Math 317 Week 09: Vector Calculus

April 3, 2014

Table of contents

References.

- (BRAND) Brand, Louis, *Vector Analysis*, Dover Publications, Inc., 2006.
- (DEMIDOVICH) Demidovich, B. etal., *Problems in Mathematical Analysis*, MIR Publishers, Moscow, Section VII.10 – 11.
- (FOLLAND) Folland, Gerald B., *Advanced Calculus*, Prentice-Hall, Inc., 2002. Chapter 5.
- (PKU3) Shen, Xiechang, *ShuXueFenXi* (Mathematical Analysis) III, Higher Education Press, 1986, Chapters 21 & 23.

1. STOKES' THEOREM

1.1. The theorem and its proof

THEOREM 1. (STOKES'S THEOREM) *Let S be a C^1 orientable surface in $\mathbb{R}^3$ with ∂S union of finitely many C^1 curves. Let a normal vector field $\boldsymbol{n}$ be chosen on S. Let ∂S be oriented such that when moving along ∂S on the positive side of S, the interior of S is to the left. Then for any C^1 function $\boldsymbol{f} = \begin{pmatrix} f \\ g \\ h \end{pmatrix} : \mathbb{R}^3 \mapsto \mathbb{R}^3$,*

$$\int_{\partial S} f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z = \int_S \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y \quad (1)$$

Remark 2. In $\mathbb{R}^3$ with Cartesian coordinates, (1) can be memorized as the formal "cross-product" between the vector operator $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ with the vector function (f, g, h). See §3.1.3.

Remark 3. If we take $f = f(x, y)$, $g = g(x, y)$, $h = 0$, then (1) reduces to Green's Theorem. On the other hand, roughly speaking (1) is simply Green's Theorem after a "change of variables". This relation can be clearly seen from the proof below.

NOTATION 4. *In many Calculus textbooks, $\oint_{\partial S}$ is used instead of $\int_{\partial S}$ to emphasize the fact that the integral is along a closed curve. We will not follow this tradition.*

The proof of the theorem in its full generality is a bit technical. Therefore we relegate it to §4.2.1 and only prove the special case when S in $\mathbb{R}^3$ is given by part of a graph: $z = \phi(x, y), (x, y) \in D$.

Proof. (WHEN S IS PART OF A GRAPH) We have

$$S: \qquad z = \phi(x, y), \qquad (x, y) \in D. \quad (2)$$

Let $(u(t), v(t)), t \in [a, b]$ be a parametrization of ∂D so that when t increases D is always to the left. Then $(u(t), v(t), \phi(u(t), v(t))), t \in [a, b]$ is a parametrization of ∂S consistent with that specified in the theorem. Now we calculate

$$\begin{aligned} \int_{\partial S} f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z &= \int_a^b \left[f\,\frac{\mathrm{d}u}{\mathrm{d}t} + g\,\frac{\mathrm{d}v}{\mathrm{d}t} + h\,\frac{\mathrm{d}\phi}{\mathrm{d}t}\right] \mathrm{d}t \\ &= \int_a^b [f\,u' + g\,v' + h\,\phi_u\,u' + h\,\phi_v\,v']\,\mathrm{d}t \\ &= \int_a^b [(f + h\,\phi_u)\,u' + (g + h\,\phi_v)\,v']\,\mathrm{d}t \\ &= \int_{\partial D} F\,\mathrm{d}u + G\,\mathrm{d}v \quad (3) \end{aligned}$$

where

$$F(u, v) := f(u, v, \phi(u, v)) + h(u, v, \phi(u, v))\,\phi_u(u, v) \quad (4)$$

and

$$G(u,v) := g(u,v,\phi(u,v)) + h(u,v,\phi(u,v))\,\phi_v(u,v). \tag{5}$$

By Green's Theorem we have

$$\int_{\partial D} F\,\mathrm{d}u + G\,\mathrm{d}v = \int_D \left[\frac{\partial G}{\partial u} - \frac{\partial F}{\partial v}\right] \mathrm{d}(u,v). \tag{6}$$

Now calculate

$$\frac{\partial G}{\partial u} = g_x + g_z\,\phi_u + (h_x + h_z\,\phi_u)\,\phi_v + h\,\phi_{uv}; \tag{7}$$

$$\frac{\partial F}{\partial v} = f_y + f_z\,\phi_v + (h_y + h_z\,\phi_v)\,\phi_u + h\,\phi_{vu}. \tag{8}$$

This gives

$$\begin{aligned}
\int_D \left[\frac{\partial G}{\partial u} - \frac{\partial F}{\partial v}\right] \mathrm{d}(u,v) &= \int_D (g_z - h_y)\,\phi_u + (h_x - f_z)\,\phi_v + (g_x - f_y)\,\mathrm{d}(u,v) \\
&= \int_D \begin{pmatrix} h_y - g_z \\ f_z - h_x \\ g_x - f_y \end{pmatrix} \cdot \begin{pmatrix} -\phi_u \\ -\phi_v \\ 1 \end{pmatrix} \mathrm{d}(u,v) \\
&= \int_S \begin{pmatrix} h_y - g_z \\ f_z - h_x \\ g_x - f_y \end{pmatrix} \cdot \mathbf{d}\boldsymbol{S} \\
&= \int_S (h_y - g_z)\,\mathrm{d}y \wedge \mathrm{d}z + (f_z - h_x)\,\mathrm{d}z \wedge \mathrm{d}x + (g_x - f_y)\,\mathrm{d}x \wedge \mathrm{d}y.
\end{aligned} \tag{9}$$

Thus ends the proof. □

1.2. Calculations with Stokes's Theorem

Example 5. (FOLLAND) Calculate

$$\int_C \sqrt{x^2+1}\,\mathrm{d}x + x\,\mathrm{d}y + 2\,y\,\mathrm{d}z \tag{10}$$

where C is the intersection of $z = x\,y$ and $x^2 + y^2 = 1$, oriented counterclockwise when viewed from above.

Exercise 1. Try to visualize the surface!

Solution. We apply Stokes's Theorem to the surface $z = x\,y$ inside $x^2 + y^2 = 1$. The natural parametrization is now $(u, v, u\,v)$ with $D = \{(u,v) | \, u^2 + v^2 \leqslant 1\}$. Then we have

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = \pm \begin{pmatrix} -y \\ -x \\ 1 \end{pmatrix}. \tag{11}$$

Since the curve is oriented counterclockwise when viewed from above, $\boldsymbol{n}$ points upward and therefore we pick the positive sign.

Straightforward calculation now gives

$$\int_C \sqrt{x^2+1}\,\mathrm{d}x + x\,\mathrm{d}y + 2\,y\,\mathrm{d}z = \int_{x^2+y^2\leqslant 1} (1-2\,y)\,\mathrm{d}(x,y) = \pi. \tag{12}$$

Example 6. (Demidovich 2356) Apply Stokes' theorem to calculate

$$\int_C (y+z)\,\mathrm{d}x + (z+x)\,\mathrm{d}y + (x+y)\,\mathrm{d}z \tag{13}$$

where C is the circle $x^2+y^2+z^2=a^2, x+y+z=0$, oriented counter-clockwise when viewing from above.

Solution. We choose S to be the intersection of $x^2+y^2+z^2 \leqslant a^2$ and $x+y+z=0$. To be consistent with the orientation of ∂S, we choose $\boldsymbol{n}$ to point upward, that is $\boldsymbol{n}=(1,1,1)/\sqrt{3}$. Simple calculation gives

$$\int_C (y+z)\,\mathrm{d}x + (z+x)\,\mathrm{d}y + (x+y)\,\mathrm{d}z = \int_S \mathbf{0}\cdot\mathbf{d}\boldsymbol{S} = 0. \tag{14}$$

Example 7. (Demidovich 2357) Calculate

$$\int_C (y-z)\,\mathrm{d}x + (z-x)\,\mathrm{d}y + (x-y)\,\mathrm{d}z \tag{15}$$

where C is the ellipse $x^2+y^2=1$, $x+z=1$. Oriented counter-clockwise when viewing from the positive x axis.

Solution. We take S to be $x^2+y^2\leqslant 1$ cut bo $x+z=1$. Then the orientation of the curve implies $\boldsymbol{n}$ pointing upward. That is $\boldsymbol{n}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$.

Now we calculate

$$I = \int_S \begin{pmatrix}-2\\-2\\-2\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}\mathrm{d}S = \int_S \left(-2\sqrt{2}\right)\mathrm{d}S = -2\sqrt{2}\cdot\text{Area of the ellipse} = -4\,\pi. \tag{16}$$

This curve is easy to parametrize so we check our result through direct calculation.

Solution. (Direct Calculation) We parametrize

$$x=\cos\theta, y=\sin\theta, z=1-\cos\theta. \qquad \theta\in[0,2\,\pi]. \tag{17}$$

Note that this is consistent with the specified orientation.

Nowe calculate

$$\begin{aligned} I &= \int_0^{2\pi} [(\sin\theta - 1 + \cos\theta)(-\sin\theta) + (1-2\cos\theta)\cos\theta + (\cos\theta - \sin\theta)\sin\theta]\,\mathrm{d}\theta \\ &= \int_0^2 (-2)\,\mathrm{d}\theta = -4\,\pi. \end{aligned} \tag{18}$$

Example 8. (Demidovich 2358) Calculate

$$\int_C x\,\mathrm{d}x + (x+y)\,\mathrm{d}y + (x+y+z)\,\mathrm{d}z \tag{19}$$

where C is given through $x=\sin t$, $y=\cos t$, $z=\sin t+\cos t$, $t\in[0,2\,\pi]$.

Solution. To apply Stokes's Theorem we need to find a surface S such that $\partial S = C$ and further choose the normal $\boldsymbol{n}$ according to the given orientation of C.

Notice that $z = x + y$ so we choose

$$S: z = x + y, \qquad D: x^2 + y^2 \leqslant 1. \tag{20}$$

The normal $\boldsymbol{n}$ should be chosen from either $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ or $\frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. We notice that as t increases, (x, y, z) moves clockwise along ∂S when viewed from above. Therefore $\boldsymbol{n}$ should point downward and we should choose $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

Now we have

$$\begin{aligned} \int_C x\,\mathrm{d}x + (x+y)\,\mathrm{d}y + (x+y+z)\,\mathrm{d}z &= \int_S \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \boldsymbol{n}\,\mathrm{d}S \\ &= \int_D \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \mathrm{d}(x, y) \\ &= -\pi. \end{aligned} \tag{21}$$

Of course the integral can be more easily calculated directly:

Solution. (DIRECT CALCULATION) We have

$$\begin{aligned} \int_C x\,\mathrm{d}x + (x+y)\,\mathrm{d}y + (x+y+z)\,\mathrm{d}z &= \int_0^{2\pi} [\sin t \cos t + (\sin t + \cos t)\,(-\sin t) + 2\,(\sin t + \cos t)\,(\cos t - \sin t)]\,\mathrm{d}t \\ &= \int_0^{2\pi} [2\,(\cos t)^2 - 3\,(\sin t)^2]\,\mathrm{d}t = -\pi. \end{aligned} \tag{22}$$

Example 9. Prove the following:

Let S be a closed surface (that is no boundary) in $\mathbb{R}^3$ with unit outward normal $\boldsymbol{n}$. Let $\begin{pmatrix} f \\ g \\ h \end{pmatrix}$ be C^1. Then

$$\int_S \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y = 0. \tag{23}$$

Proof. Take any $\boldsymbol{x}_0 \in S$. Take $\varepsilon > 0$. Denote

$$S_\varepsilon := S \cap \{\|\boldsymbol{x} - \boldsymbol{x}_0\| \geqslant \varepsilon\}. \tag{24}$$

When ε is small enough we have $\partial S_\varepsilon \subseteq S \cap \{\|\boldsymbol{x} - \boldsymbol{x}_0\| = \varepsilon\}$. Then we have, by Stokes's Theorem, the integral equals

$$\int_{\partial S_\varepsilon} f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z + \int_{-\partial S_\varepsilon} f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z = 0. \tag{25}$$

Here $-\partial S_\varepsilon$ denotes ∂S_ε with orientation reversed. $\square$

2. GAUSS'S THEOREM (DIVERGENCE THEOREM)

2.1. The theorem and its proof

Gauss's Theorem transforms between a surface integral and a volume integral. The simplest situation is on an interval.

LEMMA 10. *Let* $\boldsymbol{f} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$ *be* C^1 *on an interval* $I := [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$. *Then*

$$\int_{\partial I} f\,\mathrm{d}y \wedge \mathrm{d}z + g\,\mathrm{d}z \wedge \mathrm{d}x + h\,\mathrm{d}x \wedge \mathrm{d}y = \int_I \left[\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right] \mathrm{d}(x, y, z). \tag{26}$$

Proof. It clearly suffices to prove

$$\int_{\partial I} f\,\mathrm{d}y \wedge \mathrm{d}z = \int_I \frac{\partial f}{\partial x}\,\mathrm{d}(x, y, z); \quad \int_{\partial I} g\,\mathrm{d}z \wedge \mathrm{d}x = \int_I \frac{\partial g}{\partial y}\,\mathrm{d}(x, y, z); \quad \int_{\partial I} h\,\mathrm{d}x \wedge \mathrm{d}y = \int_I \frac{\partial h}{\partial z}\,\mathrm{d}(x, y, z) \tag{27}$$

The three identities can be proved similarly, therefore we prove the first one only.

It is clear that

$$\begin{aligned}
\int_{\partial I} f\,\mathrm{d}y\,\mathrm{d}z &= \int_{\text{top}} f\,\mathrm{d}y\,\mathrm{d}z + \int_{\text{bottom}} f\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_{[0,1]^2} [f(1, u, v) - f(0, u, v)]\,\mathrm{d}(u, v) \\
&= \int_{[0,1]^2} \left[\int_0^1 \frac{\partial f}{\partial x}(s, u, v)\,\mathrm{d}s\right] \mathrm{d}(u, v) \\
&= \int_I \frac{\partial f}{\partial x}(s, u, v)\,\mathrm{d}(s, u, v) \\
&= \int_I \frac{\partial f}{\partial x}(x, y, z)\,\mathrm{d}(x, y, z).
\end{aligned} \tag{28}$$

Note that we have used Fundamental Theorem of Calculus Version 1 and Fubini. □

Exercise 2. Without doing any differential or integral calculation, prove that if (26) holds for all $\boldsymbol{f}$ as specified in Lemma 10, then necessarily (27) holds.

Exercise 3. Prove (26) with I replaced by the unit simplex:

$$V := \{(x, y, z) | \, x \geqslant 0, y \geqslant 0, z \geqslant 0, x + y + z \leqslant 1\}. \tag{29}$$

In general, we have

THEOREM 11. (GAUSS'S THEOREM) *Let* V *be a compact region with* ∂V *a union of finitely many* C^1 *surfaces. Let* $\boldsymbol{f} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$ *be* C^1 *on* V. *Then*

$$\int_{\partial D} f\,\mathrm{d}y \wedge \mathrm{d}z + g\,\mathrm{d}z \wedge \mathrm{d}x + h\,\mathrm{d}x \wedge \mathrm{d}y = \int_D \left[\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right] \mathrm{d}(x, y, z). \tag{30}$$

Proof. See §4.2.2. □

Warning: Regularity requirement on f cannot be dropped!

It is easy to overlook the requirement that $\boldsymbol{f} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$ must be C^1 not only on the boundary ∂V, but also inside V. However this assumption is crucial.

Exercise 4. Resolve the following paradox:

We try to calculate

$$\int_S \frac{1}{z} \, \mathrm{d}x \wedge \mathrm{d}y \tag{31}$$

where S is the unit sphere oriented by the outer normal via two methods: directly or through Gauss's Theorem.

- Direction calculation: It is clear that $n_z > 0$ when $z > 0$ and <0 when $z < 0$. Therefore
$$\int_S \frac{1}{z} \, \mathrm{d}x \wedge \mathrm{d}y = \int_S \frac{n_z}{z} \, \mathrm{d}S > 0. \tag{32}$$
- Through Gauss's Theorem: We take V to be the unit ball and calculate
$$\int_S \frac{1}{z} \, \mathrm{d}x \wedge \mathrm{d}y = \int_V \frac{\partial}{\partial z}\left(\frac{1}{z}\right) \mathrm{d}(x, y, z) = -\int_V \frac{\mathrm{d}(x, y, z)}{z^2} < 0. \tag{33}$$

This understanding is also key to the theory of fundamental solutions/Green's functions in linear partial differential equations.

Remark 12. (RELATION TO GREEN'S THEOREM) If we consider a two-dimensional domain $D \subseteq \mathbb{R}^2$ and formally write the line integral

$$\int_{\partial D} \begin{pmatrix} f \\ g \end{pmatrix} \cdot \boldsymbol{n} \, \mathrm{d}s \tag{34}$$

where $\boldsymbol{n}$ is the unit normal vector pointing outward with respect to D, then since $\begin{pmatrix} n_x \\ n_y \end{pmatrix} = \begin{pmatrix} T_y \\ -T_x \end{pmatrix}$ where $\boldsymbol{T}$ is the unit tangent vector so that when moving in the direction of $\boldsymbol{T}$, D is always to the left of the boundary, we have by Green's Theorem,

$$\int_{\partial D} \begin{pmatrix} f \\ g \end{pmatrix} \cdot \begin{pmatrix} n_x \\ n_y \end{pmatrix} \mathrm{d}s = \int_{\partial D} \begin{pmatrix} -g \\ f \end{pmatrix} \cdot \begin{pmatrix} T_x \\ T_y \end{pmatrix} \mathrm{d}s = \int_{\partial D} -g \, \mathrm{d}x + f \, \mathrm{d}y = \int_D \left[\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right] \mathrm{d}(x, y) \tag{35}$$

Thus we see that Green's Theorem is formally Gauss's Theorem in dimension two.

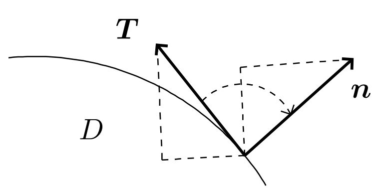

Figure 1. $\boldsymbol{n}$ is obtained from $\boldsymbol{T}$ by turning $\boldsymbol{T}$ clockwise by 90 degrees

2.2. Calculations with Gauss's Theorem

When S is the boundary of a region, $\boldsymbol{n}$ is always taken to be the outer-normal unless otherwise specified.

Example 13. Let $V \subset \mathbb{R}^3$ be smooth enough for the application of Gauss's Theorem. Then its volume is given by the following surface integral:

$$\mathrm{Vol} = \int_{\partial V} x\,\mathrm{d}y \wedge \mathrm{d}z = \int_{\partial V} y\,\mathrm{d}z \wedge \mathrm{d}x = \int_{\partial V} z\,\mathrm{d}x \wedge \mathrm{d}y = \frac{1}{3}\int_{\partial V} x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y. \tag{36}$$

Proof. We have

$$\int_{\partial V} x\,\mathrm{d}y \wedge \mathrm{d}z = \int_{\partial V} \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \cdot \mathbf{d}\boldsymbol{S} = \int_V [1+0+0]\,\mathrm{d}(x,y,z) \tag{37}$$

and the conclusion follows. □

Example 14. Calculate

$$I := \int_{\partial V} \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} \cdot \mathbf{d}\boldsymbol{S} \tag{38}$$

where V is the unit cube $\{(x,y,z) | 0 \leqslant x,y,z \leqslant 1\}$.

Solution. We have by Gauss's Theorem (and then Fubini),

$$\begin{aligned} I &= \int_V [2\,x+2\,y+2\,z]\,\mathrm{d}(x,y,z) \\ &= \int_0^1 \left[\int_0^1 \left[\int_0^1 (2\,x+2\,y+2\,z)\,\mathrm{d}z\right]\mathrm{d}y\right]\mathrm{d}x \\ &= 3. \end{aligned} \tag{39}$$

Example 15. Calculate

$$I := \int_{\partial V} x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + x^2\,\mathrm{d}x \wedge \mathrm{d}y \tag{40}$$

where V is the pyramid bounded by $x+y+z=1, x=0, y=0, z=0$.

Solution. By Gauss's Theorem we have

$$I = \int_V [1+1+0]\,\mathrm{d}(x,y,z) = 2 \times \text{Volume of the pyramid} = \frac{1}{3}. \tag{41}$$

Example 16. (DEMIDOVICH 2367) Calculate

$$I := \int_{\partial V} x^3\,\mathrm{d}y \wedge \mathrm{d}z + y^3\,\mathrm{d}z \wedge \mathrm{d}x \tag{42}$$

where V is the unit ball.

Solution. By Gauss's Theorem we have

$$I = \int_{\partial V} \begin{pmatrix} x^3 \\ y^3 \\ 0 \end{pmatrix} \cdot \mathbf{d}\boldsymbol{S} = \int_V 3\,(x^2 + y^2)\,\mathrm{d}(x, y). \tag{43}$$

Now change into spherical coordinates:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos\phi\cos\psi \\ \rho\sin\phi\cos\psi \\ \rho\sin\psi \end{pmatrix} \Longrightarrow \left|\det \frac{\partial(x, y, z)}{\partial(\rho, \phi, \psi)}\right| = \rho^2\cos\psi, \qquad \rho \in [0, 1], \phi \in [0, 2\,\pi], \psi \in [-\pi/2, \pi/2], \tag{44}$$

We have

$$I = 3\int_{[0,1]\times[0,2\pi]\times[-\pi/2,\pi/2]} \rho^2 \cdot \rho^2 \cos\psi\,\mathrm{d}(\rho, \phi, \psi) = \frac{12\,\pi}{5}. \tag{45}$$

Example 17. (DEMIDOVICH 2369) Let $S = \partial V$ be a closed surface and $\boldsymbol{l} \in \mathbb{R}^3$ be a constant vector. Prove that

$$\int_S \cos(\boldsymbol{n}, \boldsymbol{l})\,\mathrm{d}S = 0 \tag{46}$$

where $\boldsymbol{n}$ is the outernormal and $(\boldsymbol{n}, \boldsymbol{l})$ is the angle between the two vectors $\boldsymbol{n}$ and $\boldsymbol{l}$.

Proof. Denote $\boldsymbol{l} = \begin{pmatrix} l_x \\ l_y \\ l_z \end{pmatrix}$. Recall that

$$\cos(\boldsymbol{n}, \boldsymbol{l}) := \frac{\boldsymbol{n}\cdot\boldsymbol{l}}{\|\boldsymbol{n}\|\,\|\boldsymbol{l}\|} = \boldsymbol{n}\cdot\frac{\boldsymbol{l}}{\|\boldsymbol{l}\|}. \tag{47}$$

Thus we apply Gauss's Theorem:

$$\int_S \cos(\boldsymbol{n}, \boldsymbol{l})\,\mathrm{d}S = \int_S \frac{\boldsymbol{l}}{\|\boldsymbol{l}\|} \cdot \mathbf{d}\boldsymbol{S} = \int_V \left[\frac{\partial}{\partial x}\left(\frac{l_x}{\|\boldsymbol{l}\|}\right) + \frac{\partial}{\partial y}\left(\frac{l_y}{\|\boldsymbol{l}\|}\right) + \frac{\partial}{\partial z}\left(\frac{l_z}{\|\boldsymbol{l}\|}\right)\right]\mathrm{d}(x, y, z) = 0 \tag{48}$$

since $\boldsymbol{l}$ is a constant vector. □

2.3. Applications of Gauss's Theorem

Gauss's Theorem is crucial in classical mathematical physics. We will give one example here. More discussion about this can be found in the section on continuum mechanics in Week 10's notes, as well as any introductory partial differential equations book, for example Tyn Myint-U, Lokenath Debnath, *Linear Partial Differential Equations for Scientists and Engineers*, 4ed, Birkhäuser, 2007, Chapters 3 & 13.

Example 18. (DERIVATION OF HEAT EQUATION) Consider a domain Ω with boundary $\partial\Omega$. Let $u(x, y, z, t)$ be the temperature at location (x, y, z) and time t. Now consider any region $V \subseteq \Omega$. The total "heat" in V can be represented as

$$\int_V u(x, y, z, t)\,\mathrm{d}(x, y, z). \tag{49}$$

Now the change of "heat" in V is caused by "heat flows" into and out from V:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V u(x, y, z, t)\,\mathrm{d}(x, y, z) = \text{Rate of heat flow across } \partial V. \tag{50}$$

The rate of the flow of heat at every space-time point can be models as a vector function $\boldsymbol{F}(x, y, z, t) = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$. Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V u(x,y,z,t)\,\mathrm{d}(x,y,z) = -\int_{\partial V} \boldsymbol{F}(x,y,z,t)\cdot \mathbf{d}\boldsymbol{S}. \tag{51}$$

Exercise 5. Explain the negative sign.

Now we exchange the order of differentiation and integration on the left hand side and apply Gauss's Theorem on the right hand side:

$$\int_V \frac{\partial u(x,y,z,t)}{\partial t}\,\mathrm{d}(x,y,z) = \int_V -\left[\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right]\mathrm{d}(x,y,z). \tag{52}$$

If we now assume the continuity of both integrands, the arbitrariness of V leads to

$$\frac{\partial u}{\partial t} = -\left[\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right]. \tag{53}$$

To move the modeling further we need a "constitutive" relation between u and $\boldsymbol{F}$. Such a relation does not come from mathematical argument but from either experiments or more fundamental physical laws such as statistical or quantum mechanics. In the case of heat flow, we use Fourier's law:

$$\boldsymbol{F} = -\kappa\,(\mathrm{grad}\,u) \tag{54}$$

where $\kappa > 0$ is called the "thermal conductivity" of the material. Note that the negative sign is because heat flows from high temperature to low temperatute.

Putting everything together, we have

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\kappa\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa\frac{\partial u}{\partial z}\right) \tag{55}$$

which is called the "heat equation".

When the body consists of one single material, we can assume that κ is a constant and further simplify the equation to

$$\frac{\partial u}{\partial t} = \kappa\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]. \tag{56}$$

Remark 19. In modeling heat flow, it is obvious that $\kappa > 0$ should be enforced, since $\kappa < 0$ would mean "heat" flows from low temperature to high temperature – the hot gets hotter and the cold gets colder, which is ridiculous.

However once (55) has been formulated it has its own mathematical existence not confined by the physical theory of heat conduction. Thus it is not a meaningless question to study the equation with $\kappa < 0$. In fact it turned out to be very meaningful: In 1987 Pietro Perona of Caltech and his then Ph.D. student Jitendra Malik realized that (55) with $\kappa < 0$ provides a good framework to guide algorithms sharpening the edges in a digital image. Since then the "Perona-Malik equation" has become one of the most important PDEs in applied mathematics.

3. GRADIENT, CURL, DIVERGENCE IN $\mathbb{R}^3$

3.1. Gradient; Divergence; Curl

3.1.1. Gradient in $\mathbb{R}^3$

DEFINITION 20. (GRADIENT) *Let $A \subseteq \mathbb{R}^3$ be open and $f: A \mapsto \mathbb{R}$ be differentiable. Let $\boldsymbol{x}_0 \in A$. The matrix representation of the differential $(Df)(\boldsymbol{x}_0)$ is called the gradient of f at $\boldsymbol{x}_0$, denoted $(\operatorname{grad} f)(\boldsymbol{x}_0)$.*

LEMMA 21. *Let $f: \mathbb{R}^3 \mapsto \mathbb{R}$ be differentiable and let $\boldsymbol{x}: [a, b] \mapsto \mathbb{R}^3$ be a C^1 curve. Then for any $t_0 \in (a, b)$,*

$$f'(\boldsymbol{x}(t))|_{t=t_0} = (\operatorname{grad} f)(\boldsymbol{x}(t_0)) \cdot \boldsymbol{x}'(t_0). \tag{57}$$

Proof. By the chain rule we have

$$f'(\boldsymbol{x}(t))|_{t=t_0} = [Df(\boldsymbol{x}(t_0))](\boldsymbol{x}'(t_0)) = (\operatorname{grad} f)(\boldsymbol{x}(t_0)) \cdot \boldsymbol{x}'(t_0) \tag{58}$$

Thus ends the proof. □

COROLLARY 22. *Let $f: \mathbb{R}^3 \mapsto \mathbb{R}$ be differentiable. Then for any $\boldsymbol{x}_0 \in \mathbb{R}^3$,*

$$(\operatorname{grad} f)(\boldsymbol{x}_0) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} (\boldsymbol{x}_0). \tag{59}$$

Proof. Exercise. □

Remark 23. Clearly the above generalizes to $\mathbb{R}^N$, where we have

$$(\operatorname{grad} f)(\boldsymbol{x}_0) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{pmatrix} (\boldsymbol{x}_0). \tag{60}$$

3.1.2. Divergence

DEFINITION 24. (DIVERGENCE) *Let $\boldsymbol{f} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}: \mathbb{R}^3 \mapsto \mathbb{R}^3$ be C^1. Then its divergence at any $\boldsymbol{x}_0 \in \mathbb{R}^3$ is defined as,*

$$(\operatorname{div} \boldsymbol{f})(\boldsymbol{x}_0) = \frac{\partial f}{\partial x}(\boldsymbol{x}_0) + \frac{\partial g}{\partial y}(\boldsymbol{x}_0) + \frac{\partial h}{\partial z}(\boldsymbol{x}_0). \tag{61}$$

Remark 25. We note that Gauss's Theorem now can be written more compactly as

$$\int_D \operatorname{div} \boldsymbol{f} \, \mathrm{d}\boldsymbol{x} = \int_{\partial D} \boldsymbol{f} \cdot \boldsymbol{n} \, \mathrm{d}S. \tag{62}$$

LEMMA 26. *Let $A\subseteq\mathbb{R}^3$ be open and let $\boldsymbol{f}: A\mapsto\mathbb{R}^3$ be C^1. Let S be any closed C^1 surface enclosing $\boldsymbol{x}_0\in A$. Denote by V the region enclosed by S. Then we have*

$$(\operatorname{div}\boldsymbol{f})(\boldsymbol{x}_0)=\lim_{d(S)\longrightarrow 0}\frac{1}{\mu(V)}\int_S \boldsymbol{f}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x} \tag{63}$$

the divergence of $\boldsymbol{f}$ at $\boldsymbol{x}_0$.

Proof. Let $\varepsilon>0$ be arbitrary. Since $\boldsymbol{f}$ is C^1, $\operatorname{div}\boldsymbol{f}$ is continuous at $\boldsymbol{x}_0$. Consequently there is $\delta>0$ such that

$$\|\boldsymbol{x}-\boldsymbol{x}_0\|<\delta\Longrightarrow|(\operatorname{div}\boldsymbol{f})(\boldsymbol{x})-(\operatorname{div}\boldsymbol{f})(\boldsymbol{x}_0)|<\varepsilon. \tag{64}$$

Now consider any surface with $d(S)<\delta$. We have

$$\left|\frac{1}{\mu(V)}\int_S \boldsymbol{f}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x}-(\operatorname{div}\boldsymbol{f})(\boldsymbol{x}_0)\right|=\frac{1}{\mu(V)}\left|\int_D[(\operatorname{div}\boldsymbol{f})(\boldsymbol{x})-(\operatorname{div}\boldsymbol{f})(\boldsymbol{x}_0)]\right|<\varepsilon. \tag{65}$$

Thus (63) holds. □

3.1.3. Curl

DEFINITION 27. (CURL) *Let $A\subseteq\mathbb{R}^3$ be open and let $\boldsymbol{f}: A\mapsto\mathbb{R}^3$. Let $\boldsymbol{x}_0\in A$. We define*

$$(\operatorname{curl}\boldsymbol{f})(\boldsymbol{x}_0)=\begin{pmatrix}\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\\ \frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\\ \frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\end{pmatrix}. \tag{66}$$

Remark 28. (PHYSICAL MEANING OF $\operatorname{curl}\boldsymbol{f}$) Consider a vector field $\boldsymbol{f}(x,y,z)$. Fix a point (x_0, y_0, z_0). We are interested in the deviation of $\boldsymbol{f}$ from its zeroth order approximation $\boldsymbol{f}_0:=\boldsymbol{f}(x_0,y_0,z_0)$. Taylor expansion to first order gives

$$\boldsymbol{f}(x,y,z)-\boldsymbol{f}_{\mathbf{0}}\sim A\begin{pmatrix}x-x_0\\ y-y_0\\ z-z_0\end{pmatrix}=:A\,\boldsymbol{\delta x}. \tag{67}$$

where $A:=\frac{\partial(f,g,h)}{\partial(x,y,z)}(x_0,y_0,z_0)$ is the Jacobian matrix of $\boldsymbol{f}$ at (x_0,y_0,z_0). Thus if we imagine that the space is filled with particles moving with velocity $\boldsymbol{f}$, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x(t)\\ y(t)\\ z(t)\end{pmatrix}=\boldsymbol{f}(x(t),y(t),z(t)), \tag{68}$$

then modulo the uniform flow with velocity $\boldsymbol{f}_0$ (that is if we pick the particle moving with $\boldsymbol{f}_0$ as the origin of our reference frame), the particles are moving with velocity $\sim A\,\boldsymbol{\delta x}$.

Now define $V:=\frac{1}{2}(A+A^T)$ and $\Omega:=\frac{1}{2}(A-A^T)$. We clearly have $A=V+\Omega$ which means

$$\boldsymbol{f}(x,y,z)-\boldsymbol{f}_{\mathbf{0}}\sim V\boldsymbol{\delta x}+\Omega\,\boldsymbol{\delta x}. \tag{69}$$

From linear algebra we know that as V is symmetric it has the decomposition $V=O^T\,\Lambda\,O$ where O is orthogonal and $\Lambda=\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}$ is diagonal. Therefore the first part $V\boldsymbol{\delta x}$ "stretches" and/or "compresses" any volume of particles.

On the other hand,

Exercise 6. Check that for any $\boldsymbol{v} \in \mathbb{R}^3$,

$$2\,\Omega\,\boldsymbol{v} = (\text{curl}\,\boldsymbol{f}) \times \boldsymbol{v}, \tag{70}$$

thus we see that curl $\boldsymbol{f}$ represents twice the angular velocity of the flow (relative to the base point moving with $\boldsymbol{f}_0$). See http://en.wikipedia.org/wiki/Vorticity for some nice illustrations.

Example 29. (ABC FLOW) An important vector field in dynamical system and fluid mechanics is the following "ABC flow":

$$\boldsymbol{f} = \begin{pmatrix} A\sin z + C\cos y \\ B\sin x + A\cos z \\ C\sin y + B\cos x \end{pmatrix}. \tag{71}$$

We easily check

$$\text{curl}\,\boldsymbol{f} = \boldsymbol{f}. \tag{72}$$

Thus $\boldsymbol{f}$ is an "eigen-function" of the curl operator with eigenvalue 1.

Exercise 7. Find $\boldsymbol{f}: \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that curl $\boldsymbol{f} = \lambda\,\boldsymbol{f}$ for $\lambda \in \mathbb{R}$ arbitrarily given. (Hint:[1])

Remark 30. Note that Stokes's Theorem now can be written as

$$\int_S (\text{curl}\,\boldsymbol{f}) \cdot \mathbf{d}\boldsymbol{S} = \int_{\partial S} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{l}. \tag{73}$$

Exercise 8. Explain why it is a bit subtle to try to obtain a result based on Stokes's Theorem, in the same spirit as Lemma 26. (Hint:[2]) (Also see Exercise 44.)

Remark 31. We notice that formally,

$$\text{grad}\,f = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} f, \qquad \text{div}\,\boldsymbol{f} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \boldsymbol{f}, \qquad \text{curl}\,\boldsymbol{f} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \boldsymbol{f}. \tag{74}$$

Thus if we denote

$$\nabla := \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}, \tag{75}$$

we can simply write

$$\text{grad}\,f = \nabla f, \qquad \text{div}\,\boldsymbol{f} = \nabla \cdot \boldsymbol{f}, \qquad \text{curl}\,\boldsymbol{f} = \nabla \times \boldsymbol{f}. \tag{76}$$

Indeed such notations are used in much of the literature. However one should keep in mind that such formal equivalence only holds when we use the Cartesian coordinates. See §4.3 for some discussion on this issue in $\mathbb{R}^3$. A fully satisfactory theory of grad, div, and curl will be developed in the theory of differential forms.

1. Replace x, y, z by $\lambda\,x, \lambda\,y, \lambda\,z$.
2. We need some extra control over the normal $\boldsymbol{n}$ of the shrinking surface.

We emphasize again that this relation between curl and cross product is only true in $\mathbb{R}^3$ with Cartesian coordinates.

How to remember curl (How to remember the cross product formula).

Thanks to (74) we see that to remember the formula for curl, it suffices to remember the cross product formula

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2\, b_3 - a_3\, b_2 \\ a_3\, b_1 - a_1\, b_3 \\ a_1\, b_2 - a_2\, b_1 \end{pmatrix}. \tag{77}$$

There are many ways to remember (77):

- Remember that the first term in the first component is $a_2\, b_3$. Then the subscripts for the positive terms in the 2nd and 3rd components are $a_?\, b_?$ where ? appears in the order $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$.
- Let $i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, then

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \det \begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}; \tag{78}$$

- Write

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{pmatrix}. \tag{79}$$

  Then the three components of $\boldsymbol{a} \times \boldsymbol{b}$ are the determinants of 2×2 matrices formed by the 2nd,3rd; 3rd, 4th; 4th,5th columns, respectively, of (79);

3.2. Examples

Example 32. (Demidovich 2393) Evaluate the divergence and the flux of an attractive force $F = -\frac{m\,\boldsymbol{r}}{r^3}$ where $\boldsymbol{r} := \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $r := \|\boldsymbol{r}\|$ of a point mass m, located at the coordinate origin, through an arbitrary closed surface.

Solution. Let S be the closed surface, we need to calculate

$$\int_S F \cdot \mathbf{d}\boldsymbol{S}. \tag{80}$$

The plan is to apply Gauss's Theorem. Denote by V the region enclosed by S. In the following we assume the orientation of S is the outernormal of S as the boundary of V. The answer would take the opposite sign if S is oriented by the inner normal.

It is easy to see that $F \in C^1(\mathbb{R}^3 - \{0\})$ and $\operatorname{div} F = 0$ except at the origin. Therefore we need to separately discuss two cases: $0 \in V^o$ and $0 \in (\bar{V})^c$.[3]

- $0 \in V^o$. There is $\varepsilon > 0$ such that the closed ball centered at 0 with radius ε is contained in V^o. Denote this ball by B_ε and its surface with outer normal by S_ε. Denote $V_\varepsilon = V - B_\varepsilon$. Then we have

$$\begin{aligned} \int_S F \cdot \mathbf{d}\boldsymbol{S} &= \int_S F \cdot \mathbf{d}\boldsymbol{S} - \int_{S_\varepsilon} F \cdot \mathbf{d}\boldsymbol{S} + \int_{S_\varepsilon} F \cdot \mathbf{d}\boldsymbol{S} \\ &= \int_{\partial V_\varepsilon} F \cdot \mathbf{d}\boldsymbol{S} + \int_{S_\varepsilon} F \cdot \mathbf{d}\boldsymbol{S} \\ &= \int_{V_\varepsilon} [\operatorname{div} F]\, \mathrm{d}(x, y, z) + \int_{S_\varepsilon} F \cdot \mathbf{d}\boldsymbol{S} \\ &= \int_{S_\varepsilon} F \cdot \mathbf{d}\boldsymbol{S}. \end{aligned} \tag{81}$$

3. The calculation is subtle when $0 \in S$ and we do not discuss it here.

To calculate the resulting integral, notice that on S_ε, the outer normal is given by

$$\boldsymbol{n}_\varepsilon = \frac{\boldsymbol{r}}{r} = \frac{\boldsymbol{r}}{\varepsilon}. \tag{82}$$

Thus we have

$$\begin{aligned}\int_{S_\varepsilon} F \cdot \mathbf{d}\boldsymbol{S} &= \int_{S_\varepsilon} -\frac{m\,\boldsymbol{r}}{r^3} \cdot \frac{\boldsymbol{r}}{r}\,\mathrm{d}S \\ &= \int_{S_\varepsilon} -\frac{m}{r^2}\,\mathrm{d}S \\ &= -m \int_{S_\varepsilon} \frac{\mathrm{d}S}{\varepsilon^2} \\ &= -m\,\varepsilon^{-2}\,\mathrm{Area}(S_\varepsilon) \\ &= -4\,\pi\,m. \end{aligned} \tag{83}$$

Thus the flux is $-4\,\pi\,m$ for all closed surfaces enclosing 0.

- $0 \in (\bar{V})^c$. This case is simpler and left as exercise. The answer is 0.

Remark 33. Clearly the above conclusion applies to gravity and electricity fields.

On the other hand, the fact that $\operatorname{div} F = 0$ on $\mathbb{R}^3 - \{0\}$ is crucial to the above calculation. We easily check

Exercise 9. If $F: \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a central force field, that is $F(\boldsymbol{r}) = f(r)\,\boldsymbol{r}$, then $\operatorname{div} F = 0$ on $\mathbb{R}^3 - \{0\}$ only if $f(r) = k\,r^{-3}$ where k is constant. (Hint:[4])

3.3. Vector identities

THEOREM 34. *Let f, g be scalar functions and $\boldsymbol{f}, \boldsymbol{g}$ be vector functions. All of them C^1. Then*

$$\begin{aligned}\operatorname{grad}(f\,g) &= f \operatorname{grad} g + g \operatorname{grad} f; && (84)\\ \operatorname{grad}(\boldsymbol{f}\cdot\boldsymbol{g}) &= (\boldsymbol{f}\cdot\nabla)\boldsymbol{g} + \boldsymbol{f}\times(\operatorname{curl}\boldsymbol{g}) + (\boldsymbol{g}\cdot\nabla)\boldsymbol{f} + \boldsymbol{g}\times(\operatorname{curl}\boldsymbol{f}); && (85)\\ \operatorname{curl}(f\,\boldsymbol{g}) &= f \operatorname{curl}\boldsymbol{g} + (\operatorname{grad} f)\times\boldsymbol{g}; && (86)\\ \operatorname{curl}(\boldsymbol{f}\times\boldsymbol{g}) &= (\boldsymbol{g}\cdot\nabla)\boldsymbol{f} + (\operatorname{div}\boldsymbol{g})\,\boldsymbol{f} - (\boldsymbol{f}\cdot\nabla)\boldsymbol{g} - (\operatorname{div}\boldsymbol{f})\,\boldsymbol{g}; && (87)\\ \operatorname{div}(f\,\boldsymbol{g}) &= f \operatorname{div}\boldsymbol{g} + (\operatorname{grad} f)\cdot\boldsymbol{g}; && (88)\\ \operatorname{div}(\boldsymbol{f}\times\boldsymbol{g}) &= \boldsymbol{g}\cdot(\operatorname{curl}\boldsymbol{f}) - \boldsymbol{f}\cdot(\operatorname{curl}\boldsymbol{g}). && (89)\end{aligned}$$

Are you surprised by the minus sign in (89)?

Here $\boldsymbol{f}\cdot\nabla := f\,\frac{\partial}{\partial x} + g\,\frac{\partial}{\partial y} + h\,\frac{\partial}{\partial z}$.

Proof. Direct calculation. □

THEOREM 35. *Let $f, \boldsymbol{f}$ be C^2. Then*

$$\begin{aligned}\operatorname{curl}(\operatorname{grad} f) &= \mathbf{0}; && (90)\\ \operatorname{div}(\operatorname{curl}\boldsymbol{f}) &= 0; && (91)\\ \operatorname{div}(\operatorname{grad} f) &= \triangle f; && (92)\\ \operatorname{curl}(\operatorname{curl}\boldsymbol{f}) &= \operatorname{grad}(\operatorname{div}\boldsymbol{f}) - \triangle\boldsymbol{f}. && (93)\end{aligned}$$

Here $\triangle := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ *is the Laplacian.*

Proof. Direct calculation. □

4. $\operatorname{div} F = (\operatorname{grad} f)\cdot\boldsymbol{r} + f\,(\operatorname{div}\boldsymbol{r}) = r\,f'(r) + 3\,f$. Solve the ODE for $r \in (0, \infty)$.

How to Quickly Derive Vector Identities.

There are two useful tricks for deriving the above identities quickly when needed.

1. Use subscripts with Einstein's summation convention – repeated indices imply summation. For example,

Note: i is repeated!

$$[\mathrm{grad}\,(f\,g)]_i = \partial_i(f\,g) = f\,\partial_i g + g\,\partial_i f \Longrightarrow \mathrm{grad}(f\,g) = f\,\mathrm{grad}\,g + g\,\mathrm{grad}\,f; \tag{94}$$

$$\mathrm{div}\,(f\,\boldsymbol{g}) = \partial_i(f\,g_i) = (\partial_i f)\,g_i + f\,(\partial_i g_i) = f\,\mathrm{div}\,\boldsymbol{g} + (\mathrm{grad}\,f)\cdot\boldsymbol{g} \tag{95}$$

   This is enough for identities involve grad and div . To be able to deal with curl, we need the next trick.

2. Use Kronecker delta and alternating tensor.

   The Kronecker delta and alternating tensor are defined as

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}; \qquad \epsilon_{ijk} = \begin{cases} 1 & (i,j,k) = (1,2,3), (2,3,1), (3,1,2) \\ -1 & (i,j,k) = (1,3,2), (2,1,3), (3,2,1) \\ 0 & \text{otherwise} \end{cases}. \tag{96}$$

Exercise 10. Verify

$$\delta_{ij} = \delta_{ji}; \tag{97}$$

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji}. \tag{98}$$

Exercise 11. Verify:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_i\,b_i = \delta_{ij}\,a_i\,b_j; \quad (\boldsymbol{a}\times\boldsymbol{b})_i = \epsilon_{ijk}\,a_j\,b_k; \quad (\mathrm{curl}\,\boldsymbol{f})_i = \epsilon_{ijk}\,\partial_i\,f_k. \tag{99}$$

Exercise 12. Verify:

$$\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}. \tag{100}$$

With the help of (97 – 100), deriving identities involving curl becomes straightforward. For exmaple,

$$\begin{aligned} \mathrm{div}\,(\boldsymbol{f}\times\boldsymbol{g}) &= \partial_i(\epsilon_{ijk}\,f_j\,g_k) \\ &= \epsilon_{ijk}\,[(\partial_i f_j)\,g_k] + \epsilon_{ijk}\,[f_j\,(\partial_i g_k)] \\ &= g_k\,[\epsilon_{ijk}\,\partial_i\,f_j] + f_j\,[\epsilon_{ijk}\,\partial_i\,g_k] \\ &= g_k\,[\epsilon_{kij}\,\partial_i\,f_j] + f_j\,[-\epsilon_{jik}\,\partial_i\,g_k] \\ &= g_k\,[\mathrm{curl}\,f]_k + f_j\,[-\mathrm{curl}\,g]_j \\ &= \boldsymbol{g}\cdot(\mathrm{curl}\,\boldsymbol{f}) - \boldsymbol{f}\cdot(\mathrm{curl}\,\boldsymbol{g}). \end{aligned} \tag{101}$$

Exercise 13. Prove the vector identities using the two tricks introduced above.

Exercise 14. Prove

$$\epsilon_{ijk}\,\epsilon_{lmn} = \det\begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jm} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{pmatrix}. \tag{102}$$

(Hint:[5])

5. Take full advantage of (98) and the symmetry of determinant.

4. Advanced Topics, Notes, and Comments

4.1. Triangulation of C^1 surfaces

4.1.1. What is wrong with Schwarz's triangulation

We consider the possibility of approximating a surface with a union of triangles whose vertices are points in the surface. This is called "triangulation" of the surface. The goal is to approximate not only the surface, but also the normal vectors. However the example of H. A. Schwarz indicates that this cannot be accomplished by simply requiring

$$d(S_n) \longrightarrow 0 \tag{103}$$

where S_n is the approximating surface and $d(S_n)$ is the largest length of sides of all triangles in S_n.

It turns out that, there is one more requirement on the triangles. To see what it is, we first try to understand why (103) is not enough.

Consider a piece of C^1 surface S parametrized by $\boldsymbol{r}(u,v) := \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}: D \mapsto \mathbb{R}^3$. Wlog assume $(0,0) \in D$ and $(0,0,0) \in S$. Now consider the triangle given by

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} x(u_1,v_1) \\ y(u_1,v_1) \\ z(u_1,v_1) \end{pmatrix} \left(:= \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \right) - \begin{pmatrix} x(u_2,v_2) \\ y(u_2,v_2) \\ z(u_2,v_2) \end{pmatrix} \left(:= \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right). \tag{104}$$

Denote the triangle $\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$ by P. Let $h_1 := \left\| \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \right\|$ and $h_2 := \left\| \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\|$. Now clearly $d(P) \longrightarrow 0$ is equivalent to $h_1, h_2 \longrightarrow 0$. Note that no relation between the speed of $h_1, h_2 \longrightarrow 0$ is assumed.

Now we compare the normal vector of the surface at $(0,0)$:

$$\frac{\boldsymbol{r}_u(0,0) \times \boldsymbol{r}_v(0,0)}{\|\boldsymbol{r}_u(0,0) \times \boldsymbol{r}_v(0,0)\|} \tag{105}$$

and the normal vector of the triangle

$$\frac{\boldsymbol{r}_1 \times \boldsymbol{r}_2}{\|\boldsymbol{r}_1 \times \boldsymbol{r}_2\|}, \qquad \boldsymbol{r}_i := \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}. \tag{106}$$

By Taylor expansion and continuity of $D\boldsymbol{r}$ we have

$$\boldsymbol{r}_1 = \boldsymbol{r}_u(0,0)\, u_1 + \boldsymbol{r}_v(0,0)\, v_1 + \boldsymbol{R}_1; \qquad \boldsymbol{r}_2 = \boldsymbol{r}_u(0,0)\, u_2 + \boldsymbol{r}_v(0,0)\, v_2 + \boldsymbol{R}_2 \tag{107}$$

Here

$$\lim_{(u_i,v_i) \to (0,0)} \frac{\|\boldsymbol{R}_i\|}{h_i} = 0. \tag{108}$$

However note that there is no information on the directions of $\boldsymbol{R}_i$.

Now we have

$$\boldsymbol{r}_1 \times \boldsymbol{r}_2 = \boldsymbol{r}_u(0,0) \times \boldsymbol{r}_v(0,0)\, [A(P)] + \boldsymbol{R} \tag{109}$$

where

$$\lim_{d(P) \longrightarrow 0} \frac{\|\boldsymbol{R}\|}{h_1 h_2} = 0. \tag{110}$$

Thus we have

$$\boldsymbol{r}_u(0,0) \times \boldsymbol{r}_v(0,0) = [A(P)]^{-1}\, \boldsymbol{r}_1 \times \boldsymbol{r}_2 + [A(P)]^{-1}\, \boldsymbol{R}. \tag{111}$$

Now it is clear that, for $\frac{\boldsymbol{r}_1 \times \boldsymbol{r}_2}{\|\boldsymbol{r}_1 \times \boldsymbol{r}_2\|}$ to approach the true normal vector, we need

$$\lim_{d(P) \longrightarrow 0} \frac{\|\boldsymbol{R}\|}{A(P)} = 0. \tag{112}$$

As

$$A(P) = \frac{h_1 h_2 \sin\theta}{2} \tag{113}$$

where θ is the angle between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, we see that (112) does not hold if $\theta \longrightarrow 0$ which is exactly the case for Schwarz's triangulation.

Smooth surfaces are more than collection of points.

As we have shown above, the problem with Schwarz's triangulation is that, the surfaces formed by these triangles converges to the target surface as sets of points, but not as smooth surfaces. In particular, the normal vectors of the approximating surfaces do not converge to the normal vectors of the target surface.

On the other hand, once we add a condition preventing $\theta \rightarrow 0$, there is no problem approximating surfaces through triangulation.

Lemma 36. *Let S be a piece of C^1 surface parametrized by $\boldsymbol{r}(u,v) := \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}: D \mapsto \mathbb{R}^3$. Let P be a triangulation of D. Let S_P be the corresponding triangulation of S with the corresponding parametrization $\boldsymbol{r}_n(u,v): D_P \mapsto \mathbb{R}^3$. Assume that there is $\theta_0 > 0$ such that any angle in any triangle of P is between θ_0 and $\pi - \theta_0$. Then*

$$\lim_{d(P) \longrightarrow 0} \int_{D_P \cap D} (\|\boldsymbol{r}_n(u,v) - \boldsymbol{r}(u,v)\| + \|D\boldsymbol{r}_n(u,v) - D\boldsymbol{r}(u,v)\|)\, \mathrm{d}(u,v) = 0. \tag{114}$$

Proof. Exercise. □

4.1.2. Approximating area and integration

Theorem 37. *Let S_n, S be pieces of C^1 surfaces in $\mathbb{R}^3$ and $\boldsymbol{r}_n: D_n \subseteq \mathbb{R}^2 \mapsto \mathbb{R}^3, \boldsymbol{r}: D \subseteq \mathbb{R}^2 \mapsto \mathbb{R}^3$ be the corresponding C^1 parametrizations. Assume that $\mu(D_n \triangle D) \longrightarrow 0$ [6] and*

$$\lim_{n \rightarrow \infty} \int_{D_n \cap D} (\|\boldsymbol{r}_n(u,v) - \boldsymbol{r}(u,v)\| + \|D\boldsymbol{r}_n(u,v) - D\boldsymbol{r}(u,v)\|)\, \mathrm{d}(u,v) = 0. \tag{115}$$

Then

$$\mathrm{Area}(S_n) \longrightarrow \mathrm{Area}(S). \tag{116}$$

Proof. Exercise. □

Corollary 38. *Let S_n, S satisfy the conditions above. Let $\begin{pmatrix} f \\ g \\ h \end{pmatrix}: \mathbb{R}^3 \mapsto \mathbb{R}^3$ be continuous. Then*

$$\lim_{n \longrightarrow \infty} \int_{S_n} f\, \mathrm{d}y\, \mathrm{d}z + g\, \mathrm{d}z\, \mathrm{d}z + h\, \mathrm{d}x\, \mathrm{d}y = \int_S f\, \mathrm{d}y\, \mathrm{d}z + g\, \mathrm{d}z\, \mathrm{d}z + h\, \mathrm{d}x\, \mathrm{d}y. \tag{117}$$

Proof. Exercise. □

Remark 39. From the above discussion we see that it is important for the triangulation to avoid long, thin triangles. One technique of achieving this is the theory/algorithm of Delaunay triangulation. It has wide application in all kinds of computations involving surfaces, such as computational fluid mechanics and computer graphics/computer vision.[7]

6. $A \triangle B := (A - B) \cup (B - A)$.

4.2. Proofs of Stokes's and Gauss's Theorems

4.2.1. Proof of Stokes's Theorem

First as a special case of what we proved in §1.1, we

LEMMA 40. *We have, when $S \subset \mathbb{R}^3$ is a triangle, $\boldsymbol{f} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}: \mathbb{R}^3 \mapsto \mathbb{R}^3$ C^1,*

$$\int_{\partial S} f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z = \int_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x \tag{118}$$

Exercise 15. Prove the above through choosing a new orthogonal coordinate frame x', y', z' such that the triangle lies in the $x'y'$ plane. And then use Green's formula.

COROLLARY 41. *Let S be a finite union of triangles, then (118) holds.*

Proof. It suffices to prove (1) for one piece of C^1 surface given by parametrization $\boldsymbol{r}(u, v): D \mapsto \mathbb{R}^3$. Let P be any triangulation of D such that there is $\theta_0 > 0$ such that any angle in any triangle of P is between $(\theta_0, \pi - \theta_0)$, and let S_P be the corresponding triangulation of S. Then Lemma 40 implies that (1) holds for S_P. Now notice that ∂S_n is a polygonal approximation of ∂S, which means

$$\lim_{d(P) \longrightarrow 0} \int_{\partial S_P} f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z = \int_{\partial S} f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z; \tag{119}$$

On the other hand, by the theory developed in §4.1, we have

$$\lim_{d(P) \longrightarrow 0} \int_{S_P} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x \tag{120}$$

equals the right hand side of (1). Thus ends the proof. □

4.2.2. Proof of Gauss's Theorem

Proof for simple regions.

This time we consider the domain $D := I \cap P_1 \cap P_2 \cap \ldots. \cap P_m$ where I is a compact interval and $P_1, \ldots., P_m$ are half spaces. Since this can be turned into a special case of "simple regions" that are used in most advanced calculus books, we will just follow them and prove the theorem for "simple regions".

DEFINITION 42. (REGULAR REGIONS) *A region is called xy-simple if it can be written as*

$$D = \{(x, y, z) |\, (x, y) \in W, \varphi_1(x, y) \leqslant z \leqslant \varphi_2(x, y)\}. \tag{121}$$

A region is called simple if it is simultaneous xy- yz- and xz- simple.

LEMMA 43. *Let D be a compact convex set in $\mathbb{R}^3$. Then D is simple.*

Proof. Denote by W the projection of D onto the xy plane:

$$W := \{(x, y) |\, \exists z, (x, y, z) \in D\}. \tag{122}$$

We prove that for any $(x_0, y_0) \in W$, $D \cap \{(x, y, z) |\, x = x_0, y = y_0\}$ is a closed interval. Let

$$a := \min_{z \in D \cap \{(x,y,z) | x = x_0, y = y_0\}} z, \qquad b := \max_{D \cap \{(x,y,z) | x = x_0, y = y_0\}} z. \tag{123}$$

7. Triangulation is also important in Game programming. For example, compare first person shooters from early 1990s (no triangulation), late 1990s (triangulation with thousands of triangles) and after 2000 (triangulation with tens of thousands of triangles or even more) to see the big difference in the quality of the 3D graphics.

Note that the minimum and maximum exist because D is compact and is therefore closed. Thus we have

$$(x_0, y_0, a), (x_0, y_0, b) \in D. \tag{124}$$

Since D is convex, this implies

$$\{(x_0, y_0, z) |\, z \in [a, b]\} \subseteq D \cap \{(x, y, z) |\, x = x_0, y = y_0\}. \tag{125}$$

On the other hand, by (123) we have

$$D \cap \{(x, y, z) |\, x = x_0, y = y_0\} \subseteq \{(x_0, y_0, z) |\, z \in [a, b]\}. \tag{126}$$

Therefore

$$D \cap \{(x, y, z) |\, x = x_0, y = y_0\} = \{(x_0, y_0, z) |\, z \in [a, b]\}. \tag{127}$$

Now for each $(x, y) \in W$ if we define $\varphi_1(x, y), \varphi_2(x, y)$ to be the corresponding a, b, we would have (121) and consequently D is xy-simple. That it is yz- and zx- simple are proved similarly. □

Lemma 44. *Let D be a simple region. Let $\boldsymbol{f} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$ be C^1 on D. Then*

$$\int_{\partial D} f\, \mathrm{d}y \wedge \mathrm{d}z + g\, \mathrm{d}z \wedge \mathrm{d}x + h\, \mathrm{d}x \wedge \mathrm{d}y = \int_D \left[\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right] \mathrm{d}(x, y, z). \tag{128}$$

Proof. We prove

$$\int_{\partial D} h\, \mathrm{d}x \wedge \mathrm{d}y = \int_D \frac{\partial h}{\partial z}\, \mathrm{d}(x, y, z) \tag{129}$$

and leave the other two (almost identical) cases as exercises.

Note that it suffices to prove (129) for regions of the form $D = \{(x, y, z) |\, (x, y) \in W, 0 \leqslant z \leqslant \varphi(x, y)\}$. So we will restrict ourselves to this situation.

We denote

$$S_1 := \{(x, y, z) |\, (x, y) \in W, z = \varphi(x, y)\}; \quad S_2 := \{(x, y, z) |\, (x, y) \in \partial W, 0 \leqslant z \leqslant \varphi(x, y)\} \tag{130}$$

and $S_3 := W \times \{z = 0\}$.

First it is easy to calculate

$$\begin{aligned} \int_D \frac{\partial h}{\partial z}\, \mathrm{d}(x, y, z) &= \int_W \left[\int_0^{\varphi(x, y)} \frac{\partial h}{\partial z}\, \mathrm{d}z \right] \mathrm{d}(x, y) \\ &= \int_W h(x, y, \varphi(x, y))\, \mathrm{d}(x, y) - \int_W h(x, y, 0)\, \mathrm{d}(x, y). \end{aligned} \tag{131}$$

On the other hand, we have $\partial D = S_1 \cup S_2 \cup S_3$ and

$$\int_{\partial D} h\, \mathrm{d}x \wedge \mathrm{d}y = \int_{S_1} h\, \mathrm{d}x \wedge \mathrm{d}y + \int_{S_2} h\, \mathrm{d}x \wedge \mathrm{d}y + \int_{S_3} h\, \mathrm{d}x \wedge \mathrm{d}y. \tag{132}$$

Recalling the definition of the integrals, we have

$$\int_{S_2} h\, \mathrm{d}x \wedge \mathrm{d}y = \int_{S_2} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \cdot \boldsymbol{n}\, \mathrm{d}S = 0, \tag{133}$$

and

$$\int_{S_3} h\, \mathrm{d}x \wedge \mathrm{d}y = \int_{S_3} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \mathrm{d}S = -\int_W h(x, y, 0)\, \mathrm{d}(x, y). \tag{134}$$

Finally, on S_1 we have

$$\boldsymbol{n}(x,y,\varphi(x,y)) = \frac{1}{\sqrt{1+\varphi_x^2+\varphi_y^2}}\begin{pmatrix} -\varphi_x \\ -\varphi_y \\ 1 \end{pmatrix} \tag{135}$$

which gives

$$\begin{aligned}
\int_{S_1} h\,\mathrm{d}x\wedge\mathrm{d}y &= \int_{S_1}\begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\cdot\begin{pmatrix} -\varphi_x \\ -\varphi_y \\ 1 \end{pmatrix}\frac{\mathrm{d}S}{\sqrt{1+\varphi_x^2+\varphi_y^2}} \\
&= \int_{S_1} h(x,y,z)\frac{\mathrm{d}S}{\sqrt{1+\varphi_x^2+\varphi_y^2}} \\
&= \int_W h(x,y,\varphi(x,y))\sqrt{1+\varphi_x^2+\varphi_y^2}\,\frac{\mathrm{d}(x,y)}{\sqrt{1+\varphi_x^2+\varphi_y^2}} \\
&= \int_W h(x,y,\varphi(x,y))\,\mathrm{d}(x,y). \qquad (136)
\end{aligned}$$

Thus ends the proof. □

Exercise 16. Show that it suffices to prove (129) for regions of the form $D=\{(x,y,z)|(x,y)\in W, 0\leqslant z\leqslant\varphi(x,y)\}$.

Exercise 17. Finish the proof of Lemma 44.

From Lemmas 43 and 44, we easily obtain the following.

COROLLARY 45. *Let $D=I\cap P_1\cap\ldots.\cap P_m$ such that I is a compact interval and $P_1,\ldots.,P_m$ are half space. Then the formula (128) holds.*

From Corollary 45 the following is immediate:

THEOREM 46. *Let D be a compact polyhedron. Let $\boldsymbol{f}=\begin{pmatrix} f \\ g \\ h \end{pmatrix}$ be C^1 on D. Then*

$$\int_{\partial D} f\,\mathrm{d}y\wedge\mathrm{d}z + g\,\mathrm{d}z\wedge\mathrm{d}x + h\,\mathrm{d}x\wedge\mathrm{d}y = \int_D\left[\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right]\mathrm{d}(x,y,z). \tag{137}$$

Proof for the general case.

Proof. All we need to show is that there is a sequence of polyhedra D_n such that

$$\int_{\partial D_n}\boldsymbol{f}\cdot\boldsymbol{n}\,\mathrm{d}S \to \int_{\partial D}\boldsymbol{f}\cdot\boldsymbol{n}\,\mathrm{d}S,\ \int_{D_n}\left[\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right]\mathrm{d}(x,\,y,\,z) \to \int_D\left[\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right]\mathrm{d}(x,\,y,\,z). \tag{138}$$

From §4.1.2 we see that for C^1 surfaces the former is true for any triangulation S_n with $d(S_n)\longrightarrow 0$ and the smallest angle of any triangle in S_n is bounded away from 0. On the other hand, for any $\varepsilon>0$, there is are closed set E and open set F such that

$$E\subset D^o\subset\bar{D}\subset F \tag{139}$$

and $\mu(F-E)<\frac{\varepsilon}{M}$ where $M:=\sup_{x\in\bar{D}}\|\boldsymbol{f}(\boldsymbol{x})\|$. Now if $d(S_n)\longrightarrow 0$, we have $d(S_n,\partial D)\longrightarrow 0$ which means that, if we denote by D_n the region enclosed by S_n, for n large we always have

$$E\subset D_n\subset F \Longrightarrow \left|\int_{D_n}\left[\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right]\mathrm{d}(x,y,z) - \int_D\left[\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right]\mathrm{d}(x,y,z)\right| < \varepsilon. \tag{140}$$

Thus ends the proof. □

4.3. Curvilinear coordinates

4.3.1. Polar coordinates

We start by considering the simple case of 2D gradient and divergence operators. In $\mathbb{R}^2$, we have

$$\operatorname{grad} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}, \qquad \operatorname{div} \begin{pmatrix} f \\ g \end{pmatrix} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \tag{141}$$

which can be formally written as (denote $\boldsymbol{f} := \begin{pmatrix} f \\ g \end{pmatrix}$)

$$\operatorname{grad} f = \nabla f, \qquad \operatorname{div} \boldsymbol{f} = \nabla \cdot \boldsymbol{f} \tag{142}$$

where $\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$. Now what if we switch to polar coordinates? We will see that this formal correspondence between gradient and divergence disappears – there is no longer a (vector) differential operator $\boldsymbol{D}$ such that one can formally write

$$\operatorname{grad} f = \boldsymbol{D} f, \qquad \operatorname{div} \boldsymbol{f} = \boldsymbol{D} \cdot \boldsymbol{f}. \tag{143}$$

First consider gradient. Since we are using polar coordinates, the function f is represented as $f(r, \theta)$. Now the the first thing we need to clarify is that we do not use polar coordinates to represent $(\operatorname{grad} f)(r, \theta)$. Instead we establish a Euclidean coordinate system at the point (r, θ) with

$$\boldsymbol{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \boldsymbol{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} -y \\ x \end{pmatrix}. \tag{144}$$

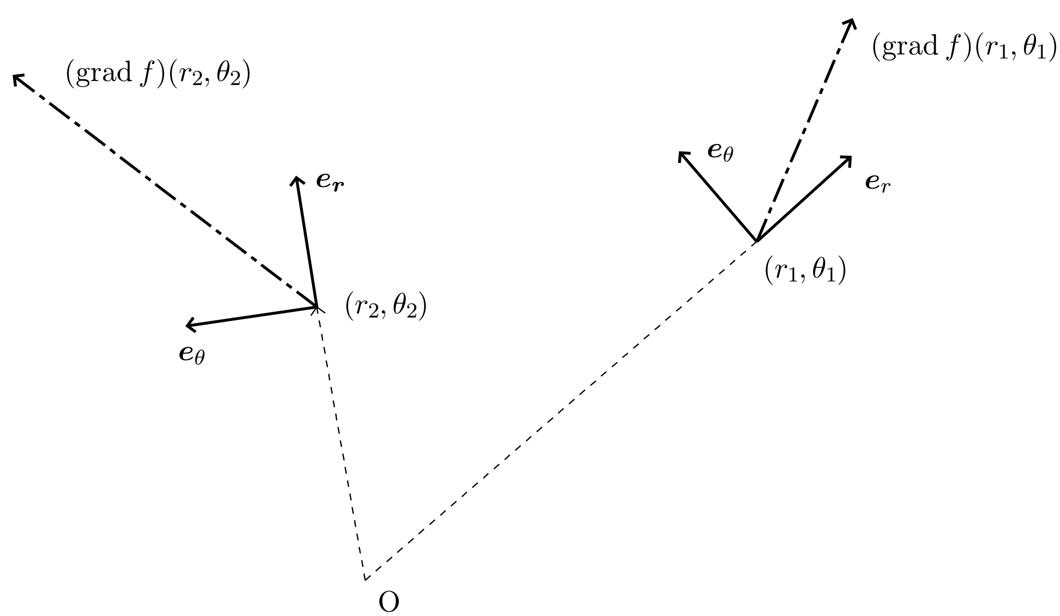

Figure 2. The natural coordinate system at (r, θ)

and try to represent

$$(\mathrm{grad}\, f)(r,\theta) = f_1(r,\theta)\, \boldsymbol{e}_r + f_2(r,\theta)\, \boldsymbol{e}_\theta. \tag{145}$$

Note that $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ change with (r,θ).

Tangent plane.

The reason why this is natural and necessary, is that the vector $\mathrm{grad}\, f$ does not "live" in the plane on which f is defined. It lives in a new plane that is centered at (r,θ) and overlaid on top of the original plane. In other words, at every (r,θ), a new "tangent plane" is defined with (r,θ) as its origin. If we consider a curve C in the underlying plane, then at $\boldsymbol{x} \in C$, the tangent vectors to C does not belong to the underlying plane but belongs to the tangent plane at $\boldsymbol{x}$.

It turns out that the coordinate system in the original plane induces a "natural" coordinate system on every tangent plane. More specifically, consider the original plane with coordinates

$$x = X(u,v), \qquad y = Y(u,v), \tag{146}$$

then at $(x_0, y_0) := (X(u_0, v_0), Y(u_0, v_0))$, the natural induced coordinate system for the tangent plane there are given by the unit tangent vectors for the curves

$$\begin{pmatrix} X(u, v_0) \\ Y(u, v_0) \end{pmatrix} \text{ and } \begin{pmatrix} X(u_0, v) \\ Y(u_0, v) \end{pmatrix}. \tag{147}$$

Exercise 18. Prove that the natural induced coordinate system for polar coordinates is given by (144).

The separation between the underlying plane and the tangent planes is much more intuitive if we replace the underlying plane by a curved surface. It turns out that even in the current situation it is still beneficial to keep this distinction.

Now we carry out the calculation. Consider $f(r,\theta) = f(r(x,y), \theta(x,y))$. Then we have

$$\mathrm{grad}\, f = \frac{\partial f}{\partial r}\,\mathrm{grad}\, r + \frac{\partial f}{\partial \theta}\,\mathrm{grad}\,\theta. \tag{148}$$

Some calculation leads to

$$\mathrm{grad}\, r = \mathrm{grad}\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{e}_r; \tag{149}$$

$$\mathrm{grad}\,\theta = \frac{1}{r}\,\boldsymbol{e}_\theta. \tag{150}$$

Exercise 19. Calculate $\mathrm{grad}\,\theta$. (Hint:[8])

This gives

$$\mathrm{grad}\, f = \frac{\partial f}{\partial r}\,\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\boldsymbol{e}_\theta. \tag{151}$$

Exercise 20. Let $(r(t), \theta(t))$ be a curve in $\mathbb{R}^2$. Let $\boldsymbol{x}_0 = \begin{pmatrix} r(t_0)\cos(t_0) \\ r(t_0)\sin(t_0) \end{pmatrix}$ be on the curve. Let f be C^1. Use

$$\frac{\mathrm{d}}{\mathrm{d}t} f(r(t)\cos\theta(t), r(t)\sin\theta(t)) = (\mathrm{grad}\, f)(\text{tangent of } C \text{ at } \boldsymbol{x}_0) \tag{152}$$

8. Differentiate $\cos\theta = x/r$ and $\sin\theta = y/r$.

to derive (151).

On the other hand, for reasons that will be clear in Differential Geometry, when considering divergence we have to consider

$$\boldsymbol{f} = f\,\boldsymbol{e}_r + g\,\boldsymbol{e}_\theta. \tag{153}$$

Treating everything as functions of x, y, we have

$$\operatorname{div}\boldsymbol{f} = (\nabla f)\cdot\boldsymbol{e}_r + f\,(\operatorname{div}\boldsymbol{e}_r) + (\nabla g)\cdot\boldsymbol{e}_\theta + g\,(\operatorname{div}\boldsymbol{e}_\theta). \tag{154}$$

Exercise 21. Prove that

$$(\nabla f)\cdot\boldsymbol{e}_r = \frac{\partial f}{\partial r}. \tag{155}$$

On the other hand, direct calculation gives

$$\operatorname{div}\boldsymbol{e}_r = \frac{1}{r}, \qquad \operatorname{div}\boldsymbol{e}_\theta = 0. \tag{156}$$

Exercise 22. Prove (156).

Summarizing, we have

$$\operatorname{div}\boldsymbol{f} = \frac{\partial f}{\partial r} + \frac{f}{r} + \frac{\partial g}{\partial \theta}. \tag{157}$$

Note that, formally,

$$\operatorname{grad} f = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \end{pmatrix} f; \qquad \operatorname{div}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial r} + \frac{1}{r} \\ \frac{\partial}{\partial\theta} \end{pmatrix}\cdot\begin{pmatrix} f \\ g \end{pmatrix} \tag{158}$$

and clearly the operators

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \end{pmatrix} \neq \begin{pmatrix} \frac{\partial}{\partial r} + \frac{1}{r} \\ \frac{\partial}{\partial\theta} \end{pmatrix}! \tag{159}$$

Definition 47. (Laplacian) *Arguably the most important differential operator, the Laplacian is defined as*

$$\triangle f := \operatorname{div}(\operatorname{grad} f). \tag{160}$$

Exercise 23. Prove that in the usual Cartesian coordinates,

$$\triangle f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_N^2}. \tag{161}$$

Exercise 24. Let $f: \mathbb{R}^2 \mapsto \mathbb{R}$ be C^2 and depending on r only. Prove that

$$\triangle f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}. \tag{162}$$

Exercise 25. Let $f: \mathbb{R}^3 \mapsto \mathbb{R}$ be C^2 and depending on r only. Prove that

$$\triangle f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}. \tag{163}$$

What is the formula for $f: \mathbb{R}^N \mapsto \mathbb{R}$? (Hint:[9])

9. $\operatorname{div}\boldsymbol{e}_r = \operatorname{div}\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right)$ depends on dimension.

grad , curl ,div : *They act on different spaces!*

Why is it that when calculating $\operatorname{grad} f$ we write $f(r,\theta)$, while when calculating $\operatorname{div} \boldsymbol{f}$ we write $\boldsymbol{f} = f(r,\theta)\,\boldsymbol{e}_r + g(r,\theta)\,\boldsymbol{e}_\theta$ instead of $\boldsymbol{f} = \begin{pmatrix} f \\ g \end{pmatrix}$ or $\boldsymbol{f} = \begin{pmatrix} R(r,\theta) \\ \Theta(r,\theta) \end{pmatrix}$? The reason is that, while grad is an operator applied to (scalar) functions defined on the (r,θ) plane, div as an operator does not act on such functions. For now let's just say it acts on vector functions defined on the tangent plane. Therefore we have to write $\boldsymbol{f} = f(r,\theta)\,\boldsymbol{e}_r + g(r,\theta)\,\boldsymbol{e}_\theta$.

More precisely,

- grad acts on functions;
- div acts on $(N-1)$-forms;
- curl acts on one-forms.

On the other hand, we can try to force these operators to act on vectors through the Hodge star operator $\star$ and the musical isomorphisms $\sharp, \flat$ as follows (although this does not change the "natural spaces" on which these operations live)

$$\begin{aligned} \operatorname{grad} f &:= (\mathrm{d}f)^\sharp; && (164)\\ \operatorname{div} \boldsymbol{f} &:= \star \mathrm{d}\big(\star \boldsymbol{f}^\flat\big); && (165)\\ \operatorname{curl} \boldsymbol{f} &:= \big[\star\big(\mathrm{d}\boldsymbol{f}^\flat\big)\big]^\sharp && (166)\\ \triangle f &:= \star \mathrm{d}(\star \mathrm{d}\boldsymbol{f}). && (167) \end{aligned}$$

Here f, $\boldsymbol{f}$ are functions. The $\sharp$ operator maps a one-form (essentially the differential of f as defined in 217) to the corresponding vector in the tangent plane; The $\flat$ operator is its dual which maps a vector to a one-form; The Hodge star operator takes a m-form and maps it to a $(N-m)$-form, through the so-called "volume form".

4.3.2. General curvilinear coordinates in $\mathbb{R}^3$

We consider general curvilinear coordinates:

$$\boldsymbol{x} = \boldsymbol{r}(\boldsymbol{u}): \qquad \begin{aligned} x &= X(u,v,w); && (168)\\ y &= Y(u,v,w); && (169)\\ z &= Z(u,v,w). && (170) \end{aligned}$$

The "natural" coordinate vectors at each point is given by:

$$\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_w. \tag{171}$$

LEMMA 48. *Let* $R := \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{pmatrix} \boldsymbol{r}_u & \boldsymbol{r}_v & \boldsymbol{r}_w \end{pmatrix}$ *and* $A := \begin{pmatrix} \nabla u & \nabla v & \nabla w \end{pmatrix}$. *We have*

$$R\,A^T = I. \tag{172}$$

Proof. Differentiate (168–170) and the result follows. $\square$

First we try to obtain the formula for gradient.

$$\operatorname{grad} f(u,v,w) = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w = U \begin{pmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \\ \frac{\partial f}{\partial w} \end{pmatrix}. \tag{173}$$

Thanks to (172) we have

$$A\,R^T = I \Longrightarrow A = R^{-T}. \tag{174}$$

Lemma 49. *Let* $A = (\, \boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3 \,)$ *be a non-singular* 3×3 *matrix. Then*

$$A^{-T} = \frac{1}{\det A}\,(\, \boldsymbol{a}_2\times\boldsymbol{a}_3 \;\; \boldsymbol{a}_3\times\boldsymbol{a}_1 \;\; \boldsymbol{a}_1\times\boldsymbol{a}_2 \,). \tag{175}$$

Exercise 26. Prove that

$$\det A = \boldsymbol{a}_1\cdot(\boldsymbol{a}_2\times\boldsymbol{a}_3) = \boldsymbol{a}_2\cdot(\boldsymbol{a}_3\times\boldsymbol{a}_1) = \boldsymbol{a}_3\cdot(\boldsymbol{a}_1\times\boldsymbol{a}_2). \tag{176}$$

Proof. Direct calculation. Left as exercise. □

Thus we have

$$\nabla u = \frac{\boldsymbol{r}_v\times\boldsymbol{r}_w}{J}, \qquad \nabla v = \frac{\boldsymbol{r}_w\times\boldsymbol{r}_u}{J}, \qquad \nabla w = \frac{\boldsymbol{r}_u\times\boldsymbol{r}_v}{J}, \tag{177}$$

where

$$J := \det\left(\frac{\partial(x,y,z)}{\partial(u,v,w)}\right). \tag{178}$$

Therefore

$$\operatorname{grad} f(u,v,w) = \frac{1}{J}\left[\frac{\partial f}{\partial u}(\boldsymbol{r}_v\times\boldsymbol{r}_w) + \frac{\partial f}{\partial v}(\boldsymbol{r}_w\times\boldsymbol{r}_u) + \frac{\partial f}{\partial w}(\boldsymbol{r}_u\times\boldsymbol{r}_v)\right]. \tag{179}$$

Similarly we calculate

$$\operatorname{div} \boldsymbol{f}(u,v,w) = \frac{1}{J}\left[\frac{\partial \boldsymbol{f}}{\partial u}\cdot(\boldsymbol{r}_v\times\boldsymbol{r}_w) + \frac{\partial \boldsymbol{f}}{\partial v}\cdot(\boldsymbol{r}_w\times\boldsymbol{r}_u) + \frac{\partial \boldsymbol{f}}{\partial w}\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)\right] \tag{180}$$

which can be further simplified to

$$\operatorname{div} \boldsymbol{f}(u,v,w) = \frac{1}{J}\left[\frac{\partial}{\partial u}[\boldsymbol{f}\cdot(\boldsymbol{r}_v\times\boldsymbol{r}_w)] + \frac{\partial}{\partial v}[\boldsymbol{f}\cdot(\boldsymbol{r}_w\times\boldsymbol{r}_u)] + \frac{\partial}{\partial w}[\boldsymbol{f}\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)]\right]. \tag{181}$$

Exercise 27. Prove through direct calculation:

$$\frac{\partial}{\partial u}(\boldsymbol{r}_v\times\boldsymbol{r}_w) + \frac{\partial}{\partial v}(\boldsymbol{r}_w\times\boldsymbol{r}_u) + \frac{\partial}{\partial w}(\boldsymbol{r}_u\times\boldsymbol{r}_v) = 0. \tag{182}$$

(Hint:[10])

4.3.3. General orthogonal curvilinear coordinates in $\mathbb{R}^3$

The situation can be significantly simplified when $\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{r}_w$ form an orthogonal set, that is

$$\boldsymbol{r}_u\cdot\boldsymbol{r}_v = \boldsymbol{r}_v\cdot\boldsymbol{r}_w = \boldsymbol{r}_w\cdot\boldsymbol{r}_u = 0. \tag{183}$$

. In this case, we write

$$\boldsymbol{r}_u = U\,\boldsymbol{e}_u, \qquad \boldsymbol{r}_v = V\boldsymbol{e}_v, \qquad \boldsymbol{r}_w = W\,\boldsymbol{e}_w \tag{184}$$

where $\boldsymbol{e}_u$, $\boldsymbol{e}_v$, $\boldsymbol{e}_w$ form an orthonormal set and furthermore $\det(\, \boldsymbol{e}_u \;\; \boldsymbol{e}_v \;\; \boldsymbol{e}_w \,) = 1$. Thus (179), (181) simplify to

$$\operatorname{grad} f(u,v,w) = \frac{1}{U}\frac{\partial f}{\partial u}\boldsymbol{e}_u + \frac{1}{V}\frac{\partial f}{\partial v}\boldsymbol{e}_v + \frac{1}{W}\frac{\partial f}{\partial w}\boldsymbol{e}_w; \tag{185}$$

and

$$\operatorname{div} \boldsymbol{f}(u,v,w) = \frac{1}{UVW}\left[\frac{\partial}{\partial u}\left(\frac{VW}{U}\,\boldsymbol{f}\cdot\boldsymbol{e}_u\right) + \frac{\partial}{\partial v}\left(\frac{UW}{V}\,\boldsymbol{f}\cdot\boldsymbol{e}_v\right) + \frac{\partial}{\partial w}\left(\frac{UV}{W}\,\boldsymbol{f}\cdot\boldsymbol{e}_w\right)\right]. \tag{186}$$

10. Remember that $\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$.

4.4. Potential field

We have the following fact.

- Let C be a closed curve that is the union of finitely many C^1 curves. Let D be the region enclosed by C. Let f be C^1 on D. Then we have

$$\boldsymbol{g} = \nabla f \Longrightarrow \int_C \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{l} = 0. \tag{187}$$

- Let A be open and $\boldsymbol{g}: A \mapsto \mathbb{R}^3$ be such that for every closed curve $C \subset A$ that is piecewise C^1, $\int_C \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{l} = 0$. Then there is $f: A \mapsto \mathbb{R}$ such that $\boldsymbol{g} = \nabla f$.

Exercise 28. Prove these facts.

Exercise 29. Show that it is not enough to assume f to be C^1 on A for some open set A containing C.

When studying a function, it is often convenient if it is the gradient of another function. In the following we will try to find sufficient conditions for this to be so. From the above fact, we see that one such sufficient condition is $\int_C \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{l} = 0$ for all closed curve C. However this is not practical to check. To find more practical conditions, we notice that

$$\nabla \times (\nabla f) = 0 \tag{188}$$

for every f that is C^2 (from now on we will make this assumption). Thus it would be great if we can prove

$$\nabla \times \boldsymbol{g} = 0 \Longrightarrow \boldsymbol{g} = \nabla f \text{ for some } f. \tag{189}$$

However this is in general not true.

Exercise 30. Show that (189) may not hold through studying

$$\boldsymbol{g}(x, y, z) = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}. \tag{190}$$

THEOREM 50. *Let $D \subseteq \mathbb{R}^3$ be convex and open. Let $\boldsymbol{g}: D \mapsto \mathbb{R}^3$ be C^1 and satisfy $\nabla \times \boldsymbol{g} = \boldsymbol{0}$ everywhere in D. Then there is a function $f: D \mapsto \mathbb{R}^3$ such that $\boldsymbol{g} = \nabla f$.*

Proof. All we need to show is that for every closed curve $C \subset A$ that is piecewise C^1, $\int_C \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{l} = 0$.

Let C be one such curve. Wlog we assume C is polygonal. Denote the vertices by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$, $\boldsymbol{v}_{m+1} = \boldsymbol{v}_1$, following an orientation of C. Now take any $\boldsymbol{a} \in A$. By Stokes's Theorem we have

$$\int_{\partial S_i} \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{l} = \int_{S_i} (\nabla \times \boldsymbol{g}) \cdot \boldsymbol{n} \, \mathrm{d}S = 0 \tag{191}$$

where S_i is the triangle formed by $\boldsymbol{a}$, $\boldsymbol{v}_i$, $\boldsymbol{v}_{i+1}$, with ∂S_i so oriented as to be consistent with the orientation of ∂S. But now we have

$$\int_{\partial S} \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{l} = \sum_{i=1}^{m} \int_{\partial S_i} \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{l} = 0. \tag{192}$$

Thus ends the proof. □

Exercise 31. What doesn't this proof work for (190)?

5. MORE EXERCISES AND PROBLEMS

For (many many) more exercises on calculation of line and surface integrals, see (DEMIDOVICH) ,(EFIMOV) .

5.1. Basic exercises

5.1.1. Stokes's Theorem

Exercise 32. Calculate

$$\int_L y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z \tag{193}$$

where L is the intersection of $\{(x,y,z)|\,x^2+y^2+z^2=R^2\}$ and $\{(x,y,z)|\,x+z=R\}$. Oriented as follows: Start from $(0,0,R)$ and moves first into $y<0$. (Hint:[11])

Exercise 33. Calculate

$$\int_L z\,\mathrm{d}x + x\,\mathrm{d}y + y\,\mathrm{d}z \tag{194}$$

where L is the intersection of $x^2+y^2+z^2=R^2$ and $x+y+z=R$. (Ans:[12])

Exercise 34. Calculate

$$\int_L y\,\mathrm{d}x - 2\,z\,\mathrm{d}y + x\,\mathrm{d}z \tag{195}$$

where L is the intersection of $y=x$ and $2\,x^2-y^2+z^2=R^2$, oriented such that the related $\boldsymbol{n}$ points into the positive x direction. (Ans:[13])

5.1.2. Gauss's Theorem

Exercise 35. Let S be a closed C^1 surface enclosing a region D. Let f,g be C^1. Prove the following (Green's formulas):

$$\int_S f\,(\mathrm{grad}\,g)\cdot\boldsymbol{n}\,\mathrm{d}S = \int_D [f\,\triangle g + (\mathrm{grad}\,f)\cdot(\mathrm{grad}\,g)]\,\mathrm{d}V; \tag{196}$$

$$\int_S [f\,(\mathrm{grad}\,g) - g\,(\mathrm{grad}\,f)]\cdot\boldsymbol{n}\,\mathrm{d}S = \int_D [f\,\triangle g - g\,\triangle f]\,\mathrm{d}V. \tag{197}$$

Exercise 36. Calculate

$$\int_S x^2\,\mathrm{d}y\wedge\mathrm{d}z + y^2\,\mathrm{d}z\wedge\mathrm{d}x + z^2\,\mathrm{d}x\wedge\mathrm{d}y \tag{198}$$

where

$$S = \{(x,y,z)|\,(x-a)^2+(y-b)^2+(z-c)^2=R^2\} \tag{199}$$

oriented by outer normal. (Answer:[14])

Exercise 37. Calculate

$$\int_{\partial V} \boldsymbol{f}\cdot\mathbf{d}\boldsymbol{S} \tag{200}$$

where $\boldsymbol{f} = \begin{pmatrix} x^3 \\ y^3 \\ R^2\,z \end{pmatrix}$ and $V = \left\{(x,y,z)|\,\frac{H}{R^2}(x^2+y^2) \leqslant z \leqslant H\right\}$ with ∂V oriented by the outer normal. (Ans:[15])

11. Pick the most convenient surface to apply Stokes's Theorem.
12. $2\,\pi\,R^2/\sqrt{3}$.
13. $3\,\pi\,R^2$.
14. $\frac{8}{3}\,\pi\,R^3\,(a+b+c)$.
15. $\pi\,H\,R^4$.

5.1.3. Gradient, curl, divergence

Exercise 38. (FOLLAND) Without going through any calculations, show that

$$\operatorname{div}(\boldsymbol{f}\times\boldsymbol{g})=\boldsymbol{g}\cdot(\operatorname{curl}\boldsymbol{f})+\boldsymbol{f}\cdot(\operatorname{curl}\boldsymbol{g}) \tag{201}$$

must be wrong. (Hint:[16])

Exercise 39. (FOLLAND) Show that for any C^2 functions f, g,

$$\operatorname{div}(\operatorname{grad} f\times\operatorname{grad} g)=0. \tag{202}$$

(Hint:[17])

5.2. More exercises

Exercise 40. Prove Stokes' Theorem for one piece of C^1 surface:

$$S:=\left\{\begin{pmatrix} x(u,v)\\ y(u,v)\\ z(u,v)\end{pmatrix} \mid (u,v)\in D\right\} \tag{203}$$

through direct calculation and Green's Theorem.

Exercise 41. Let S be a piece of C^1 surface with orientation $\boldsymbol{n}$. Let u, v be C^1 functions. Prove

$$\int_S \boldsymbol{n}\cdot(\nabla u\times\nabla v)\,\mathrm{d}S=\int_{\partial S} u\,\mathrm{d}v \tag{204}$$

where ∂S is oriented consistently with the orientation of S. (Hint:[18])

Exercise 42. (BRAND) Let S be a piece of C^1 surface with orientation $\boldsymbol{n}$. Let $f:\mathbb{R}^3\mapsto\mathbb{R}$ be C^1. Prove

$$\int_S [\boldsymbol{n}\times\nabla f]\,\mathrm{d}S=\int_{\partial S} f\,\mathrm{d}\boldsymbol{l}. \tag{205}$$

(Hint:[19])

Exercise 43. (CYLINDRICAL COORDINATES) Consider the cylindrical coordinates:

$$x=\rho\cos\varphi,\qquad y=\rho\sin\varphi,\qquad z=z. \tag{206}$$

Prove:

$$\operatorname{grad} f = \frac{\partial f}{\partial\rho}\boldsymbol{e}_\rho+\frac{1}{\rho}\frac{\partial f}{\partial\varphi}\boldsymbol{e}_\varphi+\frac{\partial f}{\partial z}\boldsymbol{e}_z; \tag{207}$$

$$\operatorname{div}\boldsymbol{f} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\,v_\rho)+\frac{1}{\rho}\frac{\partial v_\varphi}{\partial\varphi}+\frac{\partial v_z}{\partial z}; \tag{208}$$

$$\operatorname{curl}\boldsymbol{f} = \left(\frac{1}{\rho}\frac{\partial v_z}{\partial\varphi}-\frac{\partial v_\varphi}{\partial z}\right)\boldsymbol{e}_\rho+\left(\frac{\partial v_\rho}{\partial z}-\frac{\partial v_z}{\partial\rho}\right)\boldsymbol{e}_\varphi+\left(\frac{1}{\rho}\frac{\partial(\rho\,v_\varphi)}{\partial\rho}-\frac{1}{\rho}\frac{\partial v_\rho}{\partial\varphi}\right)\boldsymbol{e}_z; \tag{209}$$

$$\triangle f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2}+\frac{\partial^2 f}{\partial z^2}. \tag{210}$$

Exercise 44. Let $f, \boldsymbol{f}$ be C^1. Let $\boldsymbol{x}_0\in\mathbb{R}^3$. Let S be a closed piecewise C^1 surface enclosing a region $D\ni\boldsymbol{x}_0$. Prove

$$(\operatorname{grad} f)(\boldsymbol{x}_0) = \lim_{d(S)\longrightarrow 0}\frac{1}{\mu(D)}\int_S f\boldsymbol{n}\,\mathrm{d}S; \tag{211}$$

$$(\operatorname{div}\boldsymbol{f})(\boldsymbol{x}_0) = \lim_{d(S)\longrightarrow 0}\frac{1}{\mu(D)}\int_S \boldsymbol{f}\cdot\boldsymbol{n}\,\mathrm{d}S; \tag{212}$$

$$(\operatorname{curl}\boldsymbol{f})(\boldsymbol{x}_0) = \lim_{d(S)\longrightarrow 0}\frac{1}{\mu(D)}\int_S \boldsymbol{n}\times\boldsymbol{f}\,\mathrm{d}S. \tag{213}$$

5.3. Problems

16. Switch $\boldsymbol{f}\leftrightarrow\boldsymbol{g}$.
17. $\operatorname{grad} f\times\operatorname{grad} g=\operatorname{curl}(f\operatorname{grad} g)$.
18. $\nabla u\times\nabla v=\nabla\times[....]$.
19. Use $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{c})\,\boldsymbol{a}$ to prove $(\boldsymbol{r}_u\times\boldsymbol{r}_v)\times\nabla f=(\boldsymbol{r}_u f)_v-(\boldsymbol{r}_v f)_u$.