Math 317 Week 11: Divergent Series

March 21, 2014

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1. Two Motivating Examples

"... the theory of divergent series is another striking example of the way in which mathematics has grown. ... when a concept or technique proves to be useful even though the logic of it is confused or even nonexistent, persistent research will uncover a logical justification,..."

— Morris Kline[1]

1.1. Euler, 1754.

In 1754[2], Leonhard Euler studied the following ODE

$$x^2\, y' + y = x, \qquad y(0) = 0. \tag{1}$$

Solving it by the method of power series[3], Euler obtained

$$y(x) \sim x - (1!)\, x^2 + (2!)\, x^3 - (3!)\, x^4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)!\, x^n \tag{2}$$

Exercise 1. (If you know some ODE) Derive (2).

Exercise 2. What is the radius of convergence for the series in (2)? (Ans:[4])

Then he used the following fact.

$$(n-1)! = \Gamma(n) = \int_0^\infty t^{n-1}\, e^{-t}\, \mathrm{d}t. \tag{3}$$

Exercise 3. Prove (3).

Substituting (3) into (2), he argued

$$\begin{aligned} y(x) &\sim \sum_{n=1}^{\infty} (-1)^{n-1}\, x^n \int_0^\infty t^{n-1}\, e^{-t}\, \mathrm{d}t \\ &= \sum_{n=1}^{\infty} x \int_0^\infty (-x\, t)^{n-1}\, e^{-t}\, \mathrm{d}t \\ &\sim x \int_0^\infty \left[\sum_{n=1}^{\infty} (-x\, t)^{n-1} \right] e^{-t}\, \mathrm{d}t \\ &\sim x \int_0^\infty \frac{1}{1 + x\, t}\, e^{-t}\, \mathrm{d}t \\ &= \int_0^\infty \frac{x}{1 + x\, t}\, e^{-t}\, \mathrm{d}t. \end{aligned} \tag{4}$$

Exercise 4. Why are $\sim$ used instead of $=$ in some steps above?

1. in *Mathematical Thoughts from Ancient to Modern Times*, Oxford University Press, 1990. p. 1120 (Vol. 3).
2. Novi Comm. Acad. Sci. Petrop., 5, 1754/5, 205-37, pub. 1760.
3. Note that $x = 0$ is a irregular singular point for the equation, therefore even the theory of Frobenius does not apply.
4. 0.

Thus Euler obtained a formula for the solution

$$y(x) = \int_0^\infty \frac{x}{1+x\,t}\, e^{-t}\, \mathrm{d}t. \tag{5}$$

The function defined in (5) turns out to be well-defined and even smooth in $(0,\infty)$. The question now arises: Does $y(x)$ in (5) solve (1)?

Surprisingly it does! We calculate:

$$y'(x) = \int_0^\infty \frac{\partial}{\partial x}\left(\frac{x}{1+x\,t}\right) e^{-t}\, \mathrm{d}t = \int_0^\infty \frac{e^{-t}\, \mathrm{d}t}{(1+x\,t)^2}. \tag{6}$$

On the other hand, integration by parts gives

$$y(x) = x \int_0^\infty \frac{e^{-t}\, \mathrm{d}t}{1+x\,t} = x\left[1 + \int_0^\infty \frac{x\, e^{-t}}{(1+x\,t)^2}\, \mathrm{d}t\right] = x - x^2 \int \frac{e^{-t}\, \mathrm{d}t}{(1+x\,t)^2}. \tag{7}$$

Therefore

$$x^2\, y' + y = x. \tag{8}$$

Exercise 5. (IF YOU KNOW SOME ODE) Solve (1) to obtain

$$y(x) = e^{1/x} \int_{1/x}^\infty \frac{e^{-t}}{t}\, \mathrm{d}t. \tag{9}$$

Note that $y(x)$ is smooth in $(0,\infty)$. (Hint:[5])

Exercise 6. Prove that for every $x > 0$,

$$\int_0^\infty \frac{x}{1+x\,t}\, e^{-t}\, \mathrm{d}t = e^{1/x} \int_{1/x}^\infty \frac{e^{-t}}{t}\, \mathrm{d}t. \tag{10}$$

Thus (9) and (5) represent the same function. (Hint:[6])

1.2. Laguerre, 1879.

In 1879[7], Edmond Laguerre (1834 - 1886) "solved" the equation[8]

$$x^2\, y' + (x-1)\, y = -1. \tag{11}$$

Exercise 7. Show that the natural requirement is $y(0) = 1$.

Using the method of power series, he obtained the solutions as

$$y(x) \sim 1 + x + (2!)\, x^2 + (3!)\, x^3 + \cdots = \sum_{n=0}^\infty (n!)\, x^n. \tag{12}$$

5. It is a first order linear equation.
6. Change of variable $s = t - 1/x$ in the RHS integral.
7. Bull. Soc. Math. de France, 7, 1879, 72 - 81.
8. In (KLINE) y' is mistakenly printed as $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

Similar to what Euler did, he concluded using (3) that

$$y(x) = \int_0^\infty \frac{e^{-t}}{1 - x\,t} \tag{13}$$

which is well-defined, in fact smooth, for all $x < 0$.[9]

Exercise 8. Prove directly that (13) solves (11).

Exercise 9. Prove that (13) solves (11) through exploring the relation between (13) and (5).

Exercise 10. Show that (13) solves (11) through exploring the relation between (12) and (2). Note that this is **not a proof** of (13) solving (11), since for divergent series, the relation

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n \tag{14}$$

may not hold.

Exercise 11. Consider the following "Stieltjes series"

$$\sum_{n=0}^{\infty} (-1)^n n!\, x^n. \tag{15}$$

a) Show formally that it satisfies

$$x^2\, y'' + (1 + 3\,x)\, y' + y = 0; \tag{16}$$

b) Show formally that the series sum up to

$$y(x) = \int_0^\infty \frac{e^{-t}}{1 + x\,t}\, \mathrm{d}t; \tag{17}$$

c) Prove that (17) solves (16). (Hint:[10])

What is summability and one of the reasons for studying it.

As we have seen above, formal manipulation of divergent series may yield meaningful "sums" for such series. The theory of summability provides mathematical explanation and foundation to such formal manipulations.

In particular, the reason why the formal operations in this section work is that following:

> The series $\sum_{n=0}^{\infty} (-1)^n\, n!\, x^n$, although does not converge, is "summable" in the sense of Borel. It turns out that Borel summable series enjoy many properties of convergent series, such as addition, subtraction, and termwise differentiation. As a consequence, one could use Borel summable series to solve differential equations.

For an example of application of Borel summability to PDEs, check out the work of Prof. Saleh Tanveer of Ohio State University.

9. More precisely, defined for all $x \in \mathbb{C} - \mathbb{R}^+$.

10. Either notice that (16) is in fact a first order equation in disguise, or write the integrand for $x^2\, y'' + (1 + 3\,x)\, y' + y$ as $\frac{\mathrm{d}}{\mathrm{d}t}(\cdots)$.

2. SUMMABALE SERIES

NOTATION. *In presenting summability results, it is convenient to let the sum start at $n=0$. Therefore in the following we will use* $\sum_{n=0}^{\infty}$.

2.1. Hölder, Cesàro, and Abel summabilities

2.1.1. Hölder summability

THEOREM 1. *Let $\sum_{n=0}^{\infty} a_n$ be a infinite series. Define*

$$s_n := a_0 + \cdots + a_n. \tag{18}$$

Then

$$\sum_{n=0}^{\infty} a_n = s \Longrightarrow \lim_{n\to\infty} \frac{s_0 + \cdots + s_n}{n+1} = s. \tag{19}$$

Proof. We need to prove

$$\lim_{n\to\infty} s_n = s \Longrightarrow \lim_{n\to\infty} \frac{s_0 + \cdots + s_n}{n+1} = s. \tag{20}$$

Let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} s_n = s$ there is $N_1 \in \mathbb{N}$ such that

$$\forall n > N_1, \qquad |s_n - s| < \frac{\varepsilon}{2}. \tag{21}$$

With N_1 fixed, there is $N_2 \in \mathbb{N}$ such that

$$\left| \frac{s_0 + \cdots + s_{N_1} - (N_1+1)\, s}{N_2} \right| < \frac{\varepsilon}{2}. \tag{22}$$

Now set $N := \max\{N_1, N_2\}$. We have, for any $n > N$,

$$\left| \frac{s_0 + \cdots + s_n}{n} - s \right| \leqslant \left| \frac{s_0 + \cdots + s_{N_1} - (N_1+1)\, s}{n} \right| + \left| \frac{(s_{N_1+1} - s) + \cdots + (s_n - s)}{n} \right| < \varepsilon. \tag{23}$$

Thus ends the proof. □

DEFINITION 2. (HÖLDER SUMMABILITY) [11]*Let $\sum_{n=0}^{\infty} a_n$ be a infinite series. Define $s_n := a_0 + \cdots + a_n$ and*

$$s_n^{(0)} := s_n; \tag{24}$$

$$s_n^{(1)} := \frac{s_0^{(0)} + \cdots + s_n^{(0)}}{n+1}; \tag{25}$$

$$s_n^{(2)} := \frac{s_0^{(1)} + \cdots + s_n^{(1)}}{n+1}; \tag{26}$$

$$\vdots$$

Let $k \in \mathbb{N} \cup \{0\}$. If

$$\lim_{n\to\infty} s_n^{(k)} = s \in \mathbb{R}, \tag{27}$$

we say the series $\sum_{n=0}^{\infty} a_n$ is Hölder (H,k) summable. Denoted

$$\sum_{n=0}^{\infty} a_n = s \qquad (H, k) \tag{28}$$

11. Math. Ann. 20 1882 535-549.

Exercise 12. Prove that $(H,0)$ summability is the same as convergence of series.

Example 3. We have

$$\sum_{n=0}^{\infty} (-1)^n = \frac{1}{2} \qquad (H,1). \tag{29}$$

Example 4. The series $\sum_{n=0}^{\infty} n$ is not Hölder summable for any k.

Proof. We have

$$s_n^{(0)} = \frac{n\,(n+1)}{2} \sim \frac{n^2}{2!} \Longrightarrow s_n^{(1)} \sim \frac{n^2}{3!} \Longrightarrow s_n^{(2)} \sim \frac{n^2}{4!} \Longrightarrow s_n^{(3)} \sim \frac{n^2}{5!} \Longrightarrow \cdots \tag{30}$$

Thus

$$s_n^{(k)} = \frac{n^2}{(k+2)!}. \tag{31}$$

Whatever k is, we always have $\lim_{n\to\infty} s_n^{(k)} = \infty$. □

Exercise 13. Prove that if a series is (H,k) summable to s, then this s is unique. (Hint:[12])

Exercise 14. Let $k_2 > k_1$. Prove that if $\sum_{n=0}^{\infty} a_n$ is (H,k_1) summable to s, then it is also (H,k_2) summable with the same sum s. (Hint:[13])

Exercise 15. Find a series that is (H,2) summable but not (H,1) summable. (Hint:[14])

2.1.2. Cesàro summability

DEFINITION 5. (CESÀRO SUMMABILITY) [15]*Let* $\sum_{n=0}^{\infty} a_n$ *be a infinite series. Define* $s_n := a_0 + \cdots + a_n$ *and*

$$S_n^{(k)} := s_n + r\,s_{n-1} + \frac{r\,(r+1)}{2!}\,s_{n-2} + \cdots + \frac{r\,(r+1)\cdots(r+n-1)}{n!}\,s_0; \tag{32}$$

$$D_n^{(k)} := \frac{(r+1)\cdots(r+n)}{n!}. \tag{33}$$

If

$$\lim_{n\to\infty} \frac{S_n^{(k)}}{D_n^{(k)}} = s \in \mathbb{R}, \tag{34}$$

say $\sum_{n=0}^{\infty} a_n$ *is Cesàro* (C,k) *summable to* s*, denoted*

$$\sum_{n=0}^{\infty} a_n = s \qquad (C,k). \tag{35}$$

Exercise 16. Prove that

$$1 + r + \frac{r\,(r+1)}{2!} + \cdots + \frac{r\,(r+1)\cdots(r+n-1)}{n!} = \frac{(r+1)\cdots(r+n)}{n!}. \tag{36}$$

(Hint:[16])

THEOREM 6. (KNOPP-SCHNEE) [17]*An infinite series is* (H,k) *summable to* $s \in \mathbb{R}$ *if and only if it is* (C,k) *summable to* s.

12. Limit is unique.
13. Apply Theorem 1 repeatedly.
14. $(-1)^n n$.
15. Bull. des Sci. Math. (2) 14 1890 114-120.
16. Induction.
17. Konrad Knopp (1882 - 1957) proved $(H,k) \Longrightarrow (C,k)$ in an unpublished dissertation in 1907; Walter Schnee (1885-1958) proved $(C,k) \Longrightarrow (H,k)$ in Math. Ann. 67 1909 110 - 125.

Remark 7. As a consequence, in many textbooks the easier to understand Hölder summability is called Cesàro summability.

Exercise 17. Prove that $(H,1)$ summability is equivalent to $(C,1)$ summability. (Hint:[18])

Remark 8. We notice that it is straightforward to generalize Definition 5 to the case $k \notin \mathbb{Z}$. Thus we can talk about Cesàro summability with fractional order.

2.1.3. Abel summability

First we recall Abel's Theorem

THEOREM 9. (ABEL) *Let* $\sum_{n=0}^{\infty} a_n$ *be a infinite series with* $\limsup_{n\to\infty} |a_n|^{1/n} = 1$. *Then*

$$\sum_{n=0}^{\infty} a_n = s \Longrightarrow \lim_{x\to 1-} \left(\sum_{n=0}^{\infty} a_n x^n \right) = s. \tag{37}$$

Proof. Exercise. (Hint:[19]) □

DEFINITION 10. (ABEL SUMMABILITY) *Let* $\sum_{n=0}^{\infty} a_n$ *be a infinite series with* $\limsup_{n\to\infty} |a_n|^{1/n} = 1$. *Then if*

$$\lim_{x\to 1-} \left(\sum_{n=0}^{\infty} a_n x^n \right) = s, \tag{38}$$

we say the series is Abel summable to s, denoted

$$\sum_{n=0}^{\infty} a_n = s \qquad (A) \tag{39}$$

THEOREM 11. (FROBENIUS) [20]*Let* $\sum_{n=0}^{\infty} a_n$ *be a infinite series with* $\limsup_{n\to\infty} |a_n|^{1/n} = 1$. *Then*

$$\sum_{n=0}^{\infty} a_n = s \quad (H,1) \qquad \Longrightarrow \qquad \sum_{n=0}^{\infty} a_n = s \quad (A). \tag{40}$$

Proof. For any $|x| < 1$, $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent. Therefore we can apply Abel's re-summation trick to obtain:

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n &= s_0 + \sum_{n=1}^{\infty} (s_n - s_{n-1}) x^n \\ &= s_0 + \sum_{n=1}^{\infty} s_n x^n - \sum_{n=1}^{\infty} s_{n-1} x^n \\ &= s_0 - s_0 x + \sum_{n=1}^{\infty} s_n (x^n - x^{n+1}) \\ &= (1-x) \sum_{n=0}^{\infty} s_n x^n. \end{aligned} \tag{41}$$

18. Direct calculation.

19. Apply Abel's Test to $a_n \cdot x^n$ ($\sum a_n$ bounded, x^n decreasing) to show uniform convergence on $[0,1]$.

20. Jour. für Math. 89 1880 262-264.

Applying the same trick one more time, we have

$$\sum_{n=0}^{\infty} a_n x^n = (1-x)^2 \sum_{n=0}^{\infty} T_n x^n \tag{42}$$

where $T_n = s_0 + \cdots + s_n$. Recalling (24) we have $\lim_{n\to\infty} \frac{T_n}{n+1} = s$. Now notice that

$$\forall |x| < 1, \qquad (1-x)^2 \sum_{n=0}^{\infty} (n+1) x^n = 1. \tag{43}$$

We write

$$\sum_{n=0}^{\infty} a_n x^n - s = (1-x)^2 \sum_{n=0}^{\infty} \left(\frac{T_n}{n+1} - s \right) (n+1) x^n. \tag{44}$$

All we need to show is that the RHS $\longrightarrow 0$ as $x \longrightarrow 1$.

Let $\varepsilon > 0$ be arbitrary. Then there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$,

$$\left| \frac{T_n}{n+1} - s \right| < \frac{\varepsilon}{2}; \tag{45}$$

Now take $\delta > 0$ such that

$$\delta^2 \sum_{n=0}^{N_1} \left[\left| \frac{T_n}{n+1} \right| + s \right] (n+1) < \frac{\varepsilon}{2}. \tag{46}$$

Thus for all $x \in (1-\delta, 1)$ we have

$$\left| \sum_{n=0}^{\infty} a_n x^n - s \right| < \varepsilon. \tag{47}$$

Thus ends the proof. □

In fact we have the following more general result.

Theorem 12. *Let* $\sum_{n=0}^{\infty} a_n$ *be a infinite series with* $\limsup_{n\to\infty} |a_n|^{1/n} = 1$. *Assume there is* $k \in \mathbb{N} \cup \{0\}$ *such that the series is* (H, k) *(equivalently* (C, k)*) summable to* $s \in \mathbb{R}$, *then it is Abel summable to* s.

2.2. Beyond Abel summability

It is easy to see that a series cannot be Abel summable if $\limsup_{n\to\infty} |a_n|^{1/n} > 1$. The following are summation methods that can deal with this case.

Definition 13. (Borel Summability) *Let* $\sum_{n=0}^{\infty} a_n$ *be a infinite series. Define*

$$\psi(\lambda) := \int_0^{\lambda} e^{-t} \left[\sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \right] \mathrm{d}t. \tag{48}$$

If

$$\lim_{\lambda\to\infty} \psi(\lambda) = s \in \mathbb{R}, \tag{49}$$

we say the series is Borel summable to s, *denoted*

$$\sum_{n=0}^{\infty} a_n = s \qquad (B). \tag{50}$$

DEFINITION 14. (LEROY SUMMABILITY) *Let $\sum_{n=0}^{\infty} a_n$ be a infinite series. Define using Gamma function*

$$F(t) := \sum_{n=0}^{\infty} \frac{\Gamma(n\,t+1)}{\Gamma(n+1)} a_n. \tag{51}$$

If

$$\lim_{t \to 1} F(t) = s \in \mathbb{R}, \tag{52}$$

we say the series is Leroy summable to s.

Remark 15. Consider the power series $\sum_{n=0}^{\infty} z^n$. To fully understand convergence we have to consider $z \in \mathbb{C}$. Then the usual convergence gives

$$\forall |z| < 1, \qquad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}; \tag{53}$$

If we apply Abel summation, we obtain

$$\forall |z| \leqslant 1, z \neq 1, \qquad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad (A); \tag{54}$$

Application of Borel summation yields

$$\forall z \text{ with } \Re z < 1, \qquad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad (B); \tag{55}$$

Finally if we apply Leroy summation, the conclusion becomes

$$\forall z \in \mathbb{C} - \{z \in \mathbb{R} |\, z \geqslant 1\}, \qquad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}. \tag{56}$$

A more powerful summation theory developed by Mittag-Leffler[21] yields $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for all complex values $z \neq 1$, however the definition involves too much complex analysis so we omit it here. See (MOORE) for more details.

Why do we want the sum to be $1/(1-z)$?

The crucial fact here is that, $\frac{1}{1-z}$ is the unique analytic, that is complex differentiable, function that equals $\sum_{n=0}^{\infty} z^n$ in the disk $|z| < 1$. Therefore a summation theory is reasonable if it can yield $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

Remark 16. Let $\sum_{n=0}^{\infty} a_n z^n = f(z)$ in the sense of convergence in a neighborhood of 0. In general, The classical convergence would recover $\sum_{n=0}^{\infty} a_n z^n = f(z)$ for all $|z| < R$ such that no singular point of f is inside the circle $|z| = R$. On the other hand, Borel summation recovers the equality for all $z \in P$ where P is the polygon enclosed by lines each passing a singular point while at the same time perpendicular to the line connecting this singular point with the origin; Leroy summation recovers the equality for all z except those on the ray extending the line segment connecting the origin to every singular point; Finally Mittag-Leffler summation recovers the equality for all $z \in \mathbb{C}$ except the singular points.

21. Acta Mathematica Vol. 42, 1920.

3. DIVERGENT SERIES THROUGH "SMOOTHED" PARTIAL SUMS

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3.1. Smoothed partial sums.

Recall the definition of series convergence:

$\sum_{n=1}^{\infty} a_n = s$ if and only if $\lim_{N\to\infty} \sum_{n=1}^{N} a_n = s$.

This is equivalent to defining convergence of series through the limit

$$\lim_{N\to\infty} \sum_{n=1}^{\infty} a_n \, \eta\left(\frac{n}{N}\right) \tag{57}$$

where $\eta: [0,\infty) \mapsto \mathbb{R}$ equals the characteristic function of the interval $[0,1]$:

$$\eta(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x > 1 \end{cases}. \tag{58}$$

Now what if we consider other possible η? In particular, we can consider those η that satisfy:

i. $\eta \in C([0,\infty))$;

ii. $\exists R > 0$ such that $\eta(x) = 0$ for all $x > R$;

iii. $\eta(0) = 1$.

DEFINITION 17. (CUTOFF FUNCTION) *We will call any η satisfying the above a "cutoff" function.*

Exercise 18. Find the cutoff function corresponding to $(H,1)$ and $(H,2)$ sums.

PROPOSITION 18. *Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent*[22] *with sum $s \in \mathbb{R}$. Then for any "cutoff" function η,*

$$\lim_{N\to\infty} \sum_{n=1}^{\infty} a_n \, \eta\left(\frac{n}{N}\right) = s. \tag{59}$$

Proof. It is clear that η is bounded, there is there is $M > 0$ such that $|\eta(x)| < M$ for all x. Now for any $\varepsilon > 0$, since η is continuous, there is $\delta \in (0,1)$ such that

$$\forall x \in (0,\delta), \qquad |\eta(x) - 1| < \frac{\varepsilon}{2\,K} \tag{60}$$

where

$$K := \sum_{n=1}^{\infty} |a_n| \tag{61}$$

is finite thanks to the absolute convergence of $\sum a_n$.

22. That is $\sum |a_n|$ is also convergent.

Agains thansk to this absolute convergence, there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$,

$$\sum_{m=n+1}^{\infty} |a_n| < \frac{\varepsilon}{2\,(M+1)}. \tag{62}$$

Now take $N > N_1/\delta$. For any $n > N$, we have $\frac{n}{N} < \delta$ and consequently

$$\begin{aligned} \left| \left[\sum_{n=1}^{\infty} a_n\, \eta\Big(\frac{n}{N}\Big) \right] - s \right| &\leqslant \left| \sum_{n=1}^{N} a_n\, \eta\Big(\frac{n}{N}\Big) - \sum_{n=1}^{N} a_n \right| + \left| \sum_{n>N} a_n\, \eta\Big(\frac{n}{N}\Big) - \sum_{n>N} a_n \right| \\ &\leqslant \left| \sum_{n=1}^{N} a_n \Big(1 - \eta\Big(\frac{n}{N}\Big)\Big) \right| + \sum_{n>N} |a_n| \left[\left| \eta\Big(\frac{n}{N}\Big) \right| + 1 \right] \\ &\leqslant \frac{\varepsilon}{2\,K} K + (M+1)\, \frac{\varepsilon}{2\,(M+1)} \\ &= \varepsilon. \end{aligned} \tag{63}$$

The proposition is thus proved. □

Now we consider the situation where $\sum_{n=1}^{\infty} a_n$ is only conditionally convergent.

PROPOSITION 19. *Let $\sum_{n=1}^{\infty} a_n = s$ and let η be a cutoff function. Further assume that $\eta \in C^1$. Then*

$$\lim_{N \to \infty} \sum_{n=1}^{\infty} a_n\, \eta\Big(\frac{n}{N}\Big) = s. \tag{64}$$

Proof. Application of Abel's re-summation trick (note that for each fixed N, $\sum_{n=1}^{\infty} a_n\, \eta\big(\frac{n}{N}\big)$ only has finitely many nonzero terms. In other words for each fixed N we are dealing with a finite sum) gives

$$\left[\sum_{n=1}^{\infty} a_n\, \eta\Big(\frac{n}{N}\Big) \right] - s = \left\{ \sum_{n=1}^{\infty} s_n \left[\eta\Big(\frac{n}{N}\Big) - \eta\Big(\frac{n+1}{N}\Big) \right] \right\} - s. \tag{65}$$

Since $\eta(0) = 1$, we have

$$s = \sum_{n=0}^{\infty} s \left[\eta\Big(\frac{n}{N}\Big) - \eta\Big(\frac{n+1}{N}\Big) \right]. \tag{66}$$

Consequently

$$\left[\sum_{n=1}^{\infty} a_n\, \eta\Big(\frac{n}{N}\Big) \right] - s = s\left(\eta(0) - \eta\Big(\frac{1}{N}\Big) \right) + \sum_{n=1}^{\infty} (s_n - s) \left[\eta\Big(\frac{n}{N}\Big) - \eta\Big(\frac{n+1}{N}\Big) \right]. \tag{67}$$

Clearly $\lim_{N \to \infty} s\left(\eta(0) - \eta\big(\frac{1}{N}\big) \right) = 0$. For the second term, we split

$$\sum_{n=1}^{\infty} (s_n - s) \left[\eta\Big(\frac{n}{N}\Big) - \eta\Big(\frac{n+1}{N}\Big) \right] = \sum_{n \leqslant \sqrt{N}} + \sum_{n > \sqrt{N}}. \tag{68}$$

Denoting $M := \max\{[\max_{n \in \mathbb{N}} |s_n| + |s|], \max_{x \in \mathbb{R}} |\eta'(x)|\}$, we have

$$\left| \sum_{n \leqslant \sqrt{N}} \right| \leqslant M \int_0^{N^{-1/2}} |\eta'(x)|\, \mathrm{d}x \leqslant M^2\, N^{-1/2}; \tag{69}$$

On the other hand, recalling that $\eta\left(\frac{n}{N}\right)=0$ for all $n>R\,N$, we have, though application of MVT,

$$\left|\sum_{n>\sqrt{N}}\right|\leqslant \sup_{n>\sqrt{N}}|s_n-s|\sum_{\sqrt{N}<n<RN} M\left(\frac{n+1}{N}-\frac{n}{N}\right)\leqslant\left(\sup_{n>\sqrt{N}}|s_n-s|\right)R\,M. \tag{70}$$

We see that both $\longrightarrow 0$ as $N\longrightarrow\infty$. □

Problem 1. Can we drop the condition $\eta\in C^1$? If not can we find a counter-example?

3.2. Understanding divergent series through smoothed partial sums.

3.2.1. Grandi's series

Example 20. Consider Grandi's series $\sum_{n=1}^{\infty}(-1)^{n-1}$. Let η be a cutoff function that is furthermore C^2. Then we have, after one application of Abel's re-summation trick and then re-grouping,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\,\eta\left(\frac{n}{N}\right)=\frac{\eta(1/N)}{2}+\sum_{m=1}^{\infty}\frac{\eta\left(\frac{2\,m-1}{N}\right)-2\,\eta\left(\frac{2\,m}{N}\right)+\eta\left(\frac{2\,m+1}{N}\right)}{2}. \tag{71}$$

Exercise 19. Prove (71). (Hint:[23])

The first term clearly $\longrightarrow\frac{1}{2}$. On the other hand,

Exercise 20. Let $M:=\max_{x\in\mathbb{R}}|\eta''(x)|$. Prove that

$$\left|\eta\left(\frac{2\,m-1}{N}\right)-2\,\eta\left(\frac{2\,m}{N}\right)+\eta\left(\frac{2\,m+1}{N}\right)\right|\leqslant M\,N^{-2}. \tag{72}$$

(Hint:[24])

the absolute value of the second term is bounded by

$$\sum_{m=1}^{RN}\frac{M}{2\,N^2}=\frac{R}{2\,N}\longrightarrow 0 \text{ as } N\longrightarrow\infty. \tag{73}$$

Therefore we see that for any C^2 cut-off function η, we have

$$\lim_{N\longrightarrow\infty}\sum_{n=1}^{\infty}(-1)^{n-1}\,\eta\left(\frac{n}{N}\right)=\frac{1}{2}. \tag{74}$$

Exercise 21. Let

$$\sum_{n=1}^{\infty}a_n=s \qquad (H,1). \tag{75}$$

Prove that for any C^2 cut-off function η we have

$$\lim_{N\longrightarrow\infty}\sum_{n=1}^{\infty}a_n\,\eta\left(\frac{n}{N}\right)=s. \tag{76}$$

23. Either apply Abel's resummation once then re-group, or apply Abel's resummation once, write $s_n=\frac{1+(-1)^{n-1}}{2}$, sum the $\frac{1}{2}$ part, and then apply Abel's resummation again to $\frac{1}{2}\sum(-1)^{n-1}(\eta(n/N)-\eta((n+1)/N))$.

24. Write as $\int_{(2m-1)/N}^{2m/N}\eta'-\int_{2m/N}^{(2m+1)/N}\eta'$. Change of variable to combine into one integral. Then apply MVT.

(Hint:[25])

3.2.2. $\sum_{n=1}^{\infty} 1 = -\frac{1}{2}$, and other absurd identities.

Now we consider the obviously divergent series $\sum_{n=1}^{\infty} 1$. Note that none of the summation methods discussed in the previous section applies to this series.

Let η be a C^2 cutoff function. Then we have

$$\begin{aligned} \sum_{n=1}^{\infty} 1 \cdot \eta\Big(\frac{n}{N}\Big) &= \sum_{1 \leqslant n \leqslant RN} \eta\Big(\frac{n}{N}\Big) \\ &= N\left[\sum_{1 \leqslant n \leqslant RN} \frac{1}{N} \eta\Big(\frac{n}{N}\Big)\right] \\ &= N\left[\sum_{1 \leqslant n \leqslant RN} \int_{(n-1)/N}^{n/N} \eta\Big(\frac{n}{N}\Big) \,\mathrm{d}x\right] \\ &= N\left[\sum_{1 \leqslant n \leqslant RN} \int_{(n-1)/N}^{n/N} \Big(\eta\Big(\frac{n}{N}\Big) - \eta(x)\Big) \,\mathrm{d}x\right] + N \int_0^{\infty} \eta(x) \,\mathrm{d}x. \end{aligned} \tag{77}$$

Setting $C_{\eta,0} := \int_0^{\infty} \eta(x) \,\mathrm{d}x$, we have

$$\sum_{n=1}^{\infty} 1 \cdot \eta\Big(\frac{n}{N}\Big) = N\left[\sum_{1 \leqslant n \leqslant RN} \int_{(n-1)/N}^{n/N} \Big(\eta\Big(\frac{n}{N}\Big) - \eta(x)\Big) \,\mathrm{d}x\right] + C_{\eta,0} N. \tag{78}$$

Now we notice:

$$\begin{aligned} \int_{(n-1)/N}^{n/N} \Big(\eta\Big(\frac{n}{N}\Big) - \eta(x)\Big) \,\mathrm{d}x &= \int_{(n-1)/N}^{n/N} \left[\int_x^{n/N} \eta'(t) \,\mathrm{d}t\right] \mathrm{d}x \\ &= \int_{\frac{n-1}{N} \leqslant x \leqslant t \leqslant \frac{n}{N}} \eta'(t) \,\mathrm{d}(x,t) \\ &= \int_{(n-1)/N}^{n/N} \left[\int_{(n-1)/N}^{t} \eta'(t) \,\mathrm{d}x\right] \mathrm{d}t \\ &= \int_{(n-1)/N}^{n/N} \left[\int_{(n-1)/N}^{x} \eta'(x) \,\mathrm{d}t\right] \mathrm{d}x. \end{aligned} \tag{79}$$

Therefore

Exercise 22. Prove that

$$\int_{(n-1)/N}^{n/N} \Big(\eta\Big(\frac{n}{N}\Big) - \eta(x)\Big) \,\mathrm{d}x - \int_{(n-1)/N}^{n/N} \Big(\eta(x) - \eta\Big(\frac{n-1}{N}\Big)\Big) \,\mathrm{d}x = O\Big(\frac{1}{N^2}\Big), \tag{80}$$

that is there is $M > 0$ such that the absolute value of the LHS is bounded by M/N^2. (Hint:[26])

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25. Apply Abel's resummation trick twice.

26. By (78) the difference is bounded by

$$\int_{(n-1)/N}^{n/N} \left[\int_{(n-1)/N}^{n/N} |\eta'(x) - \eta'(t)| \,\mathrm{d}t\right] \mathrm{d}x. \tag{81}$$

Apply MVT.

and consequently

$$\begin{aligned} N \sum_{1 \leqslant n \leqslant RN} \int_{(n-1)/N}^{n/N} \left(\eta\left(\frac{n}{N}\right) - \eta(x)\right) \mathrm{d}x &= \frac{N}{2} \sum_{1 \leqslant n \leqslant RN} \int_{(n-1)/N}^{n/N} \left(\eta\left(\frac{n}{N}\right) - \eta\left(\frac{n-1}{N}\right)\right) \mathrm{d}x \\ &\quad + O\left(\frac{1}{N}\right) \\ &= \frac{1}{2} \sum_{1 \leqslant n \leqslant RN} \left[\eta\left(\frac{n}{N}\right) - \eta\left(\frac{n-1}{N}\right)\right] + O\left(\frac{1}{N}\right) \\ &= -\frac{1}{2} + O\left(\frac{1}{N}\right). \end{aligned} \tag{82}$$

Summarizing, we have proved for any C^2 cutoff function η,

$$\sum_{n=1}^{\infty} \eta\left(\frac{n}{N}\right) = -\frac{1}{2} + C_{\eta,0} N + O\left(\frac{1}{N}\right), \tag{83}$$

or equivalently

$$\lim_{N \to \infty} \left[\sum_{n=1}^{\infty} \eta\left(\frac{n}{N}\right) - C_{\eta,0} N\right] = -\frac{1}{2}. \tag{84}$$

Remark 21. We see that

$$\sum_{n=1}^{\infty} 1 = -\frac{1}{2} \tag{85}$$

in the sense that $-\frac{1}{2}$ is the “cutoff-independent” part of the sum.

Problem 2. Let η be a smooth cutoff function. Prove that there is $C_{\eta,1}, C_{\eta,2} \in \mathbb{R}$ such that

$$\sum_{n=1}^{\infty} n \, \eta\left(\frac{n}{N}\right) = -\frac{1}{12} + C_{\eta,1} N^2 + O\left(\frac{1}{N}\right); \tag{86}$$

$$\sum_{n=1}^{\infty} n^2 \, \eta\left(\frac{n}{N}\right) = C_{\eta,2} N^3 + O\left(\frac{1}{N}\right). \tag{87}$$

Thus justifying

$$1 + 2 + 3 + \cdots = -\frac{1}{12}; \qquad 1 + 2^2 + 3^2 + \cdots = 0. \tag{88}$$

Remark 22. In general we have

$$\sum_{n=1}^{\infty} n^k \, \eta\left(\frac{n}{N}\right) = -\frac{B_{k+1}}{k+1} + C_{\eta,k} N^{k+1} + O\left(\frac{1}{N}\right) \tag{89}$$

where B_{k+1} is the $(k+1)$-th Bernoulli number, and

$$C_{\eta,k} := \int_0^{\infty} x^k \, \eta(x) \, \mathrm{d}x. \tag{90}$$

Therefore in a sense

$$1 + 2^k + 3^k + \cdots = -\frac{B_{k+1}}{k+1}. \tag{91}$$

See (TAO) for more discussion on this.

4. ADVANCED TOPICS, NOTES, AND COMMENTS

4.1. Differentiability of the Riemann function

References.

- (GERVER1969) Gerver, Joseph, *The differentiability of the Riemann function at certain rational multiples of* π, PNAS, 1969, 62 (3), 668 – 670.
- (GERVER1970) Gerver, Joseph, *The differentiability of the Riemann function at certain rational multiples of* π, Amer. J. Math., 1970, 92, 33 – 55.

Riemann once proposed the function

$$f(x) := \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2} \tag{92}$$

as a candidate of "everywhere continuous but nowhere differentiable" functions. The continuity is a straightforward consequence of Weierstrass's M-test, while the differentiability turned out to be subtle. Finally it was settled by Joseph Gerver in a series of papers that the Riemann function is indeed differentiable at certain values of x. In this section we illustrate the basic ideas of Gerver's proof through studying $f(x)$ at $x=0$ and $x=\pi$.

4.1.1. $f'(0) = +\infty$.

Since $f(x)$ is clearly even, it suffices to consider the case $x \to 0+$. Thus we need to prove

$$\lim_{x\to 0+} \frac{1}{x}\left[\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2} - 0\right] = +\infty. \tag{93}$$

Now consider x very small. We split

$$\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2 x} = \sum_{n<\sqrt{\pi/x}} \frac{\sin(n^2 x)}{n^2 x} + \sum_{n\geqslant\sqrt{\pi/x}} \frac{\sin(n^2 x)}{n^2 x}. \tag{94}$$

Noticing that when $n < \sqrt{\pi/x}$, $\sin(n^2 x) \geqslant 0$, we have

$$\sum_{n<\sqrt{\pi/x}} \frac{\sin(n^2 x)}{n^2 x} \geqslant \sum_{n<\sqrt{\frac{\pi}{2x}}} \frac{\sin(n^2 x)}{n^2 x} \geqslant \sum_{n<\sqrt{\frac{\pi}{2x}}} \frac{\sin(\pi/2)}{\pi/2} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{x}}, \tag{95}$$

where the fact that $\frac{\sin x}{x}$ is decreasing on $(0, \pi/2)$ is used.

On the other hand, we have

$$\left|\sum_{n\geqslant\sqrt{\pi/x}} \frac{\sin(n^2 x)}{n^2 x}\right| \leqslant x^{-1} \sum_{n\geqslant\sqrt{\pi/x}} \frac{1}{n^2} \leqslant x^{-1} \sum_{n\geqslant\sqrt{\pi/x}} \left[\frac{1}{n-1} - \frac{1}{n}\right] \leqslant \frac{1}{\sqrt{\pi}-\sqrt{x}}\, x^{-1/2}. \tag{96}$$

Summarizing, we have

$$\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2 x} \geqslant \left[\sqrt{\frac{2}{\pi}} - \frac{1}{\sqrt{\pi}-\sqrt{x}}\right] x^{-1/2} \tag{97}$$

which clearly tends to $+\infty$ as $x \to 0+$.

4.1.2. Understanding $f'(\pi) = -1/2$ through smoothed partial sum.

Set $x = \pi + t$. Then we have

$$\frac{1}{x-\pi}\left[\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2} - \sum_{n=1}^{\infty} \frac{\sin(n^2 \pi)}{n^2}\right] = \frac{1}{t}\sum_{n=1}^{\infty} \frac{\sin(n^2\pi + n^2 t)}{n^2} = \sum_{n=1}^{\infty} (-1)^n \frac{\sin(n^2 t)}{n^2 t}. \tag{98}$$

Thus in the following we will try to prove

$$\lim_{x\to 0+} \sum_{n=1}^{\infty} (-1)^n \frac{\sin(n^2 x)}{n^2 x} = -\frac{1}{2}. \tag{99}$$

This is clearly equivalent to

$$\lim_{t\to 0+} \sum_{n=1}^{\infty} (-1)^n \frac{\sin[(n\,t)^2]}{(n\,t)^2} = -\frac{1}{2}. \tag{100}$$

Now if we denote

$$\eta(x) := \frac{\sin(x^2)}{x^2}, \qquad N := t^{-1}, \tag{101}$$

(100) becomes

$$\lim_{N\to\infty} \sum_{n=1}^{\infty} (-1)^n \eta\left(\frac{n}{N}\right) = -\frac{1}{2} \tag{102}$$

from which we see that formally the limit $-1/2$ should not be a total surprise.

However, proving (102) turns out to be quite tricky, for the following reasons:

i. The function η is not really a "cutoff" function – there is no $R > 0$ such that $\eta(x) = 0$ for $|x| > R$;

ii. The decay of η is like x^{-2}, which is not sufficient for the "tail" part to vanish;

iii. $\eta'(x)$ decays like $1/x$ and $\eta''(x)$ does not have decay anymore. In particular $|\eta'(x)|$ is not integrable on $(0, \infty)$.

It is hard to understand why ii and iii are serious problems before actually attempting to prove (102). On the other hand, some light may be shed on their effects through the following result.

Proposition 23. *Let $\eta(x) \in C^2$ satisfy the following*

- *there is $M > 0$ such that $\sup_{x\in\mathbb{R}} |\eta''(x)| < M$;*
- *there is $M' > 0$ and $p > 2$ such that $|x^p\,\eta(x)| < M'$ for all x.*

Then (102) holds.

Proof. Take $q \in \left(\frac{p}{p-1}, 2\right)$. Note that as $p > 2$, we have $\frac{p}{p-1} < 2$ so such q exists.

Similar to Example 20 we calculate

$$\sum_{n=1}^{N^q} (-1)^n \eta\left(\frac{n}{N}\right) = -\frac{\eta(1/N)}{2} - \sum_{m=1}^{N^q/2} \frac{\eta\left(\frac{2m-1}{N}\right) - 2\,\eta\left(\frac{2m}{N}\right) + \eta\left(\frac{2m+1}{N}\right)}{2}. \tag{103}$$

As $|\eta''(x)| < M$, we have

$$\left|\eta\left(\frac{2m-1}{N}\right) - 2\,\eta\left(\frac{2m}{N}\right) + \eta\left(\frac{2m+1}{N}\right)\right| < \frac{M}{N^2} \tag{104}$$

which gives

$$\left|\sum_{n=1}^{N^q} (-1)^n \eta\left(\frac{n}{N}\right) + \frac{\eta(1/N)}{2}\right| < \frac{M}{4} N^{q-2} \longrightarrow 0 \tag{105}$$

therefore

$$\lim_{N\to\infty} \sum_{n=1}^{N^q} (-1)^n \eta\left(\frac{n}{N}\right) = -\frac{1}{2}. \tag{106}$$

On the other hand we have

$$\begin{aligned}\left|\sum_{n>N^q} (-1)^n \eta\left(\frac{n}{N}\right)\right| &< \sum_{n>N^q} \left|\eta\left(\frac{n}{N}\right)\left(\frac{n}{N}\right)^p\right| \frac{N^p}{n^p} \\ &\leqslant N^p M' \sum_{n>N^q} \frac{1}{n^p} \\ &< N^p M' \frac{1}{N^{q(p-1)}} \\ &= M' N^{p-q(p-1)} \longrightarrow 0. \end{aligned} \tag{107}$$

Summarizing, we have proved (102). ☐

Exercise 23. Show that (102) still holds if the second hypothesis is relaxed to $x^2 \eta(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

Remark 24. In the case $p=2$, the above "tail estimate" becomes

$$\left|\sum_{n>N^q} (-1)^n \eta\left(\frac{n}{N}\right)\right| < M' \tag{108}$$

which does not vanish.

We may want to take advantage of the oscillating nature of $(-1)^n$ and write

$$\left|\sum_{n \text{ large}} (-1)^n \eta\left(\frac{n}{N}\right)\right| \leqslant \sum_m \left|\eta\left(\frac{2m-1}{N}\right) - \eta\left(\frac{2m}{N}\right)\right| \sim \frac{1}{N} \sum_m \left|\eta'\left(\frac{2m-1}{N}\right)\right|. \tag{109}$$

But if $\eta'(x) \sim \frac{1}{x}$, we would reach

$$\frac{1}{N} \sum_m \frac{N}{2m-1} = \infty \tag{110}$$

which is even worse than (108)! On the other hand,

Exercise 24. Prove (102) for $\eta(x) = \frac{1}{x^2+1}$.

4.1.3. Estimating the tail: idea

Now we have convinced ourselves that the vanishing of the tail terms in the estimate for $f'(\pi) = -1/2$ does not easily follow from the "smoothed partial sum" framework that we have established, and furthermore it is not a consequence of the "oscillation" $(-1)^n$. Then where does the cancelation come from? The following is a revealing example, the motivation comes from writing

$$\sum_{n>N^q} (-1)^n \frac{\sin(n^2 x^2)}{n^2 x^2} = \sum_{n \text{ odd}} \frac{\sin(n^2 x^2)}{n^2 x^2} - \sum_{n \text{ even}} \frac{\sin(n^2 x^2)}{n^2 x^2} \tag{111}$$

Now we have

$$\sum_{n \text{ odd}} \frac{\sin(n^2 x^2)}{n^2 x^2} = (2x)^{-1} \sum_{n \text{ odd}} \frac{\sin(n^2 x^2)}{n^2 x^2} [(n+2)x - nx] \sim (2x)^{-1} \int_{N^q x} \frac{\sin(\xi^2)}{\xi^2} \mathrm{d}\xi. \tag{112}$$

Example 25. We show that, if $q>3/2$, then

$$x^{-1}\int_{N^q x}^{\infty}\frac{\sin(\xi^2)}{\xi^2}\,\mathrm{d}\xi\longrightarrow 0. \tag{113}$$

Let $k\in\mathbb{N}$ be large. We have

$$\begin{aligned}\int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}}\frac{\sin(\xi^2)}{\xi^2}\,\mathrm{d}\xi+\int_{\sqrt{(k+1)\pi}}^{\sqrt{(k+2)\pi}}\frac{\sin(\xi^2)}{\xi^2}\,\mathrm{d}\xi &= \int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}}\frac{\sin(\xi^2)}{\xi^2}\,\mathrm{d}\xi\\ &\quad+\int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}}\frac{\sin(\xi^2+\pi)}{\xi^2+\pi}\,\mathrm{d}\sqrt{\xi^2+\pi}\\ &= \int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}}\left[\frac{\sin(\xi^2)}{\xi^2}-\frac{\xi\sin(\xi^2)}{(\xi^2+\pi)^{3/2}}\right]\mathrm{d}\xi \end{aligned}\tag{114}$$

which can be bounded by

$$C\int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}}\frac{1}{\xi^4}\,\mathrm{d}\xi\sim\frac{C}{k^2}. \tag{115}$$

This means

$$x^{-1}\int_{N^q x}^{\infty}\frac{\sin(\xi^2)}{\xi^2}\,\mathrm{d}\xi\sim x^{-1}\sum_{k>(N^q x)^2}\frac{C}{k^2}\sim\frac{1}{N^{2q}x^3}\sim\frac{1}{N^{2q-3}}. \tag{116}$$

Remark 26. Note that (112) per se does not help in the proof, as the "$\sim$" actually has $O(1)$ error due to the slow decay of $\eta'=\left(\frac{\sin(x^2)}{x^2}\right)'$.

Thus we see that, to show $\sum_{n>N^q}(-1)^n\frac{\sin(n^2x^2)}{n^2x^2}\longrightarrow 0$ with $x\sim N^{-1}$, we should try to show that $\sum_{n\ \mathrm{odd},n>N^q}\frac{\sin(n^2x^2)}{n^2x^2}$ and $\sum_{n\ \mathrm{even},n>N^q}\frac{\sin(n^2x^2)}{n^2x^2}$ both $\longrightarrow 0$, taking advantage of the "long range" cancellation in the sine function and ignore the $(-1)^n$ which is essential for small n but becomes useless when $n\longrightarrow\infty$.

4.1.4. Estimating the tail: Gerver's proof

Basic idea.

To carry out the plan we following Gerver's proof. First at this stage writing $\frac{\sin(n^2x)}{n^2x}$ as $\frac{\sin(n^2x^2)}{n^2x^2}$ does not have any advantage anymore so we return to the original setting: Show

$$\lim_{x\longrightarrow 0}\sum_{n=1}^{\infty}(-1)^n\frac{\sin(n^2x)}{n^2x}=-\frac{1}{2}. \tag{117}$$

We have seen that, using "smoothed partial sum", for any $N(x)\gg x$ (that is $\lim_{x\to 0}\frac{N(x)}{x}=\infty$,

$$\lim_{x\to 0}\sum_{n=1}^{N(x)}(-1)^n\frac{\sin(n^2x)}{n^2x}=-\frac{1}{2}. \tag{118}$$

Therefore the task is to show

$$\sum_{n>N(x)}(-1)^n\frac{\sin(n^2x)}{n^2x}\longrightarrow 0 \tag{119}$$

or equivalently

$$\sum_{n>N(x)} (-1)^n \frac{\sin(n^2 x)}{n^2} = o(x) \qquad \text{as } x \to 0. \tag{120}$$

As we have discussed earlier, the $(-1)^n$ does not matter here. Therefore we need to show

$$\sum_{n>N(x), n \text{ even}} \frac{\sin(n^2 x)}{n^2} = o(x) \quad \text{and} \quad \sum_{n>N(x), n \text{ odd}} \frac{\sin(n^2 x)}{n^2} = o(x). \tag{121}$$

In the following we will simply try to prove

$$\sum_{n>N(x)} \frac{\sin(n^2 x)}{n^2} = o(x) \tag{122}$$

for appropriate $N(x) \ll x^{-1}$.

The basic idea is to exploit the "long range" cancellation between $\sin(\xi)$ and $\sin(\xi + \pi)$. Assume that $n_1^2 x + \pi = n_2^2 x$. The observation is that since $N(x)$ can be taken "almost as large as" x^{-1}, for $n > N(x)$, $n^2 x$ is almost as large as $N(x)$. Therefore $n_1 \sim n_2$. Thus

$$\frac{\sin(n_1^2 x)}{n_1^2} + \frac{\sin(n_2^2)}{n_2^2} \ll \frac{1}{n_1^2}. \tag{123}$$

Now the idea is to consider the chain

$$\frac{\sin(n^2 x)}{n^2} + \frac{\sin((n+1)^2 x)}{(n+1)^2} + \cdots \tag{124}$$

and find for each n a "pair" n^* such that $n^2 x \sim (n^*)^2 x + k\pi$ with some odd number k. To make this possible we need $\sin(n^2 x), \sin((n+1)^2 x), \ldots.$ to "fill" $[-1, 1]$, the range of $\sin \xi$. For most n this is true but for some it is not. The proof then divides into two parts. In the first part we show that those "bad" n's are so few that they do not cause trouble; In the second part we prove that for other n's the desired "pairing" can indeed be carried out.

In the following we take $N(x) \sim x^{-13/14}$.

Bad n's.

We characterize the bad n's by

$$n \in K_{\text{bad}} \Longleftrightarrow \exists i, j \in \mathbb{N} \text{ co-prime}, j < x^{-1/7}, \qquad \left| n - \frac{2\pi i}{x j} \right| < x^{-4/7}. \tag{125}$$

Note that the condition can be written as

$$|n x j - 2\pi i| < x^{3/7} j < x^{2/7}. \tag{126}$$

In other words this implies (remember that x is very small)

$$(n+j)^2 x - n^2 x \sim 2\pi i \tag{127}$$

which means $\sin((n+j)^2 x)$ and $\sin(n^2 x)$ are very close. Recalling that $j < x^{-1/7}$, this means that $\sin((n+k)^2 x)$ "almost has period" $j < x^{-1/7}$. Such a small period implies that the distribution of

$$\sin(n^2 x), \sin((n+1)^2 x), \ldots, \sin((n+j)^2 x) \tag{128}$$

in $[-1, 1]$ are not very dense and the chance of finding cancellation "pairing" within them is slim.

Now we show that for any subset $Q \subseteq K_{\text{bad}}$,

$$\left| \sum_{n > N(x), n \in Q} \frac{\sin(n^2 x)}{n^2} \right| \lesssim x^{15/14} \ll x. \tag{129}$$

$\lesssim, \gtrsim, \sim, \ll, \gg, o(\cdot), O(\cdot)$.

In the study of limits, we often need to compare the "limiting behavior" of functions or sequences. It is convenient to use these symbols.

- $f \lesssim g$ means there is $C > 0$ such that $f \leqslant C g$;
- $f \sim g$ means $f \lesssim g$ and $g \lesssim f$;
- $f \ll g$ (or $f = o(g)$) means $\lim \frac{f}{g} = 0$;
- $f = O(g)$ means there is $C > 0$ such that $|f| \leqslant C g$;

Exercise 25. Prove the following:

a) $f \lesssim g, g \lesssim h \Longrightarrow f \lesssim h$;

b) $|f| \ll |g|, |g| \lesssim |h| \Longrightarrow f \ll h$;

c) $f \sim g, g \ll h \Longrightarrow f \ll h$.

Remark 27. It is important to understand why we do not just prove

$$\left| \sum_{n > N(x), n \in K_{\text{bad}}} \frac{\sin(n^2 x)}{n^2} \right| \lesssim x^{15/14} \ll x. \tag{130}$$

Exercise 26. Let $\sum_{n=1}^{\infty} a_n$ be convergent. Let $\{n_k\}$ be a subset of $\mathbb{N}$. Does it follow that $\sum_{k=1}^{\infty} a_{n_k}$ also converges? (Hint:[27])

Now we prove (129). Since we do not expect any cancellation, we simply try to prove that the bad n's are few and far between and $\sum \frac{1}{n^2}$ is already $\ll x$.

First notice that, for any two pairs of integers i_1, j_1, i_2, j_2 satisfying $j_i \lesssim x^{-1/7}$ with each pair co-prime, we have

$$\left| \frac{2\pi i_1}{x j_1} - \frac{2\pi i_2}{x j_2} \right| \geqslant \frac{2\pi}{x} \frac{1}{j_1 j_2} \gtrsim x^{-5/7}. \tag{131}$$

On the other hand, for each fixed (i, j), there are at most $x^{-4/7}$ k's satisfying (125) and thus in K_{bad}. Thus we see that K_{bad} are distributed around each $\frac{2\pi i}{x j}$ while occupies only $x^{-4/7}/x^{-5/7} = x^{1/7}$ of the whole $\{n > N(x)\}$. Consequently (a bit handwaving here)

$$\sum_{n > N(x), n \in K_{\text{bad}}} \frac{1}{n^2} \sim x^{1/7} \sum_{n > N(x)} \frac{1}{n^2} \sim x^{1/7} N(x)^{-1} \sim x^{15/14} \ll x. \tag{132}$$

Remark 28. For rigorous implementation of the above idea, see (GERVER1970) pp.43 – 45.

27. No. $\frac{(-1)^n}{n}$.

Good n's.

Keep this picture in mind.

The set $\{n > N(x)\}$ consists of long "chains" of "good n's" interrupted by "a few" "bad n's' clustered around every $\frac{2\pi i}{x j}$ with $j < x^{-1/7}$.

We have seen that the "bad n" terms sum up to $o(x)$ because there are very few of them. In the following we show that the "good n" terms also sum up to $o(x)$, for a different reason: There is much cancellation within each "chain".

Now we try to define the "chains". Let t_1 be the first number $>N(x)$ that is not in K_{bad}. Then $t_1, t_1+1, t_1+2, \ldots$ all belong to "good n's" until we reach j_1 such that

$$\left| j_1 t_1 - \frac{2\pi i}{x} \right| \leqslant x^{-4/7}. \tag{133}$$

Then take $t_2 \geqslant t_1 + j_1$ to be the first that is not in K_{bad}, and so on.

Note that as $t_u \notin K_{\text{bad}}$ for every $u = 1, 2, 3, \ldots.$, we must have $j_u \geqslant x^{-1/7}$. Now consider

$$j t_u \mod \frac{2\pi}{x}, \tag{134}$$

by Pigeon hold principle we have

$$j_u \leqslant \frac{2\pi/x}{x^{-4/7}} \sim x^{-3/7}. \tag{135}$$

We see that each "chain" contains at least $x^{-1/7}$ and at most $x^{-3/7}$ numbers.

In the following we focus on one chain and omit the subscripts. Thus we study $n = t, t+1, \ldots, t+j-1$.

The first observation is that since $n > N(x) = x^{-13/14}$, $n^2 x \gg 1$. Therefore trying to find cancellation through the study of $n^2 x$, $(n^2+1) x$, ... is not efficient as the numbers are too wide apart. Thus we try to replace $(n+p)^2 x$ by smaller numbers.

For any $0 \leqslant p < j$, there is a unique $\alpha_p \in \left[0, \frac{2\pi}{(t+p)^2}\right) \subseteq \left[0, \frac{2\pi}{t^2}\right)$ such that

$$(t+p)^2 x = (t+p)^2 \alpha_p + 2k\pi \qquad \text{for some } k \in \mathbb{Z} \tag{136}$$

Now we consider the relation between the ψ_p's.

If we denote

$$\beta_p := (x - \alpha_p) \left(1 - \frac{(t+p)^2}{(t+p+1)^2} \right), \tag{137}$$

then we have

$$x - \alpha_p - \beta_p = (x - \alpha_p) \frac{(t+p)^2}{(t+p+1)^2} \tag{138}$$

and therefore

$$(t+p+1)^2 (x - \alpha_p - \beta_p) = (t+p)^2 (x - \alpha_p) = 2k\pi. \tag{139}$$

Now we start from $\alpha := \alpha_0$ and obtain $\beta_1, \beta_2, \ldots, \beta_{p-1}$ successively as above. We have

$$(t+p)^2 \left(x - \alpha - \sum_1^{p-1} \beta_i \right) = (t+p)^2 (x - \alpha_p) + 2k\pi. \tag{140}$$

Denoting $\psi_p := \alpha_p - \alpha$ and $\theta_p := \sum_1^{p-1} \beta_i$, we have

$$\theta_p = \psi_p + \frac{2\pi k}{(t+p)^2} \tag{141}$$

for some $k \in \mathbb{Z}$.

Now we estimate the size of each β_p using (137). We have

$$\left| \beta_p - x \frac{2(t+p)+1}{(t+p+1)^2} \right| = |\alpha_p| \left| \frac{2(t+p)+1}{(t+p+1)^2} \right| \tag{142}$$

which gives

$$\left| \beta_p - \frac{2x}{t} \right| \lesssim t^{-2} x^{4/7}. \tag{143}$$

Exercise 27. Obtain (143). (Hint:[28])

This gives (recall that $j \lesssim x^{-3/7}$)

$$\left| \theta_p - \frac{2px}{t} \right| \lesssim t^{-2} x^{1/7}. \tag{144}$$

Summarizing what we have so far: There is $k \in \mathbb{Z}$ such that

$$\left| \psi_p + \frac{2k\pi}{(t+p)^2} - \frac{2px}{t} \right| \lesssim t^{-2} x^{1/7}. \tag{145}$$

Recalling $\psi_p \lesssim t^{-2}$, we have

$$\left| \frac{2k\pi}{(t+p)^2} \right| \lesssim \frac{1}{p(t+p)} x^{1/7} \tag{146}$$

which leads to

$$\left| \frac{2k\pi}{(t+p)^2} - \frac{2k\pi}{t^2} \right| \lesssim t^{-2} x^{1/7} \tag{147}$$

which in turn gives

$$\left| \psi_p + \frac{2k\pi}{t^2} - \frac{2px}{t} \right| \lesssim t^{-2} x^{1/7}. \tag{148}$$

This finally gives

$$\left| \frac{t^2 \psi_p}{2\pi} + k - \frac{tpx}{\pi} \right| \lesssim x^{1/7}. \tag{149}$$

Recalling that there is i such that

$$\left| jt - \frac{2\pi i}{x} \right| \leqslant x^{-4/7} \Longrightarrow \left| t - \frac{2\pi i}{jx} \right| \leqslant x^{-4/7} j^{-1} \lesssim x^{-3/7}. \tag{150}$$

Therefore

$$\left| \frac{tpx}{\pi} - \frac{2pi}{j} \right| \lesssim x^{1/7}. \tag{151}$$

Combining (149) and (151) we have

$$\left| \frac{t^2 \psi_p}{2\pi} + k - \frac{2pi}{j} \right| \lesssim x^{1/7}. \tag{152}$$

As $\frac{t^2 \psi_p}{2\pi} \in [0,1)$ we have

$$-C x^{1/7} \leqslant \frac{2pi}{j} - k \leqslant 1 + C x^{1/7}. \tag{153}$$

28. $\alpha_p \sim t^{-2}$ and $t \gg p$.

for some constant C.

Summarize: For every $p=0,1,\ldots,j-1$, we have found i,k such that (153) holds.

To see why this is good, we check the situation $0<\frac{2\,p\,i}{j}-k<1$. In this case denote $q_p=2\,p\,i-k\,j\in\{0,1,\ldots,j-1\}$. Then we have

$$\left|\frac{t^2\,\psi_p}{2\,\pi}-\frac{q_p}{j}\right|\lesssim x^{1/7}. \tag{154}$$

As j is the least integer such that (133) holds, i,j are co-prime. As a consequence we can pair up each q_p with $q_{p'}$ such that

$$|q_p-(q_{p'}+1/2)|\leqslant 1 \tag{155}$$

which gives

$$|t^2\,\psi_p-(t^2\,\psi_{p'}+\pi)|\lesssim x^{1/7}. \tag{156}$$

Application of MVT now gives (keep in mind that $t\gg p$):

$$\begin{aligned}\left|\frac{\sin(t+p)^2\,x}{(t+p)^2}+\frac{\sin(t+p')^2\,x}{(t+p')^2}\right| &= \left|\frac{\sin[(t+p)^2\,(\alpha+\psi_p)]}{(t+p)^2}+\frac{\sin(t+p')^2\,(\alpha+\psi_{p'})}{(t+p')^2}\right|\\ &= \left|\frac{\sin[t^2\,(\alpha+\psi_p)]}{t^2}+\frac{\sin[t^2\,(\alpha+\psi_{p'})]}{t^2}\right|+O\big(t^{-2}\,x^{3/7}\big)\\ &\lesssim t^{-2}\,x^{1/7}.\end{aligned} \tag{157}$$

Therefore in this particular chain (denote it by R_u, $u=1,2,3,\ldots$), we have

$$\sum_{n\in R_u}\frac{\sin n^2 x}{n^2}\lesssim\sum_{n\in R_u}t^{-2}\,x^{1/7}\lesssim x^{1/7}\sum_{n\in R_u}\frac{1}{n^2}. \tag{158}$$

Note that the last inequality is due to $n\in R_u\Longrightarrow n=t,t+1,\ldots,t+j-1$ but $j\ll t$ so $t^{-2}\sim(t+j)^{-2}$ for every j involved.

Now notice that $R_u\cap R_v=\varnothing$ whenever $u\neq v$, we have

$$\sum_{n\in\cup_{u=1}^{\infty}R_u}\frac{\sin n^2 x}{n^2}\lesssim x^{1/7}\sum_{n>N(x)}\frac{1}{n^2}\leqslant x^{1/7}\,N(x)^{-1}\sim x^{15/14}\ll x. \tag{159}$$

Finally, as

$$Q:=\{n|\,n>N(x)\}-\cup_{u=1}^{\infty}R_u\subseteq K_{\text{bad}}, \tag{160}$$

we have, following (132),

$$\sum_{n\in Q}\frac{\sin n^2 x}{n^2}\leqslant\sum_{n\in K_{\text{bad}}}\frac{1}{n^2}\lesssim x^{15/14}\ll x. \tag{161}$$

Therefore

$$\sum_{n>N(x)}\frac{\sin n^2\,x}{n^2}\lesssim x^{15/14}\ll x\Longrightarrow\lim_{x\to 0}\sum_{n>N(x)}\frac{\sin n^2\,x}{n^2\,x}\lesssim x^{1/14}\longrightarrow 0. \tag{162}$$

The estimate for the tail is complete.

4.2. Tauberian theorems

THEOREM 29. (TAUBER) *Let $\sum_{n=0}^{\infty}a_n$ be Abel summable. Further assume $\lim_{n\to\infty}n\,a_n=0$. Then $\sum_{n=0}^{\infty}a_n$ is convergent.*

Proof. Let

$$\sum_{n=0}^{\infty} a_n = s \qquad (A). \tag{163}$$

Then we have

$$\lim_{x \to 1} \left(\sum_{n=0}^{\infty} a_n x^n \right) = s. \tag{164}$$

Now let N be large and set $x_N = 1 - \frac{1}{N}$. It suffices to prove

$$\lim_{N \to \infty} \left| \sum_{n=0}^{N} a_n - \sum_{n=0}^{\infty} a_n x_N^n \right| = 0. \tag{165}$$

We denote

$$m_N := \max_{n \geqslant N} \{ |n\, a_n| \} \tag{166}$$

which satisfies $\lim_{N \to \infty} m_N = 0$.

Now calculate

$$\begin{aligned} \left| \sum_{n=0}^{N} a_n - \sum_{n=0}^{\infty} a_n x_N^n \right| &\leqslant \left| \sum_{n=0}^{N} a_n (1 - x_N^n) \right| + \left| \sum_{n=N+1}^{\infty} a_n x_N^n \right| \\ &\leqslant \left| \sum_{n=0}^{N} a_n (1 - x_N)\, n \right| + m_{N+1} \left| \sum_{n=N+1}^{\infty} \frac{x_N^n}{n} \right| \\ &\leqslant \left| \frac{\sum_{n=0}^{N} a_n\, n}{N} \right| + m_{N+1} \frac{1}{N} \left| \sum_{n=0}^{\infty} x_N^n \right| \\ &= \left| \frac{\sum_{n=0}^{N} a_n\, n}{N} \right| + m_{N+1}. \end{aligned} \tag{167}$$

Clearly the 2nd term $\longrightarrow 0$. That the first term $\longrightarrow 0$ is left as an exercise. □

Exercise 28. Prove that if $x_n \longrightarrow 0$ then $\lim_{N \to \infty} \frac{x_1 + \cdots + x_N}{N} = 0$. (Hint:[29])

Remark 30. D. E. Littlewood proved that the same conclusion still holds if $a_n = O(n^{-1})$, that is there is $M > 0$ such that $|a_n\, n| < M$ for all n.

Tauberian Theorems

The above theorem was proved by Tauber, which is the first theorem of the form:

> If $\sum_{n=0}^{\infty} a_n$ is summable (according to certain summability) and a_n satisfies some further decay conditions, then $\sum_{n=0}^{\infty} a_n$ is convergent.

Such theorems are called "Tauberian theorems".

29. Any ε, there is N_1 such that $|x_n| < \varepsilon$ whenever $n > N_1$. Now split

$$\frac{x_1 + \cdots + x_N}{N} = \frac{x_1 + \cdots + x_{N_1}}{N} + \frac{x_{N_1+1} + \cdots + x_N}{N}. \tag{168}$$

5. More Exercises and Problems

Problem 3. If $\sum_{n=0}^{\infty} a_n$ is (H,k) (equivalently (C,k)) summable, then $a_n = O(n^k)$.

Problem 4. Let

$$\sum_{n=1}^{\infty} a_n = s \qquad (H,1). \tag{169}$$

Prove that for any C^1 cut-off function η we have

$$\lim_{N \longrightarrow \infty} \sum_{n=1}^{\infty} a_n \, \eta\left(\frac{n}{N}\right) = s. \tag{170}$$

Note that the regularity assumption is C^1 instead of C^2.

Problem 5. Let $\sum_{n=1}^{\infty} a_n$ be a positive series. Assume there is a cutoff function η such that

$$\lim_{N \to \infty} \sum_{n=1}^{\infty} a_n \, \eta\left(\frac{n}{N}\right) = s \in \mathbb{R}. \tag{171}$$

Prove or disprove:

$$\sum_{n=1}^{\infty} a_n = s. \tag{172}$$

What if $s = +\infty$?