n$ in the definition. (Hint:[35])

DEFINITION 32. (BOREL SET) *The set of Borel sets, $\mathcal{B}$, is the smallest σ-algebra on $[0,1]$ containing all open sets.*

THEOREM 33. *$\mathcal{B}$ exists.*

Proof. Clearly $\mathcal{P}([0,1])$ is a σ-algebra, that is the set W of all σ-algebras on $[0,1]$ is not empty. Now take

$$\mathcal{B} := \cap_{\mathcal{X} \in W} \mathcal{X}. \tag{44}$$

We can prove straightforwardly that $\mathcal{B}$ is also a σ-algebra. □

Exercise 34. Prove that $\mathcal{B}$ is the smallest σ-algebra containing all open intervals.

Exercise 35. Prove that $\mathcal{B}$ is the smallest σ-algebra containing all closed sets.

Taking the subscript σ to mean the operation of taking countable union and δ to mean the operation of taking countable intersection, we can form two sequences:

$$F_\sigma, F_{\sigma\delta}, F_{\sigma\delta\sigma}, \ldots.; G_\delta, G_{\delta\sigma}, G_{\delta\sigma\delta}, \ldots. \tag{45}$$

Exercise 36. Let $\{f_n\}$ be a sequence of continuous functions. Let $C := \{x \in \mathbb{R} | f_n(x) \text{ converges}\}$. Prove that C is $F_{\sigma\delta}$. (Hint:[36])

3.2.2. Borel hierarchy

We can construct $\mathcal{B}$ as follows.

DEFINITION 34. *We define*

$$F_0 := \{\textit{closed sets}\}, \qquad F_1 := F_\sigma, \qquad F_2 := F_{\sigma\delta}, \ldots. \tag{46}$$

34. Intersection of $\omega(f) < 1/n$.
35. $\cap A_n = (\cup A_n^c)^c$.
36. Following Cauchy criterion, $C = \cap_{n=1}^\infty [\cup_{m=1}^\infty [\cap_{k=m}^\infty \{x | |f_k(x) - f_m(x)| \leqslant 1/n\}]]$.

iteratively so that for any ordinal number α, if it is even[37] *then*

$$F_\alpha := \{\cap_{n=1}^\infty A_n |\, A_n \in F_{\alpha_n}, \alpha_n < \alpha\}; \tag{47}$$

If α is odd, then

$$F_\alpha := \{\cup_{n=1}^\infty A_n |\, A_n \in F_{\alpha_n}, \alpha_n < \alpha\}. \tag{48}$$

We can similarly define $G_0, G_1, \ldots$.

Theorem 35. *$\mathcal{B} = \cup_{\alpha<\omega_1} F_\alpha = \cup_{\alpha<\omega_1} G_\alpha$. Here ω_1 is the first uncountable ordinal.*

Exercise 37. Prove that

$$\omega_1 = \{\alpha |\, \alpha \text{ is a countable ordinal}\}. \tag{49}$$

Alternatively, we can define the hierarchy as follows.

Definition 36. (Borel hierarchy) *We define $\Sigma_\alpha^0, \Pi_\alpha^0, \Delta_\alpha^0$ iteratively for all ordinal numbers α as follows.*

- $\Sigma_1^0 := \{$*open sets*$\}$;
- $\Pi_\alpha^0 := \{A^c |\, A \in \Sigma_\alpha^0\}$;
- $\Sigma_\alpha^0 := \{A |\, A = \cup_{n=1}^\infty A_n, A_n \in \Pi_{\alpha_n}^0$ *with* $\alpha_n < \alpha\}$;
- $\Delta_\alpha^0 := \Sigma_\alpha^0 \cap \Pi_\alpha^0$.

Exercise 38. Prove that

$$F_\sigma = \Sigma_2^0; \qquad G_\delta = \Pi_2^0. \tag{50}$$

Theorem 37. *$\mathcal{B} = \cup_{\alpha<\omega_1} \Sigma_\alpha^0 = \cup_{\alpha<\omega_1} \Pi_\alpha^0 = \cup_{\alpha<\omega_1} \Delta_\alpha^0$.*

3.2.3. Measurability of Borel sets

Theorem 38. *Borel sets are measurable.*

Proof. Exercise. (Hint:[38]) □

Theorem 39. *The cardinality of $\mathcal{B}$ is $\mathfrak{c}$, that is $\mathcal{B} \sim \mathbb{R}$.*

Proof. We can prove that for each fixed $\alpha < \omega_1$, $\Sigma_\alpha^0 \sim \mathbb{R}$. On the other hand we know that the cardinality of ω_1 is $\aleph_1 \leqslant 2^{\aleph_0} = \mathfrak{c}$. Therefore the cardinality of $\mathcal{B}$ is no more than $\aleph_1 \cdot \mathfrak{c} = \mathfrak{c}$. On the other hand clearly $\mathcal{B} \gtrsim \mathbb{R}$. The conclusion now follows from Schroeder-Bernstein. □

Corollary 40. *There are measurable but non-Borel sets.*

Proof. Because the Cantor set C has zero Jordan measure and therefore has zero measure. Thus all of its subsets are measurable. Therefore

$$\{\text{Measurabe sets}\} \gtrsim \mathcal{P}(C) \sim \mathcal{P}(\mathbb{R}) > \mathbb{R} \sim \mathcal{B}. \tag{51}$$

Therefore $\{\text{Measurable sets}\} - \mathcal{B} \neq \varnothing$. □

37. If α is a limit ordinal, treat it as odd.

38. Use Definition 34 or 36.

Remark 41. In constrast to non-measurable sets, the explicit construction of Borel but measurable sets is difficult but not impossible. The following is given by Nikolai Nikolaevich Luzin (1883 - 1950) in 1927:

$$A = \left\{ x \in \mathbb{Q}^c | \, x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}, a_0 \in \mathbb{Z}, a_n \in \mathbb{N}, n \geqslant 1 \right\} \tag{52}$$

such that there is a subsequence n_k satisfying

$$a_{n_1} | \, a_{n_2} | \, a_{n_3} | \cdots \tag{53}$$

Here $a | \, b$ means a is a factor of b (as in elementary number theory).

Remark 42. In 1905, Lebesgue claimed in a paper to have proved the existence of measurable non-Borel sets. Later in 1915 the article was assigned to Mikhail Yakovlevich Suslin (1894 - 1919)[39] by N. N. Luzin. Suslin soon discovered that Lebesgue had made a mistake. In particular, Lebesgue thought that the image of Borel sets through a continuous function – called "analytic" later by Luzin – is still Borel. In 1917, Luzin proved that analytic sets are measurable. The A in Remark 41 is analytic.

Remark 43. The complements for analytic sets are called co-analytic sets.

Exercise 39. Given that analytic sets are measurable, prove that co-analytic sets are also measurable.

Now naturally one would ask, what about sets of the form $f(A)$ where f is continuous and A is co-analytic? Are they all measurable?

Exercise 40. Why didn't anybody study the measurability of $f(A)$ where f is continuous and A is analytic? (Hint:[40])

Surprisingly the answer is that this is undecidable in ZFC. More specifically, the answer is negative in L, the "Goedel Universe", while positive in a different model of ZFC.

In 1969 Solovay proved that if one assumes the existence of measurable cardinals (See Remark 25), then the answer is positive.

Remark 44. There is a surprising relation between analytic sets (see Remark 42) and summability of infinite series (see Week 11's notes).

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. Its *Riemann convergence set* $\mathcal{R}(\sum_{n=1}^{\infty} a_n)$ is defined to be the set of all sums of convergent re-arrangements of the series. We have seen in Week 1 that $\mathcal{R}(\sum_{n=1}^{\infty} a_n)$ is either empty (for example when $\lim_{n \rightarrow \infty} a_n = 0$ does not hold), or a single number (when $\sum a_n$ is absolutely convergent), or the whole $\mathbb{R}$. In other words, we have

$$\cup_{\text{All } \sum_{n=1}^{\infty} a_n} \mathcal{R}\left(\sum_{n=1}^{\infty} a_n \right) = \{\varnothing\} \cup \{\{s\} | \, s \in \mathbb{R}\} \cup \mathbb{R}. \tag{54}$$

Now consider any summation method C. We can define

$$\mathcal{R}\left(C, \sum_{n=1}^{\infty} a_n \right) := \{s \in \mathbb{R} | \, \text{There is a rearrangement summable to } s \text{ with method } C\}. \tag{55}$$

39. He had health problems and Russia had food problems. Mathematics was only the first stage of Suslin's grand life plan. It was to be followed by research in physics, chemistry, biology, and finally medicine. See http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Suslin.html.

40. f, g continuous $\Longrightarrow f \circ g$ continuous.

It is proved in 1958[41] that when we consider all possible summation methods, then the corresponding $\mathcal{R}(C, \sum_{n=1}^{\infty} a_n)$ coincide with the analytics sets of $\mathbb{R}$, that is

$$\cup_{\text{All summation methods}}\left[\cup_{\text{All }\sum_{n=1}^{\infty} a_n}\mathcal{R}\left(C, \sum_{n=1}^{\infty} a_n\right)\right] = \{A \subseteq \mathbb{R} | A \text{ is analytic}\}. \tag{56}$$

Exercise 41. Prove that if $a_n \to 0$ does not hold, then $\mathcal{R}(\sum_{n=1}^{\infty} a_n) = \varnothing$, that is no matter how we re-arrange the series, the resulting series is not convergent. (Hint:[42])

Exercise 42. Find an infinite series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} a_n = 0$ but $\mathcal{R}(\sum_{n=1}^{\infty} a_n) = \varnothing$. (Hint:[43])

3.3. Littlewood's three principles

Littlewood's Three Principles of Real Analysis.

In his 1944 *Lectures on the Theory of Functions*, John Edensor Littlewood stated the following "three principles of real analysis".

i. Every measurable set is nearly a finite sum of intervals;
ii. Every measurable function is nearly continuous;
iii. Every convergent sequence of measurable functions is nearly uniformly convergent.

Here "nearly" means the claim holds except on a set of (arbitrarily) small Lebesgue measure. These three principles make it easy to understand some important results in real analysis. For example Lebesgue's Dominated Convergence Theorem, which roughly says, if $f_n \longrightarrow f$ and there is $g > 0$ integrable, such that $|f_n| \leqslant g$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x. \tag{57}$$

To see why this should be true, we apply Principle iii: $f_n \to f$ uniformly except on a set E whose measure is as small as we want. But on this set we have

$$\left|\int_E f_n(x)\,\mathrm{d}x\right| \leqslant \left|\int_E g(x)\,\mathrm{d}x\right| \tag{58}$$

which can be made arbitrarily small.

Remark 45. In Principles ii and iii we restrict ourselves to "measurable functions". The reason is that we need to guarantee that all the sets involved, in particular those of the form $\{x \in \mathbb{R}| a < (\leqslant) f(x) < (\leqslant) b\}$, are measurable.

41. Lorentz, G. G. and Zeller, K., *Series rearrangements and analytic sets*, Acta Math., 100 (1958), pp. 149 – 169.

42. Prove that no matter how we re-arrange, $a_{n_k} \to 0$ still fails.

43. $\sum 1/n$.

4. Advanced Topics, Notes, and Comments

4.1. Proof of Theorem 19.

Proof. "If" is obvious as we can simply take $E=[0,1]$. In the following we prove the "only if" part.

We will prove that, if $A\subseteq[0,1]$ satisfies $m^*(A)+m^*(A^c)=1$, then it satisfies

$$m^*(A\cap E)+m^*(A^c\cap E)=m^*(E) \tag{59}$$

for every $E\subseteq[0,1]$.

From now we we assume A satisfies $m^*(A)+m^*(A^c)=1$.

- (59) holds when $E=(a,b)$ is an open interval.

  All we need to show is $m^*(A\cap(a,b))+m^*(A^c\cap(a,b))\leqslant b-a$. Let $\varepsilon>0$ be arbitrary. We take disjoint open intervals $(c_{1i},d_{1i}),(c_{2i},d_{2i})$ such that

  $$A\subseteq\cup_{i=1}^{\infty}(c_{1i},d_{1i}),\qquad A^c\subseteq\cup_{i=1}^{\infty}(c_{2i},d_{2i}) \tag{60}$$

  and furthermore $m^*(A)\leqslant\sum(d_{1i}-c_{1i})\leqslant m^*(A)+\varepsilon$, $m^*(A^c)\leqslant\sum(d_{2i}-c_{2i})\leqslant m^*(A^c)+\varepsilon$. Now as $\sum(d_{1i}-c_{1i})$ and $\sum(d_{2i}-c_{2i})$ are convergent, there is $N\in\mathbb{N}$ such that

  $$\sum_{i>N}(d_{1i}-c_{1i}),\sum_{i>N}(d_{2i}-c_{2i})<\varepsilon. \tag{61}$$

  Further as $[0,1]=A\cup A^c\subseteq\cup_{i=1}^{\infty}[(c_{1i},d_{1i})\cup(c_{2i},d_{2i})]$, there is a finite subcovering. Thus we can take N such that not only (61) holds, but also $[0,1]\subseteq\cup_{i=1}^{N}[(c_{1i},d_{1i})\cup(c_{2i},d_{2i})]$.

  Denote $B_N:=\cup_{i=1}^{N}(c_{1i},d_{1i})$; $C_N:=\cup_{i=1}^{N}(c_{2i},d_{2i})$. Then we have

  $$m^*(A-B_N),m^*(A^c-C_N)<\varepsilon\text{ and }m^*(B_N\cap C_N)=\mu(B_N\cap C_N)<2\,\varepsilon \tag{62}$$

  where the last relation comes from

  $$\mu(B_N\cap C_N)+\mu(B_N\cup C_N)=\mu(B_N)+\mu(C_N)\leqslant m^*(A)+m^*(A^c)+2\,\varepsilon. \tag{63}$$

Now we have

$$\begin{aligned} m^*(A\cap E)+m^*(A^c\cap E) &\leqslant m^*(B_N\cap E)+m^*(A-B_N) \\ &\quad +m^*(C_N\cap E)+m^*(A^c-C_N) \\ &\leqslant m^*(B_N\cap E)+m^*([C_N-B_N]\cap E)+m^*(C_N\cap B_N)+2\,\varepsilon \\ &\leqslant b-a+4\,\varepsilon. \end{aligned} \tag{64}$$

Here $m^*(B_N\cap E)+m^*([C_N-B_N]\cap E)=m^*(E)=b-a$ because all the sets involved consist of finitely many disjoint intervals.

As $\varepsilon>0$ is arbitrary, it follows that $m^*(A\cap(a,b))+m^*(A^c\cap(a,b))\leqslant b-a$.

- (59) holds when $E=\cup_{i=1}^n(a_i,b_i)$ where the open intervals are disjoint.

  It is a straightforward adaptation of the previous case. Left as exercise.

- (59) holds for every $E\subseteq[0,1]$.

  By the same argument as in the first case, for every $\varepsilon>0$ there is an open set E_ε which is a finite unions of disjoint open intervals such that

$$m^*(E-E_\varepsilon)<\varepsilon; \qquad m^*(E_\varepsilon)<m^*(E)+\varepsilon. \tag{65}$$

Now we have

$$\begin{aligned} m^*(A\cap E)+m^*(A^c\cap E) &\leqslant m^*(A\cap E_\varepsilon)+m^*(E-E_\varepsilon) \\ &\quad +m^*(A^c\cap E_\varepsilon)+m^*(E-E_\varepsilon) \\ &= m^*(E_\varepsilon)+2\,\varepsilon \\ &\leqslant m^*(E)+3\,\varepsilon. \end{aligned} \tag{66}$$

The conclusion now follows from the arbitrariness of ε. □

Remark 46. Note that (65) cannot be replaced by the stronger requirement $m^*(E\triangle E_\varepsilon)<2\,\varepsilon$ where the symmetric difference $E\triangle E_\varepsilon:=(E-E_\varepsilon)\cup(E_\varepsilon-E)$. The problem is that it may not be possible to make $m^*(E_\varepsilon-E)$ small as E could be non-measurable. More specifically, if $E_\varepsilon-E$ could be made arbitrarily small also, then following $E^c-E_\varepsilon^c=E_\varepsilon-E$ we would have $m^*(E^c-E_\varepsilon^c)<\varepsilon$ which gives

$$m^*(E)+m^*(E^c)\leqslant m^*(E_\varepsilon)+m^*(E-E_\varepsilon)+m^*(E_\varepsilon^c)+m^*(E^c-E_\varepsilon^c)<1+2\,\varepsilon. \tag{67}$$

The arbitrariness of ε then gives $m^*(E)+m^*(E^c)=1$ from which the measurability of E follows.

5. Exercises and Problems

5.1. Basic exercises

Exercise 43. (Bear) Let $T \subseteq [0,1]$ be any set. Let $E_n, n = 1, 2, 3, \ldots$ be measurable. Prove that

$$m^*(T \cap \cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(T \cap E_n). \tag{68}$$

(Hint:[44])

5.2. More exercises

5.3. Problems

Problem 1. When Riemann was minus three years old, Cauchy proposed the following criterion for "integrability":

Let $f: [a, b] \mapsto \mathbb{R}$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$. Define the sum:

$$\Sigma(f, P) := \sum_{i=1}^{n} f(x_{i-1})\,(x_i - x_{i-1}). \tag{69}$$

Say f is integrable if and only if

$$\lim_{d(P) \longrightarrow 0} \Sigma(f, P) \text{ exists and is finite,} \tag{70}$$

where $d(P) := \max_{i \in \{0,1,\ldots,n-1\}} (x_i - x_{i-1})$.

Prove or disprove: The above integrability is equivalent to Riemann integrability. (Hint:[45])

Problem 2. Prove or disprove:

$A \subseteq \mathbb{R}^N$ is Jordan measurable if and only if for every $E \subseteq \mathbb{R}^N$, $\mu_{\text{out}}(E) = \mu_{\text{out}}(A \cap E) + \mu_{\text{out}}(A^c \cap E)$.

Problem 3. Let $A \subset [0,1]$ be such that $m^*(A) > 0$. Prove or disprove: $A \sim \mathbb{R}$. Also discuss the relation between this problem and Continuum Hypothesis.

Problem 4. What are the outer and inner measures of Vitali's non-measurable set? Justify.

Problem 5. What is the cardinality of Vitali's non-measurable set? (Hint:[46])

44. Prove LHS $\geqslant \sum_1^N m^*(T \cap E_n)$ for any $N \in \mathbb{N}$.

45. If f is Riemann integrable obviously it is integrable according to Cauchy's definition. On the other hand, wlog let $[a, b] = [0, 1]$. assume f is not Riemann integrable. Then there is $\varepsilon > 0$ such that $\mu_{\text{out}}(\{\omega_f > \varepsilon\}) = \delta > 0$. Now modify the partition $P = \left\{0 < \frac{1}{n} < \cdots < \frac{n-1}{n} < 1\right\}$ to find two partitions P_1, P_2 such that $d(P_i) < \frac{2}{n}$ but $|\Sigma(f, P_1) - \Sigma(f, P_2)| > \frac{\varepsilon\,\delta}{2}$.

46. $\aleph_0 \cdot \mathfrak{x} = 2^{\aleph_0}$. If $\mathfrak{x} \leqslant \aleph_0$ then the LHS $\leqslant \aleph_0$. Therefore $\mathfrak{x} > \aleph_0$ but then $\aleph_0 \cdot \mathfrak{x} = \mathfrak{x}$. Therefore $\mathfrak{x} = 2^{\aleph_0} = \mathfrak{c}$.