Math 317 Week 07: Integration Along Curves

March 30, 2014

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1. Arc Length of Curves

1.1. C^1 curves in $\mathbb{R}^N$

Definition 1. (Parametrized Curve; Trace of a Curve)

- *A* ***parametrized curve*** *in* $\mathbb{R}^N$ *is a continuous mapping*

$$\boldsymbol{x}\colon [a,b] \mapsto \mathbb{R}^N, \qquad \boldsymbol{x}(t) := \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{pmatrix} \tag{1}$$

- *The variable t is called the* ***parameter****, the representation (1) is called a* ***parametrization****, the functions $x_n(t)$ are called* ***parametrization functions****.*
- *Let $\boldsymbol{x}(t)$ be a parametrized curve. Its image $\boldsymbol{x}([a,b])$ is called the* ***trace*** *of the curve.*

Example 2. Consider the parametrized curves:

$$(\cos t, \sin t) \qquad t \in [0, 2\pi]; \qquad (\cos 2t, \sin 2t) \qquad t \in [0, 2\pi]. \tag{2}$$

They are two different parametrized curves, but have the same trace.

Remark 3. In some books a distinction is made between an "arc" and a "curve". We will not make such distinction.

Remark 4. We will use other types of intervals in place of $[a,b]$ when it is convenient and will not cause any confusion.

Exercise 1. Recall what it means to say $\boldsymbol{x}$ is continuous. Prove that the continuity of $\boldsymbol{x}$ is equivalent to the continuity of every $x_n(t)$, $n = 1, 2, 3, \ldots, N$.

Counter-examples such as Peano's curve reveals that requiring only continuity of $\boldsymbol{x}$ is far from enough to guarantee the curve to fit our intuition. Stronger restriction on $\boldsymbol{x}$ than only continuity is necessary when discussing many properties of the curves. The most convenient assumptions for our purposes are as follows.

Definition 5. (C^1 curve) *A curve $\boldsymbol{x}\colon [a,b] \mapsto \mathbb{R}^N$ is called C^1 if the following are satisfied.*

- $\boldsymbol{x} \in C^1([a, b])$, that is $\boldsymbol{x}$ is differentiable on (a, b) and both $\boldsymbol{x}$ and its derivative $\boldsymbol{x}'$ are continuous on $[a,b]$ (for $\boldsymbol{x}'$ we require the one-sided limits $\lim_{t \longrightarrow b-} \boldsymbol{x}'(t)$ and $\lim_{t \to a+} \boldsymbol{x}'(t)$ both exist);

  Exercise 2. Prove that this is equivalent to the same requirements for every $x_n(t)$ on $[a,b]$.

- The curve is simple, that is not self-crossing, that is

$$t_1 \neq t_2 \Longrightarrow \boldsymbol{x}(t_1) \neq \boldsymbol{x}(t_2). \tag{3}$$

- The curve is regular, that is $\boldsymbol{x}'(t) \neq \mathbf{0}$ for all $t \in [a,b]$.

  Exercise 3. Does this condition imply the previous one ($t_1 \neq t_2 \Longrightarrow \boldsymbol{x}(t_1) \neq \boldsymbol{x}(t_2)$)? Justify your answer.

Remark 6. It will be clearly seen that many of the following results still hold for union of finitely many C^1 curves.

Example 7. The following are C^1 curves

$$(2\cos(2t), 2\sin(2t)), \qquad t \in [0, \pi]; \tag{4}$$

$$(3\cos t, 2\sin t, e^t), \qquad t \in \mathbb{R}; \tag{5}$$

while the following are not

$$(t^3, t^2), \qquad t \in \mathbb{R}; \tag{6}$$

$$(t^3 - 4t, t^2 - 4), \qquad t \in \mathbb{R}; \tag{7}$$

$$(t, |t|). \qquad t \in \mathbb{R} \tag{8}$$

Exercise 4. Plot the above curves.

Remark 8. We notice that whether a curve is C^1 or not depends not only on the trace, but also on the parametrization. For example,

$$(\cos t, \sin t) \text{ and } (\cos t^3, \sin t^3), \qquad t \in [0, 1] \tag{9}$$

parametrize the same curve, but the first is C^1 while the second is not. In the following, when we say a curve is C^1, we mean there is a parametrization $\boldsymbol{x}(t)$ of this curve that is C^1. This is justified by the following theorem which basically says that all regular parametrizations of one same curve are kind of equivalent.

THEOREM 9. (EQUIVALENCE OF PARAMETRIZATION) *Let $\boldsymbol{x}(t): [a, b] \mapsto \mathbb{R}^N$ and $\boldsymbol{y}(s): [c, d] \mapsto \mathbb{R}^N$ be two C^1 curves having the same trace L in $\mathbb{R}^N$. Then there is a C^1 bijection $T: [c, d] \mapsto [a, b]$ such that*

$$\boldsymbol{x}(T(s)) = \boldsymbol{y}(s). \tag{10}$$

Proof. See §4.2.1. $\square$

Exercise 5. Prove that in fact T is either strictly increasing or strictly decreasing, and furthermore

$$T'(s_0) = \begin{cases} \frac{|\boldsymbol{y}'(s_0)|}{|\boldsymbol{x}'(s_0)|} & \boldsymbol{y}(c) = \boldsymbol{x}(a) \\ -\frac{|\boldsymbol{y}'(s_0)|}{|\boldsymbol{x}'(s_0)|} & \boldsymbol{y}(c) = \boldsymbol{x}(b) \end{cases}. \tag{11}$$

DEFINITION 10. (ORIENTATION) *When T is strictly increasing we say $\boldsymbol{x}, \boldsymbol{y}$ have the same orientations; When T is strictly decreasing we say $\boldsymbol{x}, \boldsymbol{y}$ have opposite orientations.*

1.2. Arc Length of C^1 curves

1.2.1. Definition and properties

To define length of curves, we recall the notion of "partition":

(PARTITION) A partition P of a compact interval $[a, b]$ is a finite set of points:

$$P = \{a = t_0 < t_1 < \cdots < t_m = b\}. \tag{12}$$

We denote the diameter of the partition as

$$d(P) := \max\{(t_1 - t_0), \ldots, (t_m - t_{m-1})\}. \tag{13}$$

DEFINITION 11. (LENGTH OF CURVE) *Let* $\boldsymbol{x}(t)$: $[a,\ b] \mapsto \mathbb{R}^N$ *represent a curve* $L \subseteq \mathbb{R}^N$*. Let* $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ *be any partition of* $[a, b]$*. Then we write*

$$L(\boldsymbol{x}, P) := \sum_{k=0}^{m-1} \|\boldsymbol{x}(t_{k+1}) - \boldsymbol{x}(t_k)\| = \sum_{k=0}^{m-1} [(x_1(t_{k+1}) - x_1(t_k))^2 + \cdots + (x_N(t_{k+1}) - x_N(t_k))^2]^{1/2}. \quad (14)$$

If

$$\lim_{d(P) \longrightarrow 0} L(\boldsymbol{x}, P) \quad (15)$$

exists and is finite, we say the curve $\boldsymbol{x}(t)$ *is rectifiable, and say the limit* l *is its arc length.*

Remark 12. Here is limit is a kind of "net convergence", defined as follows:

We say

$$l = \lim_{d(P) \longrightarrow 0} L(\boldsymbol{x}, P) \quad (16)$$

if and only if, for every $\varepsilon > 0$, there is $\delta > 0$, such that for any partition P with $d(P) < \delta$, there holds

$$|l - L(\boldsymbol{x}, P)| < \varepsilon. \quad (17)$$

Remark 13. Note that the definition relies on the parametrization $\boldsymbol{x}$. We will try to get rid of this dependence soon.

Example 14. The straight line $\boldsymbol{x}(t) := \boldsymbol{u} + \boldsymbol{v}\, t$, $t \in [a, b]$, is rectifiable, with $l(\boldsymbol{x}) = (b - a)\, \|\boldsymbol{v}\|$. In particular, any compact interval $[a, b]$ is rectifiable with $l([a, b]) = b - a$.

Example 15. The curve $\boldsymbol{x}\colon [0, 1] \mapsto \mathbb{R}^2$

$$\boldsymbol{x}(t) := \begin{cases} \left(t, t \sin\left(\frac{\pi}{t}\right)\right) & t > 0 \\ (0, 0) & t = 0 \end{cases} \quad (18)$$

is not rectifiable.

Proof. For any $n \in \mathbb{N}$, we make the partition:

$$t_0 = 0, \quad t_1 = \frac{2}{2\,n+1}, \quad \ldots \quad , t_{n-1} = \frac{2}{3}, \quad t_n = 1. \quad (19)$$

Then we have

$$\boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}(t_1) = \begin{pmatrix} t_1 \\ \frac{2\,(-1)^n}{2\,n+1} \end{pmatrix}, \ldots \quad (20)$$

It is easy to check that

$$\sum_{m=0}^{n-1} \|\boldsymbol{x}(t_{m+1}) - \boldsymbol{x}(t_m)\| > \frac{2}{3} + \cdots + \frac{2}{2\,n+1} \quad (21)$$

and the limit cannot be finite. □

Exercise 6. Prove that Koch curve (Koch snowflake: http://en.wikipedia.org/wiki/Koch_snowflake) is not rectifiable.

THEOREM 16. *The following are equivalent for a* C^1 *curve* $\boldsymbol{x}\colon [a, b] \mapsto \mathbb{R}^N$*:*

A) It is rectifiable with length $l < \infty$*;*

B) $\sup_P L(\boldsymbol{x}, P) = l < \infty$*;*

C) $\int_a^b \|\boldsymbol{x}'(t)\|\,\mathrm{d}t = l < \infty$. *Recall that the length of a vector is defined as* $\|\boldsymbol{u}\| := \sqrt{u_1^2 + \cdots + u_N^2}$.

Proof. See §4.2.2. □

COROLLARY 17. *Let L be a C^1 rectifiable curve. Then its length is independent of parametrization.*

Exercise 7. Prove the corollary (Hint:[1]).

COROLLARY 18. *Let $f: [a,b] \mapsto \mathbb{R}$ be C^1. Then the graph of f, as a curve in $\mathbb{R}^2$, is a C^1 rectifiable curve with*

$$l = \int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x. \tag{22}$$

Exercise 8. Prove the corollary. (Hint:[2])

Exercise 9. Generalize the above corollary to $\boldsymbol{f}: [a,b] \mapsto \mathbb{R}^{N-1}$.

In practice we often need to deal with segments or unions of C^1 curves. The following theorem basically says that nothing can go wrong.

THEOREM 19. *Let L, L_1, L_2 be rectifiable C^1 curves. Then*

a) Any segment of L is also rectifiable;

b) $L_1 \cup L_2$ is rectifiable and $l(L_1 \cap L_2) = l(L_1) + l(L_2)$.

Remark 20. Note that $L_1 \cup L_2$ may not be C^1.

1.2.2. Calculation of arc length

Arc length formulas

If the curve is parametrized in the general form $(x_1(t), \ldots, x_N(t))$, $t \in [a,b]$, then

$$\text{Length} = \int_a^b \sqrt{x_1'(t)^2 + \cdots + x_N'(t)^2}\,\mathrm{d}t; \tag{23}$$

If the curve is given as the graph of a function $y = \phi(x)$, $x \in [a,b]$, then

$$\text{Length} = \int_a^b \sqrt{1 + \phi'(x)^2}\,\mathrm{d}x \tag{24}$$

Example 21. Calculate the arc length of the parabola $y = \frac{x^2}{2}$ from $x = -1$ to $x = 1$.

Solution. The natural parametrization is

$$\begin{pmatrix} t \\ t^2/2 \end{pmatrix}, \qquad t \in [-1,1]. \tag{25}$$

Then the arc length can be calculated as

$$\begin{array}{rcl} \displaystyle\int_{-1}^1 \sqrt{1+t^2}\,\mathrm{d}t & = & \displaystyle 2\int_0^1 \sqrt{1+t^2}\,\mathrm{d}t \\ t = \dfrac{e^x - e^{-x}}{2} & = & \displaystyle 2\int_0^{x_0} \left(\frac{e^x + e^{-x}}{2}\right)^2 \mathrm{d}x \qquad \left(\frac{e^{x_0} - e^{-x_0}}{2} = 1\right) \\ & = & x_0 + \dfrac{1}{4}\,[e^{2x_0} - e^{-2x_0}]. \end{array} \tag{26}$$

1. Use formula C) in Theorem 16.
2. Parametrization $(t, f(t))$.

To find out the value we calculate

$$e^{x_0} - e^{-x_0} = 2 \Longrightarrow (e^{x_0} + e^{-x_0})^2 = 8 \Longrightarrow e^{x_0} + e^{-x_0} = 2\sqrt{2} \Longrightarrow e^{2x_0} - e^{-2x_0} = 4\sqrt{2}. \tag{27}$$

Therefore the arc length is given by

$$x_0 + \sqrt{2} = \sqrt{2} + \sinh^{-1}(1). \tag{28}$$

Example 22. Calculate the arc length of the graph of $\sin x$ from $x = 0$ to $x = 2\pi$.

Solution. The natural parametrization of the curve is

$$\begin{pmatrix} t \\ \sin t \end{pmatrix}, \qquad t \in [0, 2\pi]. \tag{29}$$

Then we have the arc length to be

$$\int_0^{2\pi} \sqrt{1 + (\cos t)^2}\, \mathrm{d}t. \tag{30}$$

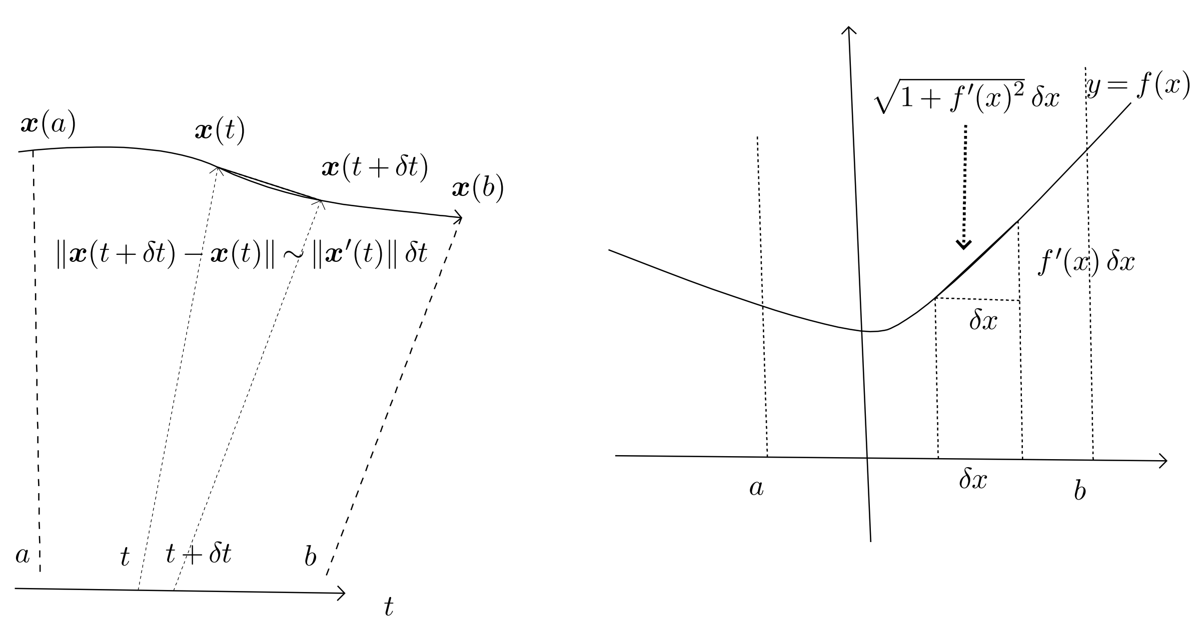
It turns out that this integral does not have a closed form solution.

Example 23. Calculate the arc length of the cycloid $\begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}$, $t \in [0, 2\pi]$.

Solution. The parametrization is already given. We calculate

$$\begin{aligned} \int_0^{2\pi} \sqrt{[(t - \sin t)']^2 + [(1 - \cos t)']^2} &= \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2} \\ &= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, \mathrm{d}t \\ &= \int_0^{2\pi} \sqrt{2 - 2\left(1 - 2\left(\sin\frac{t}{2}\right)^2\right)}\, \mathrm{d}t \\ &= 2\int_0^{2\pi} \left|\sin\left(\frac{t}{2}\right)\right| \mathrm{d}t \\ &= 4\int_0^{\pi} \sin u\, \mathrm{d}u \\ &= 8 \end{aligned} \tag{31}$$

Understanding the arc length formulas



2. Integration Along Curves I: Line Integral of Scalar Functions

2.1. Definition and properties

Let $f: \mathbb{R}^N \mapsto \mathbb{R}$. Let $\boldsymbol{x}: [a, b] \mapsto \mathbb{R}^N$ representing a curve L. We would like to define

$$\int_L f(\boldsymbol{x})\,\mathrm{d}s. \tag{32}$$

To define this integral, we need to first generalize the idea of partition to rectifiable curves.

> (Partition along curves) Let L be a rectifiable curve with parametrization $\boldsymbol{x}$: $[a, b] \mapsto \mathbb{R}^N$. A partition P of L is a set of points along L $\boldsymbol{x}_0 := \boldsymbol{x}(t_0), \boldsymbol{x}_1 := \boldsymbol{x}(t_1), \ldots,$ $\boldsymbol{x}_m = \boldsymbol{x}(t_m)$, such that
>
> i. $\boldsymbol{x}(t_0)$ and $\boldsymbol{x}(t_m)$ are the two end points of L;
>
> ii. $\{t_n\}$ is either strictly increasing or strictly decreasing.
>
> We will denote by L_k the segment between $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+1}$, and denote by l_k the arc length of L_k. We define the diameter of P as
>
> $$d(P) := \max\{l_0, l_1, \ldots, l_{m-1}\} \tag{33}$$

Definition 24. (Line integral of the first kind/type) *We define the line integral as follows. Let P be an arbitrary partition. Let $\Xi := \{\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_{m-1}\} \subset L$ be such that*

$$\boldsymbol{\xi_i} \in L_i. \tag{34}$$

We form the sum

$$I(f, \Xi, P) := \sum_{i=0}^{m-1} f(\boldsymbol{\xi}_i)\, l_i. \tag{35}$$

If

$$\lim_{d(P) \longrightarrow 0} I(f, \Xi, P) \tag{36}$$

exists and is finite, we say f is integrable along L, and denote this limit by $\int_L f(\boldsymbol{x})\,\mathrm{d}s$.

Remark 25. The limit above is defined by:

The limit is s if and only if for any $\varepsilon > 0$, there is $\delta > 0$ such that for any partition P with $d(P) < \delta$, and any choice of Ξ satisfying $\boldsymbol{\xi}_i \in L_i$, we have

$$|s - I(f, \Xi, P)| < \varepsilon. \tag{37}$$

Exercise 10. Let L be rectifiable with length l. Then the constant function $f(\boldsymbol{x}) \equiv c$ is integrable along L and

$$\int_L f(\boldsymbol{x})\,\mathrm{d}s = c\,l. \tag{38}$$

THEOREM 26. *Let f be integrable on L_1, L_2. Further assume that $L_1 \cap L_2$ has arc length 0. Then f is integrable on $L_1 \cup L_2$ and*

$$\int_{L_1 \cup L_2} f(\boldsymbol{x})\,\mathrm{d}s = \int_{L_1} f(\boldsymbol{x})\,\mathrm{d}s + \int_{L_2} f(\boldsymbol{x})\,\mathrm{d}s. \tag{39}$$

Exercise 11. Prove the above theorem.

2.2. Integration along C^1 curves

THEOREM 27. *Let $\boldsymbol{x}(t)$: $[a, b] \mapsto \mathbb{R}^N$ parametrize a C^1 curve. Let f: $\mathbb{R}^N \mapsto \mathbb{R}$ be continuous on a compact interval I satisfying $L \subset I^o$. Then*

$$\int_L f(\boldsymbol{x})\,\mathrm{d}s = \int_a^b f(\boldsymbol{x}(t))\,\|\boldsymbol{x}'(t)\|\,\mathrm{d}t. \tag{40}$$

Proof. See §4.2.3. □

COROLLARY 28. *Let the curve L be given by $x \in [a, b]$, $y = \phi(x)$, then*

$$\int_L f(x, y)\,\mathrm{d}s = \int_a^b f(x, \phi(x))\,\sqrt{1 + \phi'(x)^2}\,\mathrm{d}x. \tag{41}$$

Summary:

Line integral of the first type/kind

If the curve L is parametrized in the general form $(x_1(t), \ldots, x_N(t))$, $t \in [a, b]$, then

$$\int_L f(\boldsymbol{x})\mathrm{d}s = \int_a^b f(x_1(t), \ldots, x_N(t))\,\sqrt{x_1'(t)^2 + \cdots + x_N'(t)^2}\,\mathrm{d}t; \tag{42}$$

If the curve is given as the graph of a function $y = \phi(x)$, $x \in [a, b]$, then

$$\int_L f(\boldsymbol{x})\mathrm{d}s = \int_a^b f(x, \phi(x))\,\sqrt{1 + \phi'(x)^2}\,\mathrm{d}x \tag{43}$$

Example 29. Calculate

$$\int_L x^2 \, \mathrm{d}s \tag{44}$$

where L is the unit circle.

Solution. We have the parametrization $(\cos t, \sin t)$ with $t \in [0, 2\pi]$. Calculate:

$$\int_L x^2 \, \mathrm{d}s = \int_0^{2\pi} (\cos t)^2 \, \mathrm{d}t = \pi. \tag{45}$$

Exercise 12. Explain why $\boldsymbol{x}(0) = \boldsymbol{x}(2\pi)$ is not a problem in the above example.

Example 30. Calculate

$$\int_L \sqrt{1+y} \, \mathrm{d}s \tag{46}$$

where L is the boundary of the shape enclosed by the parabola $y = x^2$ and $y = 1$.

Solution. We write $L = L_1 \cup L_2$ with

$$L_1: (t, t^2), \qquad t \in [-1, 1]; \qquad L_2: (t, 1), \qquad t \in [-1, 1]. \tag{47}$$

Then we have

$$\begin{aligned} \int_L \sqrt{1+y} \, \mathrm{d}s &= \int_{L_1} \sqrt{1+y} \, \mathrm{d}s + \int_{L_2} \sqrt{1+y} \, \mathrm{d}s \\ &= \int_{-1}^{1} \sqrt{1+t^2} \sqrt{1+t^2} \, \mathrm{d}t + \int_{-1}^{1} \sqrt{2} \, \mathrm{d}t \\ &= 2 + \frac{2}{3} + 2\sqrt{2}. \end{aligned} \tag{48}$$

Example 31. (DEMIDOVICH, NO. 2309) With what force will a mass M distributed with uniform density over the circle $x^2 + y^2 = 1, z = 0$, act on a mass m located at the point $A: (0, 0, b)$?

Solution. Denote the circle by C. Since M is uniformly distributed along C, the density is $\rho = \frac{M}{2\pi}$. Intuitively, along an infinitesimal segment $\mathrm{d}s$, the mass is $\rho \, \mathrm{d}s$. Therefore the force on m is given by the following integral (note that obviously the x, y components of the force are 0 so we only calculate the z component.)

$$\int_C \left[-\frac{G m \rho \, \mathrm{d}s}{x^2 + y^2 + (b-z)^2} \right] = -G m \frac{M}{2\pi} \frac{1}{1+b^2} \int_C \mathrm{d}s = -\frac{G m M}{1+b^2}. \tag{49}$$

3. Integration Along Curves II: Line Integral of Vector Functions

3.1. Definition and properties

There is a trivial generalization to integrating a vector functions $\boldsymbol{f}: \mathbb{R}^N \mapsto \mathbb{R}^M$ along a curve $\boldsymbol{x}$: $[a, b] \mapsto \mathbb{R}^N$:

$$\int_L \boldsymbol{f}(\boldsymbol{x})\,\mathrm{d}s := \begin{pmatrix} \int_L f_1(\boldsymbol{x})\,\mathrm{d}s \\ \vdots \\ \int_L f_M(\boldsymbol{x})\,\mathrm{d}s \end{pmatrix}. \tag{50}$$

We are interested in the following non-trivial one.

Definition 32. (Line integral of the second kind/type) *Let $\boldsymbol{f}: \mathbb{R}^N \mapsto \mathbb{R}^N$ and let $L \subset \mathbb{R}^N$. Let $\boldsymbol{x}_i$ and $\boldsymbol{\xi}_i$ be as in §2.1. Denote $A := \boldsymbol{x}_0$ and $B := \boldsymbol{x}_m$. We say the second type of line integral (from A to B) exists, if the limit*

$$\lim_{d(P) \longrightarrow 0} \sum \left(\boldsymbol{f}(\boldsymbol{\xi}_i) \cdot (\boldsymbol{x}_{i+1} - \boldsymbol{x}_i)\right) \tag{51}$$

exists and is finite. We denote the limit as

$$\int_{AB} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{l}. \tag{52}$$

or as the less compact notation

$$\int_{AB} f_1(\boldsymbol{x})\,\mathrm{d}x_1 + \cdots + f_N(\boldsymbol{x})\,\mathrm{d}x_N. \tag{53}$$

Exercise 13. Prove that

$$\left| \int_{AB} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{l} \right| \leqslant \left(\max_{\boldsymbol{x} \in L} \|\boldsymbol{f}(\boldsymbol{x})\| \right) \cdot l \tag{54}$$

where l is the arc-length of the curve $L := AB$.

Theorem 33. *Let $\boldsymbol{f}$ be interable along $L = AB$ from A to B. Then it is also integrable along L from B to A and furthermore*

$$\int_{AB} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{l} = -\int_{BA} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{l}. \tag{55}$$

Proof. Exercise. □

Theorem 34. *We have*

$$\int_{AB} f_1(\boldsymbol{x})\,\mathrm{d}x_1 + \cdots + f_N(\boldsymbol{x})\,\mathrm{d}x_N = \left[\int_{AB} f_1(\boldsymbol{x})\,\mathrm{d}x_1\right] + \cdots + \left[\int_{AB} f_N(\boldsymbol{x})\,\mathrm{d}x_N\right]. \tag{56}$$

Proof. Exercise. Note that each term on the RHS is also a line integral of the second kind. □

Theorem 35. *Let $L_1 := AB$ and $L_2 := BC$. Define $L = L_1 \cup L_2$ with orientation ABC. Then*

$$\int_{ABC} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{l} = \int_{AB} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{l} + \int_{BC} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{l}. \tag{57}$$

Proof. Exercise. □

3.2. Integration along C^1 curves

When the curve is C^1, line integrals of the second kind have a simpler equivalent definition.

DEFINITION 36. (LINE INTEGRAL OF THE SECOND KIND ALONG C^1 CURVES) *Let L be C^1 with end points A, B and orientation $A \longrightarrow B$. Then we define*

$$\int_{AB} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{l} := \int_L [\boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{T}(\boldsymbol{x})]\, \mathrm{d}s \tag{58}$$

where $\boldsymbol{T}(\boldsymbol{x})$ is the unit tangent vector at $\boldsymbol{x} \in L$ whose direction is consistent with the orientation of the curve.

Remark 37. Note that the right hand side of (58) is line integral of the first kind, and therefore is already well-defined.

We know that

$$\boldsymbol{T}(\boldsymbol{x}(t)) = \frac{\boldsymbol{x}'(t)}{\|\boldsymbol{x}'(t)\|} \tag{59}$$

if $\boldsymbol{x}(t)$: $[a, b]$ is a parametrization of L such that $\boldsymbol{x}(a) = A$, $\boldsymbol{x}(b) = B$. This immediately gives the following formula for calculation purpose:

$$\begin{aligned} \int_{AB} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{l} &= \int_L [\boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{T}(\boldsymbol{x})]\, \mathrm{d}s \\ \text{(Calculate line integral of the 1st kind)} &= \int_a^b [\boldsymbol{f}(\boldsymbol{x}(t)) \cdot \boldsymbol{T}(\boldsymbol{x}(t))]\, \|\boldsymbol{x}'(t)\|\, \mathrm{d}t \\ &= \int_a^b \left[\boldsymbol{f}(\boldsymbol{x}(t)) \cdot \frac{\boldsymbol{x}'(t)}{\|\boldsymbol{x}'(t)\|}\right] \|\boldsymbol{x}'(t)\|\, \mathrm{d}t \\ &= \int_a^b [\boldsymbol{f}(\boldsymbol{x}(t))] \cdot \boldsymbol{x}'(t)\, \mathrm{d}t \\ &= \int_a^b [f_1(\boldsymbol{x}(t))\, x_1'(t) + \cdots + f_N(\boldsymbol{x}(t))\, x_N'(t)]\, \mathrm{d}t. \end{aligned} \tag{60}$$

That Definition 36 is equivalent to Definition 32 for C^1 curves follows from the theorem below.

THEOREM 38. *Let $\boldsymbol{x}(t)$: $[a, b] \mapsto \mathbb{R}^N$ parametrize a C^1 curve $L = AB$ and furthermore $\boldsymbol{x}(a) = A$, $\boldsymbol{x}(b) = B$. Let $f: \mathbb{R}^N \mapsto \mathbb{R}$ be continuous on a compact interval I satisfying $L \subset I^o$. Then the integral*

$$\int_{AB} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathbf{d}\boldsymbol{l} \tag{61}$$

exists in the sense of Definition 32 and can be calculated as

$$\int_{AB} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{l} = \int_a^b [f_1(\boldsymbol{x}(t))\, x_1'(t) + \cdots + f_N(\boldsymbol{x}(t))\, x_N'(t)]\, \mathrm{d}t. \tag{62}$$

Proof. See §4.2.4. □

Line integral of the second type/kind

If the curve L is parametrized in the general form $(x_1(t), \ldots, x_N(t))$, $t \in [a, b]$, then

$$\int_{AB} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathbf{d}\boldsymbol{l} = \int_a^b [f_1(\boldsymbol{x}(t))\, x_1'(t) + \cdots + f_N(\boldsymbol{x}(t))\, x_N'(t)]\, \mathrm{d}t; \tag{63}$$

If the curve is given as the graph of a function $y = \phi(x)$, $x \in [a, b]$, then

$$\int_{AB} \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} \cdot \mathbf{d}\boldsymbol{l} = \int_a^b [f(x, \phi(x)) + g(x, \phi(x)) \phi'(x)]\, \mathrm{d}x. \tag{64}$$

It should be emphasized that the parametrization must be consistent with the orientation of the curve: $\boldsymbol{x}(a) = A, \boldsymbol{x}(b) = B$, or in the graph situation, $(a, \phi(a)) = A, (b, \phi(b)) = B$.

Example 39. Calculate

$$\int_{L_i} (x^2 + y^2)\, \mathrm{d}x + (x^2 - y^2)\, \mathrm{d}y \tag{65}$$

where L_1 is given by the line segments $(0, 0)$ to $(1, 1)$ and then to $(2, 0)$; L_2 is given by the line segment $(0,0)$ to $(2,0)$.

Solution.

- L_1.

  We parametrize L_1: $t \in [0, 2]$,

  $$(x(t), y(t)) := \begin{cases} (t, t) & t \in [0, 1] \\ (t, 2 - t) & t \in [1, 2] \end{cases}. \tag{66}$$

  Then

  $$\begin{aligned} \int_{L_1} (x^2 + y^2)\, \mathrm{d}x + (x^2 - y^2)\, \mathrm{d}y &= \int_0^2 [(x(t)^2 + y(t)^2)\, x'(t) + (x(t)^2 - y(t)^2)\, y'(t)]\, \mathrm{d}t \\ &= \int_0^1 [(t^2 + t^2) \cdot 1 + (t^2 - t^2) \cdot 1]\, \mathrm{d}t \\ &\quad + \int_1^2 [(t^2 + (2 - t)^2) \cdot 1 + (t^2 - (2 - t)^2) \cdot (-1)]\, \mathrm{d}t \\ &= \frac{4}{3}. \end{aligned} \tag{67}$$

- L_2.

  We first parametrize L_2: $t \in [0, 2]$

  $$(x(t), y(t)) := (t, 0). \tag{68}$$

  Then

  $$\int_{L_2} (x^2 + y^2)\, \mathrm{d}x + (x^2 - y^2)\, \mathrm{d}y = \int_0^2 [(t^2 + 0^2) \cdot 1 + (t^2 - 0^2) \cdot 0]\, \mathrm{d}t = \frac{8}{3}. \tag{69}$$

Example 40. Calculate

$$\int_L \frac{x\, \mathrm{d}x + y\, \mathrm{d}y}{\sqrt{x^2 + y^2}} \tag{70}$$

where $L \subset \{(x, y) |\, x > 0\}$ is any C^1 curve connecting $(1, 1)$ and $(2, 2)$.

Solution. Let $(x(t), y(t))$ be any C^1 parametrization. Then we have

$$\begin{aligned} \int_L \frac{x\, \mathrm{d}x + y\, \mathrm{d}y}{\sqrt{x^2 + y^2}} &= \int_a^b \frac{x(t)\, x'(t) + y(t)\, y'(t)}{\sqrt{x(t)^2 + y(t)^2}}\, \mathrm{d}t \\ &= \frac{1}{2} \int_a^b \frac{(x(t)^2 + y(t)^2)'}{\sqrt{x(t)^2 + y(t)^2}}\, \mathrm{d}t \\ &= \int_a^b \left(\sqrt{x(t)^2 + y(t)^2}\right)'\, \mathrm{d}t \end{aligned}$$

$$\begin{aligned} &= \sqrt{x(b)^2 + y(b)^2} - \sqrt{x(a)^2 + y(a)^2} \\ &= \sqrt{2}. \end{aligned} \tag{71}$$

Exercise 14. What would happen if we drop the assumption $L \subset \{(x, y) | x > 0\}$?

Example 41. Calculate

$$\int_L (y - z)\, \mathrm{d}x + (z - x)\, \mathrm{d}y + (x - y)\, \mathrm{d}z \tag{72}$$

where L is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 0$. The orientation is such that when seen from the top the curve runs counter-clockwise.

Solution. First we parametrize the curve

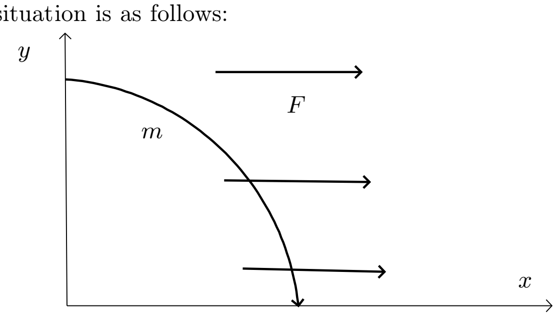
$$(x(\theta), y(\theta), z(\theta)) := (\cos\theta, \sin\theta, -\cos\theta - \sin\theta) \tag{73}$$

with $\theta \in [0, 2\pi]$. Then

$$\begin{aligned} \int_L (y - z)\, \mathrm{d}x + (z - x)\, \mathrm{d}y + (x - y)\, \mathrm{d}z &= \int_0^{2\pi} (2\sin\theta + \cos\theta)\, (\cos\theta)'\, \mathrm{d}\theta \\ &\quad + \int_0^{2\pi} (-2\cos\theta - \sin\theta)\, (\sin\theta)'\, \mathrm{d}\theta \\ &\quad + \int_0^{2\pi} (\cos\theta - \sin\theta)\, (-\cos\theta - \sin\theta)'\, \mathrm{d}\theta \\ &= -6\pi. \end{aligned} \tag{74}$$

Example 42. (DEMIDOVICH NO.2343) A field is generated by a force of constant magnitude F in the positive x-direction. Find the work that the field does when a material point traces clockwise a quarter of the circle $x^2 + y^2 = R^2$ lying in the first quadrant.

Solution. The situation is as follows:



We parametrized the curve:

$$(R\sin\theta, R\cos\theta), \qquad \theta \in [0, \pi/2]. \tag{75}$$

Now calculate (denote the components of F by $F_x, 0$)

$$\begin{aligned} \text{Work} &= \int_C F \cdot \mathbf{d}\boldsymbol{l} \\ &= R \int_0^{\pi/2} \begin{pmatrix} F_x \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} \mathrm{d}\theta \\ &= R \int_0^{\pi/2} [F_x \cos\theta]\, \mathrm{d}\theta \\ &= R\, F_x. \end{aligned} \tag{76}$$

Remark 43. Intuitively, we can think of $\mathbf{d}\boldsymbol{l}$ as an "infinitesimal vector" and interpret $\boldsymbol{f} \cdot \mathbf{d}\boldsymbol{l}$ as "dot product" between two vectors. Following this interpretation we can write $\mathrm{d}s := \|\mathbf{d}\boldsymbol{l}\|$ and thus $\mathbf{d}\boldsymbol{l} = \boldsymbol{T}\, \mathrm{d}s$. From which (58) formally follows.

4. Advanced Topics, Notes, and Comments

4.1. Green's theorem

LEMMA 44. *Let D be the compact triangle with ∂D oriented counter-clockwise. Let $f(x, y)$, $g(x, y) \in C^1(G)$ for some open set $G \supset D$. Then*

$$\int_{\partial D} f \, \mathrm{d}x + g \, \mathrm{d}y = \int_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] \mathrm{d}(x, y). \tag{77}$$

Proof. We prove the special case $D = ABO$ with $A = (1, 0)$, $B = (0, 1)$, and O being the origin. Then we have

$$\int_{OA} f \, \mathrm{d}x = \int_0^1 f(t, 0) \, \mathrm{d}t; \qquad \int_{BO} f \, \mathrm{d}x = 0; \tag{78}$$

and

$$\int_{AB} f \, \mathrm{d}x = -\int_0^1 f(1 - t, t) \, \mathrm{d}t = -\int_0^1 f(t, 1 - t) \, \mathrm{d}t. \tag{79}$$

Thus we have

$$\begin{aligned} \int_{\partial D} f \, \mathrm{d}x &= \int_0^1 [f(t, 0) - f(t, 1 - t)] \, \mathrm{d}t \\ &= \int_0^1 \left[-\int_0^{1-t} \frac{\partial f}{\partial y}(t, s) \, \mathrm{d}s \right] \mathrm{d}t \\ &= -\int_D \frac{\partial f(t, s)}{\partial y} \, \mathrm{d}(t, s) = -\int_D \frac{\partial f(x, y)}{\partial y} \, \mathrm{d}(x, y). \end{aligned} \tag{80}$$

In the above the second equality is due to fundamental theorem of calculus version 1, and the third equality is due to Fubini. The application of both theorems is justified by the hypothesis that $f(x, y) \in C^1(G)$.

Similarly we have

$$\int_{\partial D} g \, \mathrm{d}y = \int_D \frac{\partial g(x, y)}{\partial x} \, \mathrm{d}(x, y) \tag{81}$$

and the lemma is proved. □

Exercise 15. Prove the lemma for general triangles.

COROLLARY 45. *Let D be a polygon. Let ∂D be its boundary oriented counter-clockwise. Let $f(x, y)$, $g(x, y) \in C^1(G)$ for some open set $G \supset D$. Then*

$$\int_{\partial D} f \, \mathrm{d}x + g \, \mathrm{d}y = \int_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] \mathrm{d}(x, y). \tag{82}$$

LEMMA 46. *Let AB be a C^1 curve. Let $P = \{A = \boldsymbol{x}_0, \ldots, \boldsymbol{x}_m = B\}$ be a partition. Let L_P be the union of the line segments $\overline{\boldsymbol{x}_i \boldsymbol{x}_{i+1}}$. Let $f, g \in C^1(G)$ for an open set $G \supset AB \cup L_P$ for all P. Then*

$$\lim_{d(P) \longrightarrow 0} \left| \int_{AB} f \, \mathrm{d}x + g \, \mathrm{d}y - \int_{L_P} f \, \mathrm{d}x + g \, \mathrm{d}y \right| = 0. \tag{83}$$

Exercise 16. Prove the above lemma. (Hint:[3])

LEMMA 47. *Let D be Jordan measurable and such that ∂D is the union of finitely many C^1 curves. Let P be any partition along ∂D following the specified orientation. Let L_P be the union of the segments $\overline{\boldsymbol{x}_i \boldsymbol{x}_{i+1}}$. Let D_P be the domain enclosed by L_P such that D_P^o is to the left of L_P when moving along L_P. Let $f: G \mapsto \mathbb{R}$ be continuous for some open set G containing $\bar{D}$. Then*

$$\lim_{d(P) \longrightarrow 0} \int_{D_P} f(x, y) \, \mathrm{d}(x, y) = \int_D f(x, y) \, \mathrm{d}(x, y). \tag{84}$$

Exercise 17. Prove that there is an open set V such that $\bar{D} \subset V \subset \bar{V} \subset G$, and $\delta_0 > 0$, such that $L_P \subset V$ when $d(P) < \delta_0$.

Proof. Let $\bar{V}, \delta_0$ be as in the above exercise. By assumption there is $M > 0$ such that $|f(x, y)| < M$ on $\bar{V}$. All we need to prove is

$$\lim_{d(P) \longrightarrow 0} \int_{D_P \triangle D} |f(x, y)| \, \mathrm{d}(x, y) = 0 \tag{85}$$

where $D_P \triangle D := (D_P - D) \cup (D - D_P)$. This is guaranteed if

$$\lim_{d(P) \longrightarrow 0} \mu(D_P \triangle D) = 0. \tag{86}$$

This is left as exercise. □

Exercise 18. Prove (86).

THEOREM 48. *Let D be Jordan measurable and such that ∂D is the union of finitely many C^1 curves, oriented such that when moving along ∂D, D^o is always on the left hand side. Then we have*

$$\int_{\partial D} f \, \mathrm{d}x + g \, \mathrm{d}y = \int_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] \mathrm{d}(x, y). \tag{87}$$

Proof. Let P be any partition along ∂D following the specified orientation. Let L_P be the union of the segments $\overline{\boldsymbol{x}_i \boldsymbol{x}_{i+1}}$. Let D_P be the domain enclosed by L_P such that D_P^o is to the left of L_P when moving along L_P. Then by Lemma 46 and 47 we have

$$\int_{L_P} f \, \mathrm{d}x + g \, \mathrm{d}y = \int_{D_P} \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] \mathrm{d}(x, y). \tag{88}$$

Taking limit $d(P) \longrightarrow 0$ we obtain (87). □

Remark 49. One way to remember the formula, in particular the signs, is to test it on the unit square. In this case, if we denote $A_1 = (0, 0)$, $A_2 = (1, 0)$, $A_3 = (1, 1)$, $A_4 = (0, 1)$, then we have

$$\int_{A_1 A_2} f \, \mathrm{d}x + g \, \mathrm{d}y = \int_0^1 f(x, 0) \, \mathrm{d}x. \tag{89}$$

3. Check Definition 32.

Doing similar things for the other three, we have

$$\begin{aligned}\int_{\partial D} f\,\mathrm{d}x + g\,\mathrm{d}y &= \int_0^1 [f(x,0) - f(x,1)]\,\mathrm{d}x + \int_0^1 [g(1,y) - g(0,y)]\,\mathrm{d}y \\ &= -\int_0^1 \left[\int_0^1 \frac{\partial f(x,y)}{\partial y}\,\mathrm{d}y\right]\mathrm{d}x + \int_0^1 \left[\int_0^1 \frac{\partial g(x,y)}{\partial x}\,\mathrm{d}x\right]\mathrm{d}y \\ &= -\int_D \frac{\partial f(x,y)}{\partial y}\,\mathrm{d}(x,y) + \int_0^1 \left[\int_0^1 \frac{\partial g(x,y)}{\partial x}\,\mathrm{d}x\right]\mathrm{d}y. \end{aligned} \tag{90}$$

Example 50. (Area) From (87) it is clear that if we choose $g(x,y) = x$ and $f(x,y) = 0$, then we have

$$\mu(D) = \int_D \mathrm{d}(x,y) = \int_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right]\mathrm{d}(x,y) = \int_{\partial D} x\,\mathrm{d}y. \tag{91}$$

Alternatively, we can take $g(x,y) = 0$ and $f(x,y) = -y$. This leads to

$$\mu(D) = -\int_{\partial D} y\,\mathrm{d}x. \tag{92}$$

From the above it is also clear that

$$\mu(D) = \frac{1}{2}\int_{\partial D} x\,\mathrm{d}y - y\,\mathrm{d}x. \tag{93}$$

Example 51. (Isoperimetric inequality) We consider the following problem:

$$\max \mu(D) \qquad \text{subject to } l(\partial D) = l \tag{94}$$

where $l > 0$ is fixed.

Jakob Steiner in 1838 showed that if the solution exists, then ∂D must be a circle. The first complete proof was by Adolf Hurwitz in 1902 using Fourier series. The following proof was given by E. Schmidt in 1938. We follow the presentation in (do Carmo) .

Denote $2\,r := \operatorname{diam}(D) := \max\{\|\boldsymbol{x} - \boldsymbol{y}\| \,|\, \boldsymbol{x}, \boldsymbol{y} \in D\}$. Then wlog we can assume D is contained between $x = -r$ and $x = r$. Let $S := \{(x,y) \,|\, x^2 + y^2 \leqslant r\}$. We further assume that $D \cap S = \varnothing$ and D is above S.

First notice that it suffices to consider D's that are convex. In this case $\partial D - \{(x,y) \,|\, x = \pm r\}$ is separated into exactly two parts: those points above and those points blow.

Let $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}: t \in [a,b]$ be a C^1 parametrization of ∂D such that D is always to the left of ∂D. We parametrize ∂S as follows:

$$\forall t \in [a,b], \qquad X(t) = x(t), \qquad Y(t) = \pm\sqrt{1 - x(t)^2} \tag{95}$$

where $+$ is chosen if $(x(t), y(t))$ lies "above" and $-$ is chosen if $(x(t), y(t))$ lies "below". Note that $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ may **not** be a C^1 parametrization. However one can check that it is the union of finitely many C^1 curves.

Now we have

$$\mu(D) = \int_{\partial D} x\,\mathrm{d}y = \int_a^b x(t)\,y'(t)\,\mathrm{d}t = \int_a^b X(t)\,y'(t)\,\mathrm{d}t; \tag{96}$$

$$\pi\,r^2 = \int_{\partial S} -y\,\mathrm{d}x = -\int_a^b Y(t)\,X'(t)\,\mathrm{d}t = -\int_a^b Y(t)\,x'(t)\,\mathrm{d}t. \tag{97}$$

Thus

$$\begin{aligned} \mu(D) + \pi\,r^2 &= \int_a^b X(t)\,y'(t) - Y(t)\,x'(t)\,\mathrm{d}t \\ &\leqslant \int_a^b \sqrt{X(t)^2 + Y(t)^2}\,\sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t \\ &= r\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t \\ &= r\,l. \end{aligned} \tag{98}$$

Now notice

$$r\,l \geqslant \mu(D) + \pi\,r^2 \geqslant 2\,\sqrt{\mu(D)}\,\sqrt{\pi\,r^2} \Longrightarrow \mu(D) \leqslant \frac{l^2}{4\,\pi}. \tag{99}$$

Since clearly when D is a circle we have $\mu(D) = \frac{l^2}{4\,\pi}$, the proof is complete.

4.2. Proofs of Some Theorems

4.2.1. Proof of Theorem 9

Proof. Since $\boldsymbol{x}, \boldsymbol{y}$ have the same trace, for every $s \in [c, d]$ there is at least on $t \in [a, b]$ such that

$$\boldsymbol{y}(s) = \boldsymbol{x}(t). \tag{100}$$

Now we prove that such t is unique. Otherwise there are $t_1 \neq t_2$ such that $\boldsymbol{x}(t_1) = \boldsymbol{x}(t_2) = \boldsymbol{y}(s)$. Contradiction to the regularity of $\boldsymbol{x}$.

Thus we have a one-to-one mapping T: $[c, d] \mapsto [a, b]$ such that $\boldsymbol{x}(T(s)) = \boldsymbol{y}(s)$. But clearly this mapping has to be onto since otherwise there would be a $t_0 \in [a, b]$ such that

$$\boldsymbol{x}(t_0) \notin \{\boldsymbol{x} \in \mathbb{R}^N | \, \boldsymbol{x} = \boldsymbol{y}(s) \text{ for some } s \in [c, d]\}. \tag{101}$$

This contradicts the assumption that $\boldsymbol{x}, \boldsymbol{y}$ have the same trace.

Next we prove that T is continuous. Assume otherwise, that there is $s_0 \in [a, b]$ such that T is not continuous there. Thus we have two sequences

$$s_n', s_n'' \longrightarrow s_0 \text{ but } T(s_n') \longrightarrow t_1, T(s_n'') \longrightarrow t_2, t_1 \neq t_2. \tag{102}$$

But

$$\boldsymbol{x}(t_1) = \lim_{n\to\infty} \boldsymbol{x}(T(s_n')) = \lim_{n\to\infty} \boldsymbol{y}(s_n') = \boldsymbol{y}(s_0) \tag{103}$$

and similarly

$$\boldsymbol{x}(t_2) = \boldsymbol{y}(s_0). \tag{104}$$

Therefore $\boldsymbol{x}(t_1) = \boldsymbol{x}(t_2)$. Contradiction.

Finally we prove differentiability. Fix a point $s_0 \in (c,d)$ and denote $t_0 := T(s_0)$. By differentiability of $\boldsymbol{x}, \boldsymbol{y}$ we know there are vectors $\boldsymbol{u}_0, \boldsymbol{v}_0 \in \mathbb{R}^N$ such that

$$\lim_{t \to t_0} \frac{\boldsymbol{x}(t) - \boldsymbol{x}(t_0)}{t - t_0} = \boldsymbol{u}_0, \qquad \lim_{s \to s_0} \frac{\boldsymbol{y}(s) - \boldsymbol{y}(s_0)}{s - s_0} = \boldsymbol{v}_0. \tag{105}$$

This leads to

$$\lim_{s \to s_0} \left[\frac{\boldsymbol{x}(T(s)) - \boldsymbol{x}(T(s_0))}{T(s) - T(s_0)} \cdot \frac{T(s) - T(s_0)}{s - s_0} \right] = \boldsymbol{v}_0. \tag{106}$$

Since

$$\lim_{s \to s_0} \frac{\boldsymbol{x}(T(s)) - \boldsymbol{x}(T(s_0))}{T(s) - T(s_0)} = \boldsymbol{u}_0 \neq \boldsymbol{0}, \tag{107}$$

there is at least one $i \in \{1, 2, \ldots, N\}$ such that

$$\lim_{s \to s_0} \frac{x_i(T(s)) - x_i(T(s_0))}{T(s) - T(s_0)} = u_{0i} \neq 0 \Longrightarrow \lim_{s \to s_0} \frac{T(s) - T(s_0)}{s - s_0} \text{ exists and equals } \frac{v_{0i}}{u_{0i}}. \tag{108}$$

This also gives continuity of T' and ends the proof. □

Exercise 19. Compare the last step of the above proof with the proof of Chain rule in single variable calculus. Do they follow the same idea? Is there any difference?

4.2.2. Proof of Theorem 16

Proof.

- We first prove $B \Longleftrightarrow C$ by showing

$$\sup_P L(\boldsymbol{x}, P) = \int_a^b \|\boldsymbol{x}'(t)\| \, \mathrm{d}t \tag{109}$$

(if one side is ∞ then so is the other side).

On one hand for any partition P,

$$L(\boldsymbol{x}, P) \leqslant \int_a^b \|\boldsymbol{x}'(t)\| \, \mathrm{d}t. \tag{110}$$

This follows immediately from

$$\|\boldsymbol{x}(t_{i+1}) - \boldsymbol{x}(t_i)\| = \left\| \int_{t_i}^{t_{i+1}} \boldsymbol{x}'(t) \, \mathrm{d}t \right\| \leqslant \int_{t_i}^{t_{i+1}} \|\boldsymbol{x}'(t)\| \, \mathrm{d}t. \tag{111}$$

Therefore we have

$$\sup_P L(\boldsymbol{x}, P) \leqslant \int_a^b \|\boldsymbol{x}'(t)\| \, \mathrm{d}t. \tag{112}$$

On the other hand, since $\boldsymbol{x} \in C^1$, for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\forall |t - s| < \delta, \qquad \|\boldsymbol{x}'(t) - \boldsymbol{x}'(s)\| < \frac{\varepsilon}{2\,(b-a)}. \tag{113}$$

Now take $n \in \mathbb{N}$ such that $h := \frac{b-a}{n} < \delta$, and define partition

$$P_n := \{t_0 = a, t_1 = a + h, t_2 = a + 2\,h, \ldots, t_n = a + n\,h = b\}. \tag{114}$$

We have

$$\begin{aligned}\left|\|\boldsymbol{x}(t_{i+1})-\boldsymbol{x}(t_i)\|-\int_{t_i}^{t_{i+1}}\|\boldsymbol{x}'(t)\|\,\mathrm{d}t\right| &\leqslant \left|\left\|\int_{t_i}^{t_{i+1}}\boldsymbol{x}'(t)\,\mathrm{d}t\right\|-\int_{t_i}^{t_{i+1}}\|\boldsymbol{x}'(t_i)\|\,\mathrm{d}t\right| \\ &\quad+\left|\int_{t_i}^{t_{i+1}}\|\boldsymbol{x}'(t_i)\|\,\mathrm{d}t-\int_{t_i}^{t_{i+1}}\|\boldsymbol{x}'(t)\|\,\mathrm{d}t\right| \\ &= \left|\left\|\int_{t_i}^{t_{i+1}}\boldsymbol{x}'(t)\,\mathrm{d}t\right\|-\left\|\int_{t_i}^{t_{i+1}}\boldsymbol{x}'(t_i)\,\mathrm{d}t\right\|\right| \\ &\quad+\left|\int_{t_i}^{t_{i+1}}\|\boldsymbol{x}'(t_i)\|\,\mathrm{d}t-\int_{t_i}^{t_{i+1}}\|\boldsymbol{x}'(t)\|\,\mathrm{d}t\right| \\ &\leqslant \left\|\int_{t_i}^{t_{i+1}}[\boldsymbol{x}'(t)-\boldsymbol{x}'(t_i)]\,\mathrm{d}t\right\| \\ &\quad+\int_{t_i}^{t_{i+1}}|\|\boldsymbol{x}'(t_i)\|-\|\boldsymbol{x}'(t)\||\,\mathrm{d}t \\ &\leqslant 2\int_{t_i}^{t_{i+1}}\|\boldsymbol{x}'(t)-\boldsymbol{x}'(t_i)\|\,\mathrm{d}t \\ &< 2\,(t_{i+1}-t_i)\,\frac{\varepsilon}{2\,(b-a)}=\frac{t_{i+1}-t_i}{b-a}\,\varepsilon. \end{aligned} \tag{115}$$

This leads to

$$\left|L(\boldsymbol{x},P_n)-\int_a^b\|\boldsymbol{x}'(t)\|\,\mathrm{d}t\right|<\varepsilon \tag{116}$$

and the conclusion follows from the arbitrariness of ε.

- $A\Longrightarrow B$ is left as exercise.
- $C\Longrightarrow A$ is left as exercise. □

Exercise 20. Prove that if $\lim_{d(P)\longrightarrow 0}L(\boldsymbol{x},P)=l<\infty$, then $\sup_P L(\boldsymbol{x},P)=l$.

Exercise 21. Prove that if $\int_a^b\|\boldsymbol{x}'(t)\|\,\mathrm{d}t=l<\infty$ then the limit $\lim_{d(P)\longrightarrow 0}L(\boldsymbol{x},P)$ exists and equals l.

Exercise 22. Prove that $\boldsymbol{x}\colon[0,1]\mapsto\mathbb{R}^2$

$$\boldsymbol{x}(t):=\begin{cases}\left(t,t^2\sin\left(\frac{\pi}{t}\right)\right) & t>0\\ (0,0) & t=0\end{cases} \tag{117}$$

is rectifiable. Note that $\boldsymbol{x}(t)$ is not C^1.

Remark 52. Note that $\boldsymbol{x}'(t)\neq\mathbf{0}$ is not necessary for the above theorem.

Exercise 23. Prove this claim.

4.2.3. Proof of Theorem 27

Proof. Denote by l the arc length of the curve L.

Let $\varepsilon>0$ be arbitrary. Since f is continuous on a compact set, it is uniformly continuous and there is $\delta_1>0$ such that

$$\forall\boldsymbol{x},\boldsymbol{y}\in L,\qquad\|\boldsymbol{x}-\boldsymbol{y}\|<\delta\Longrightarrow|f(\boldsymbol{x})-f(\boldsymbol{y})|<\frac{\varepsilon}{l}. \tag{118}$$

Now we take any partition $P=\{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_m\}\subset L$ with $d(P)<\delta$.

First notice that,

$$\|\boldsymbol{x}_{i+1}-\boldsymbol{x}_i\| = \left\|\int_{t_i}^{t_{i+1}} \boldsymbol{x}'(t)\,\mathrm{d}t\right\| \leqslant \int_{t_i}^{t_{i+1}} \|\boldsymbol{x}'(t)\|\,\mathrm{d}t = l_i. \tag{119}$$

Thus $l_i < \delta \Longrightarrow \|\boldsymbol{x}_{i+1}-\boldsymbol{x}_i\| < \delta$.

Now for any $\Xi := \{\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_{m-1}\} \subset L$ satisfying $\boldsymbol{\xi_i} \in L_i$ we have

$$\begin{aligned}
\left|I(f,\Xi,P) - \int_a^b f(\boldsymbol{x}(t))\,\|\boldsymbol{x}'(t)\|\,\mathrm{d}t\right| &= \left|\sum_{i=0}^{m-1}\left[f(\boldsymbol{\xi}_i)\,l_i - \int_{t_i}^{t_{i+1}} f(\boldsymbol{x}(t))\,\|\boldsymbol{x}'(t)\|\,\mathrm{d}t\right]\right| \\
&\leqslant \sum_{i=0}^{m-1}\left|\int_{t_i}^{t_{i+1}} [f(\boldsymbol{\xi}_i) - f(\boldsymbol{x}(t))]\|\boldsymbol{x}'(t)\|\,\mathrm{d}t\right| \\
&\leqslant \sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} |f(\boldsymbol{\xi}_i) - f(\boldsymbol{x}(t))|\|\boldsymbol{x}'(t)\|\,\mathrm{d}t \\
&< \frac{\varepsilon}{l}\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} \|\boldsymbol{x}'(t)\|\,\mathrm{d}t \\
&= \frac{\varepsilon}{l}\cdot l = \varepsilon.
\end{aligned} \tag{120}$$

Thus we have

$$\lim_{d(P)\longrightarrow 0} I(f,\Xi,P) = \int_a^b f(\boldsymbol{x}(t))\,\|\boldsymbol{x}'(t)\|\,\mathrm{d}t \tag{121}$$

and the proof ends. □

Exercise 24. Prove that the line integral is independent of parametrization.

4.2.4. Proof of Theorem 38.

Proof. Let $\varepsilon > 0$ be arbitrary. By continuity of $\boldsymbol{f}$ there is $\delta > 0$ such that

$$\forall \|\boldsymbol{x}-\boldsymbol{y}\| < \delta, \qquad \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| < \frac{\varepsilon}{l}. \tag{122}$$

Here l is the arc length of L.

Now let $P = \{A = \boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m = B\}$ be a partition of AB and let $\boldsymbol{\xi_i} \in \boldsymbol{x_i}\,\boldsymbol{x_{i+1}}$ be arbitrary. Then we have

$$\begin{aligned}
&\left|\sum_{i=0}^{m-1}(\boldsymbol{f}(\boldsymbol{\xi}_i)\cdot(\boldsymbol{x}_{i+1}-\boldsymbol{x}_i)) - \int_a^b [f_1(\boldsymbol{x}(t))\,x_1'(t) + \cdots + f_N(\boldsymbol{x}(t))\,x_N'(t)]\,\mathrm{d}t\right| \\
\leqslant\ & \sum_{i=0}^{m-1}\left|(\boldsymbol{f}(\boldsymbol{\xi}_i)\cdot(\boldsymbol{x}_{i+1}-\boldsymbol{x}_i)) - \int_{t_i}^{t_{i+1}} [f_1(\boldsymbol{x}(t))\,x_1'(t) + \cdots + f_N(\boldsymbol{x}(t))\,x_N'(t)]\,\mathrm{d}t\right| \\
=\ & \sum_{i=0}^{m-1}\left|\int_{t_i}^{t_{i+1}} [\boldsymbol{f}(\boldsymbol{\xi}_i) - \boldsymbol{f}(\boldsymbol{x}(t))]\cdot\boldsymbol{x}'(t)\,\mathrm{d}t\right| \\
\leqslant\ & \sum_{i=0}^{m-1}\left|\int_{t_i}^{t_{i+1}} \|\boldsymbol{f}(\boldsymbol{\xi}_i) - \boldsymbol{f}(\boldsymbol{x}(t))\|\,\|\boldsymbol{x}'(t)\|\,\mathrm{d}t\right| \\
<\ & \frac{\varepsilon}{l}\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} \|\boldsymbol{x}'(t)\|\,\mathrm{d}t = \frac{\varepsilon}{l}\cdot l = \varepsilon.
\end{aligned} \tag{123}$$

Thus the limit exists and equals $\int_a^b [f_1(\boldsymbol{x}(t))\, x_1'(t) + \cdots + f_N(\boldsymbol{x}(t))\, x_N'(t)]\, \mathrm{d}t$. □

4.3. Arc length parametrization

For a rectifiable curve, there is a standard parametrization using arc length: $\boldsymbol{x}\colon [0, l] \mapsto \mathbb{R}^N$ such that s equals the arc length from $\boldsymbol{x}(0)$ to $\boldsymbol{x}(s)$.

THEOREM 53. *Let $\tilde{\boldsymbol{x}}\colon [a, b] \mapsto \mathbb{R}^N$ be a C^1 curve. Then we can parametrize it using arc length: $\boldsymbol{x}(s)$: $[0, l] \mapsto \mathbb{R}^N$. Then $\boldsymbol{x}(s)$ is also C^1 and we have for all $x \in (0, l)$*

$$\|\boldsymbol{x}'(s)\| = 1. \tag{124}$$

Proof. Assume wlog that $\tilde{\boldsymbol{x}}(a) = \boldsymbol{x}(0)$. Let $S(t)$ be the arc length of the curve from $\tilde{\boldsymbol{x}}(a)$ to $\tilde{\boldsymbol{x}}(t)$, that is

$$\boldsymbol{x}(S(t)) = \tilde{\boldsymbol{x}}(t). \tag{125}$$

Then we have

$$S(t) = \int_a^t \|\tilde{\boldsymbol{x}}'(u)\|\, \mathrm{d}u. \tag{126}$$

Since $\tilde{\boldsymbol{x}}'$ is continuous, so is $\|\tilde{\boldsymbol{x}}'\|$ and it follows from Fundamental Theorem of Calculus (2nd version) that

$$S'(t) = \|\tilde{\boldsymbol{x}}'(t)\|. \tag{127}$$

By the inverse function theorem we conclude that $S(t)$ has a inverse function $T(s)$ satisfying

$$T'(s) = \frac{1}{\|\tilde{\boldsymbol{x}}'(T(s))\|}. \tag{128}$$

Finally, fix any $s_0 \in (0, l)$. Denote $t_0 = T(s_0)$. Then we have

$$\lim_{s \longrightarrow s_0} \frac{\boldsymbol{x}(s) - \boldsymbol{x}(s_0)}{s - s_0} = \lim_{s \to s_0} \left[\frac{\tilde{\boldsymbol{x}}(T(s)) - \tilde{\boldsymbol{x}}(T(s_0))}{T(s) - T(s_0)} \cdot \frac{T(s) - T(s_0)}{s - s_0} \right] = \frac{\tilde{\boldsymbol{x}}'(T(s))}{\|\tilde{\boldsymbol{x}}'(T(s))\|}. \tag{129}$$

Note that the last equality holds because the limits of both ratios exist. □

DEFINITION 54. (TANGENT VECTOR) *Let L be a C^1 curve and let $\boldsymbol{x}(t)$ be a C^1 parametrization of it. Then we define the tangent vector at $\boldsymbol{x}(t)$ to be*

$$\boldsymbol{t}(t) := \frac{\boldsymbol{x}'(t)}{\|\boldsymbol{x}'(t)\|}. \tag{130}$$

Remark 55. It should be emphasized that the direction of $\boldsymbol{\tau}$ depends on the orientation of the curve.

Exercise 25. What is the relation between the tangent vectors of the same curve with two parametrizations of opposite orientations?

Exercise 26. When L is parametrized by arc length, we have

$$\boldsymbol{t}(s) = \boldsymbol{x}'(s). \tag{131}$$

Exercise 27. Prove that

$$\int_{AB} \boldsymbol{f}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{l} = \int_L [\boldsymbol{f} \cdot \boldsymbol{t}]\, \mathrm{d}s. \tag{132}$$

DEFINITION 56. (NORMAL VECTOR; CURVATURE) *Let L be a C^1 curve and let $\boldsymbol{x}(s)$ be its arc length parametrization. Define the normal vector and curvature of L at $\boldsymbol{x} = \boldsymbol{x}(s)$ to be*

$$\boldsymbol{n}(\boldsymbol{x}) := \frac{\boldsymbol{x}''(s)}{\|\boldsymbol{x}''(s)\|}; \qquad \kappa(s) := \|\boldsymbol{x}''(s)\|. \tag{133}$$

Exercise 28. Derive the formula for $\boldsymbol{n}$ for arbitrary C^1 parametrization $\boldsymbol{x}(t)$.

Exercise 29. Using Green's theorem to prove: If $\boldsymbol{f} = \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right): \bar{D} \mapsto \mathbb{R}^2$ is C^1,

$$\int_{\partial D} \boldsymbol{f} \cdot \boldsymbol{n} \, \mathrm{d}s = \int_D \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right] \mathrm{d}(x, y). \tag{134}$$

DEFINITION 57. (BINORMAL VECTOR; OSCULATING PLANE) *When $N = 3$, the vector $\boldsymbol{b}(s) := \boldsymbol{t}(s) \times \boldsymbol{n}(s)$ is called the binormal vector of the curve at $\boldsymbol{x}(s)$. The plane spanned by $\boldsymbol{t}, \boldsymbol{n}$ is called the osculating plane at $\boldsymbol{x}(s)$. Thus $\boldsymbol{b}$ is a normal vector to the osculating plane.*

Exercise 30. Prove that $\boldsymbol{b}'(s) \parallel \boldsymbol{n}$.

DEFINITION 58. (TORSON) *The factor τ in*

$$\boldsymbol{b}'(s) = \tau(s) \, \boldsymbol{n}(s) \tag{135}$$

is called the torsion of the curve at $\boldsymbol{x}(s)$.

Exercise 31. Prove that the torsion is given by

$$\tau(s) = -\frac{(\boldsymbol{x}'(s) \times \boldsymbol{x}''(s)) \cdot \boldsymbol{x}'''(s)}{|\kappa(s)|^2}. \tag{136}$$

Here s is the arc length parameter.

Remark 59. The curvature κ measures how fast the curve is leaving the tangent line; The torsion τ measures how fast the curve is leaving the osculating plane.

Exercise 32. Calculate the tangent, normal, binormal, curvature, torsion at every point for the curve

$$(\cos t, \sin t, t), \qquad t \in \mathbb{R}. \tag{137}$$

Exercise 33. Prove that $\boldsymbol{n}'(s) = -\kappa(s) \, \boldsymbol{t}(s) - \tau(s) \, \boldsymbol{b}(s)$. Therefore the evolution of the $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ coordinate system is governed by

$$\boldsymbol{t}' = \kappa \boldsymbol{n}; \tag{138}$$

$$\boldsymbol{n}' = -\kappa \boldsymbol{t} - \tau \boldsymbol{n}; \tag{139}$$

$$\boldsymbol{b}' = \tau \boldsymbol{n}. \tag{140}$$

4.4. The Cauchy-Crofton formula

Consider the set of all straight lines in $\mathbb{R}^2$. It is clear that there is a bijection between it and the set $\{(p, \theta) \in D := [0, \infty) \times [0, 2\pi)\}$ where p is the distance from the line to the origin and θ is the angle between the x-axis and the vector starting from the origin and perpendicular to the line.

THEOREM 60. (CAUCHY-CROFTON) *Let C be a C^1 curve in $\mathbb{R}^2$. Let a function $N(p, \theta)$ be defined as: $N(p, \theta) = \#$ of times the line (p, θ) intersects C. Then $N(p, \theta)$ is Riemann integrable and furthermore*

$$l(C) = \frac{1}{2} \int_D N(p, \theta) \, \mathrm{d}(p, \theta). \tag{141}$$

Exercise 34. Prove that if $l(C)$ is finite, then there is $R>0$ such that $N(p,\theta)=0$ for all $p>R$. Thus the integration in (141) is in fact over a finite interval.

Exercise 35. Prove that $N(p,\theta)$ is Riemann integrable.

Exercise 36. Prove that is suffices to prove (141) for all piecewise linear curves. Then conclude that it suffices to prove the formula for one single straight line segment.

Exercise 37. Prove that the integral in (141) is invariant under rigit motion of C: That is if C' can be obtained from C by translation, rotation, and flipping, then $\int_D N(p,\theta)\,\mathrm{d}(p,\theta)=\int_D N'(p,\theta)\,\mathrm{d}(p,\theta)$, where $N'(p,\theta)$ is the intersection counting function for C'.

Proof. Thanks to the above exercises, it suffices to prove (141) for the case C is a straight line segment with ends $(-l/2,0)$ and $(l/2,0)$.

Note that in this case $N(p,\theta)$ takes either 0 or 1. Therefore all we need to figure out are those (p,θ) such that the corresponding line intersects C.

This is equivalent to

$$\left\{\left[\begin{pmatrix} -l/2 \\ 0 \end{pmatrix} - p\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right]\cdot\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right\}\cdot\left\{\left[\begin{pmatrix} l/2 \\ 0 \end{pmatrix} - p\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right]\cdot\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right\}\leqslant 0. \tag{142}$$

This simplifies to

$$\left(p+\frac{l\cos\theta}{2}\right)\cdot\left(p-\frac{l\cos\theta}{2}\right)\leqslant 0 \Longleftrightarrow p\leqslant\frac{l}{2}\,|\cos\theta|. \tag{143}$$

Now we integrate:

$$\int_{0\leqslant p\leqslant\frac{l}{2}|\cos\theta|}\mathrm{d}(p,\theta)=\int_0^{2\pi}\frac{l}{2}\,|\cos\theta|\,\mathrm{d}\theta=2\,l. \tag{144}$$

Thus ends the proof. □

Example 61. (141) and similar formulas have many applications such as medical imaging. In particular, it can be used to estimate the length of curves.

Let $r>0$. Consider a family of parallel straight lines with distance r. Count the number of intersections. Rotate the lines by $\pi/4$, $\pi/2$, $3\,\pi/4$ and count the number of intersections. Add the numbers up to get n. Then

$$l(C)\approx\frac{1}{2}\,n\,r\,\frac{\pi}{4}. \tag{145}$$

For example we apply this method to the unit circle. Consider all the horizontal lines with distance r. Each will have two intersections so that total is $4/r$. The other three families give the same numbers. Therefore we have

$$2\,\pi\approx\frac{1}{2}\,\frac{16}{r}\,r\,\frac{\pi}{4}=2\,\pi. \tag{146}$$

So the estimate is in fact accurate for circles.

Remark 62. This method has been applied to estimate the length of DNA molecules. See p. 46 of (do Carmo) .

Problem 1. Explain why the above method of estimation makes sense. Prove that if C is a circle then the estimate is accurate. Try to obtain error estimates for general curves.

Problem 2. Find a non-rectifiable curve C such that the integral in (141) is well-defined. Thus the formula can be used to define arc length for non-rectifiable curves.

5. MORE EXERCISES AND PROBLEMS

For (many many) more exercises on calculation of line and surface integrals, see (DEMIDOVICH) ,(EFIMOV) ,(PKUB) .

5.1. Basic exercises

5.1.1. Curves in $\mathbb{R}^N$

Exercise 38. Let $\boldsymbol{x}: [a,b] \mapsto \mathbb{R}^N$ be a C^1 curve. Prove that

$$\boldsymbol{x}(t) \cdot \boldsymbol{x}'(t) = 0 \Longleftrightarrow \|\boldsymbol{x}(t)\| \text{ is constant.} \tag{147}$$

(Hint:[4])

Exercise 39. (DO CARMO) Show that the tangent lines to the regular parametrized curve $\boldsymbol{x}(t) := (3\,t, 3\,t^2, 2\,t^3)$ make a constant angle with the line $y = 0, z = x$.

Exercise 40. (DO CARMO) Let $\boldsymbol{x}: [a,b] \mapsto \mathbb{R}^N$ be continuous. Let $t_0 \in (a,b)$. We say the curve has a *weak tangent* at t_0 if the line determined by $\boldsymbol{x}(t_0 + h)$ and $\boldsymbol{x}(t_0)$ has a limit position as $h \longrightarrow 0$. We say it has a *strong tangent* at t_0 if the line determined by $\boldsymbol{x}(t_0 + h)$ and $\boldsymbol{x}(t_0 + h')$ has a limit position as $h, h' \longrightarrow 0$.

a) Prove that (t^3, t^2) has weak tangent at $t = 0$ but not strong tangent.

b) If $\boldsymbol{x}$ is a regular curve then it has strong tangent at every $t_0 \in (a,b)$.

Exercise 41. (DO CARMO) Consider the curve $(a\,e^{bt}\cos t, a\,e^{bt}\sin t)$, $t \in [0, \infty)$ with $a > 0, b < 0$. Prove that its arc length is finite.

Exercise 42. Let a curve be given in polar coordinates by $\rho = \mathrm{P}(\theta)$ for $\theta \in [a,b]$.

a) Give sufficient conditions on $\mathrm{P}(\theta)$ such that the curve is C^1? (Hint:[5])

b) Prove that, when the curve is C^1, its arc length is given by

$$l = \int_a^b \sqrt{\mathrm{P}(\theta)^2 + \mathrm{P}'(\theta)^2}\,\mathrm{d}\theta. \tag{148}$$

Exercise 43. Let a curve be given in cylindrical coordinates by

$$(\rho(t), \theta(t), z(t)), \qquad t \in [a,b]. \tag{149}$$

Prove that its arc length is given by

$$\int_a^b \sqrt{\rho'(t)^2 + \rho(t)^2\,\theta'(t)^2 + z'(t)^2}\,\mathrm{d}t. \tag{150}$$

5.1.2. Line integral of the first type/kind (scalar function)

Exercise 44. (DEMIDOVICH, CH. 7 EXAMPLE 1) Calculate the line integral

$$\int_C (x + y)\,\mathrm{d}s \tag{151}$$

where C is the contour of the triangle ABO with vertices $A(1,0), B(0,1)$ and $O(0,0)$. (Ans:[6])

Exercise 45. Calculate

$$\int_C \frac{\mathrm{d}s}{\sqrt{x^2 + y^2 + 5}} \tag{152}$$

4. $\|\boldsymbol{x}(t)\|$ is constant $\Longleftrightarrow \boldsymbol{x}(t) \cdot \boldsymbol{x}(t) = \|\boldsymbol{x}(t)\|^2$ is constant.

5. $x(\theta) = P(\theta)\cos\theta,\ y(\theta) = P(\theta)\sin\theta$.

6. $\sqrt{2} + 1$.

where C is the segment of the straight line connecting $O(0,0)$ and $A(1,2)$. (Hint:[7])

Exercise 46. (DEMIDOVICH, NO. 2301) Calculate

$$\int_C \frac{\mathrm{d}s}{x^2+y^2+z^2} \tag{154}$$

where C is the first turn of the screw-line $x=a\cos t$, $y=a\sin t$, $z=b\,t$.

Exercise 47. (DEMIDOVICH, NO. 2302) Calculate

$$\int_C \sqrt{2\,y^2+z^2}\,\mathrm{d}s \tag{155}$$

where C is the circle $x^2+y^2+z^2=a^2$, $x=y$. (Hint:[8])

5.1.3. Line integral of the second type/kind (vector function)

Exercise 48. Calculate

$$\int_L y^2\,\mathrm{d}x - x^2\,\mathrm{d}y \tag{156}$$

for

a) $L=$boundary of the triangle $(1,0)\to(0,1)\to(-1,0)\to(1,0)$;

b) $L=$the circle $x=\cos t$, $y=1+\sin t$.

(Answer:[9])

Exercise 49. Calculate

$$\int_L x\,\mathrm{d}y - y\,\mathrm{d}x \tag{157}$$

with L from $(0,0)$ to $(1,1)$ given by

a) the straight line $x=y$;

b) polygonal curve $(0,0)\to(1,0)\to(1,1)$;

c) polygonal curve $(0,0)\to(0,1)\to(1,1)$;

d) $y=x^2$;

e) $x=\cos t$, $y=1+\sin t$.

(Answers:[10])

Exercise 50. Calculate

$$\int_L y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z \tag{158}$$

for

a) L is the polygonal curve $(0,0,0)\to(1,0,0)\to(1,1,0)\to(1,1,1)$;

b) L is the polygonal curve $(0,0,0)\to(0,1,0)\to(1,1,0)\to(1,1,1)$;

c) L is the straight line segment $(0,0,0)\to(1,1,1)$;

d) L is $x=t$, $y=t^2$, $z=t^3$, $t\in[0,1]$.

(Answers:[11])

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7. Parametrization $(t,2\,t)$, $t\in[0,1]$. Now calculate

$$\int_0^1 \frac{1}{\sqrt{t^2+(2\,t)^2+4}}\sqrt{1^2+2^2}\,\mathrm{d}t = \int_0^1 \frac{\mathrm{d}t}{\sqrt{t^2+1}}. \tag{153}$$

Change of variables: Either $t=\tan u$ or $t=\sinh u$.

8. Note that on this curve $x=y$, so $\sqrt{2\,y^2+z^2}=\sqrt{x^2+y^2+z^2}=a$. Answer is thus $2\,\pi\,a^2$.

9. $-2/3, -2\,\pi$.

10. $0, 1, -1, 1/3, \pi/2-1$.

11. $1, 2, 3/2, 89/60$.

Exercise 51. Calculate

$$I = \int_S \begin{pmatrix} y \\ -z \\ x \end{pmatrix} \cdot \mathbf{d}\boldsymbol{S} \tag{159}$$

where S is the intersection of $\frac{1}{2}(x^2+y^2)+z^2=a^2$ with $y=x$. The orientation is such that it is counter-clockwise when seen from the positive x direction. (Ans:[12])

Exercise 52. Let $F: \mathbb{R}^N \mapsto \mathbb{R}$ be C^1. Let L be a C^1 curve connecting two points $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$. The orientation is from $\boldsymbol{x}_A$ to $\boldsymbol{x}_B$. Prove

$$\int_L (\nabla F) \cdot \mathrm{d}\boldsymbol{l} = F(\boldsymbol{x}_B) - F(\boldsymbol{x}_A). \tag{160}$$

Here $\nabla F = \begin{pmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \frac{\partial F}{\partial x_N} \end{pmatrix}$.

Then use this result to calculate

$$\int_L 2\,x\,y\,z^3\,\mathrm{d}x + x^2\,z^3\,\mathrm{d}y + 3\,x^2\,y\,z^2\,\mathrm{d}z \tag{161}$$

where L is any C^1 curve connected $(1,2,-1)$ and $(2,3,1)$.

Exercise 53. (BRAND) Let $\boldsymbol{f} = \begin{cases} (x,y,z) & z \geqslant 0 \\ (x,y,-z) & z<0 \end{cases}$. Prove that $\int_L \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{l} = 0$ for any closed C^1 curve L. (Hint:[13])

5.1.4. Green's Theorem

Exercise 54. Let D be a regular region with C^1 boundary. Let f, g be C^1. Prove

$$\int_D \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \int_{\partial D} f\,\mathrm{d}y - g\,\mathrm{d}x. \tag{162}$$

Exercise 55. Prove that

$$\int_L (x+y^2)\,\mathrm{d}x + 2\,x\,y\,\mathrm{d}y = 48 \tag{163}$$

for every C^1 curve L from $(1,2)$ to $(3,4)$.

Exercise 56. Use Green's Theorem to calculate the area of ellipsis $\left\{(x,y)|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1\right\}$.

Exercise 57. Use Green's Theorem to calculate

$$\int_D x^2\,\mathrm{d}(x,y) \tag{164}$$

where D is the triangle with vertices $(x_1,y_1)-(x_2,y_2)-(x_3,y_3)$. (Hint:[14])

Exercise 58. Calculate

$$\int_L (2\,x+y)\,\mathrm{d}x + (2\,x-y)\,\mathrm{d}y \tag{166}$$

using Green's formula. Here L is the ellipsis $x=2\cos t$, $y=\sin t$, $t\in[0,2\,\pi]$, oriented counter-clockwise.

Exercise 59. Find the area enclosed by $4\,y=x^2$ and $4\,x=y^2$.

Exercise 60. Let $L=\{(x,y)|\, x^2+y^2=R^2\}$ be the boundary of the disk centered at the origin and with radius R. Explain why

$$\frac{1}{R}\int_L x^2\,\mathrm{d}s = \pi\,R^2. \tag{167}$$

Note that the integral is Line integral of first type. (Hint:[15])

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12. $2\,\pi\,a^2$.
13. Consider the parts of L above and below $z=0$, connect the intersections.
14. Take $f=0, g=x^3/3$. Answer:

$$\frac{1}{12}[(y_2-y_1)(x_2+x_1)(x_2^2+x_1^2)+(y_1-y_3)(x_1+x_3)(x_1^2+x_3^2)+(y_3-y_2)(x_3+x_2)(x_3^2+x_2^2)]. \tag{165}$$

15. Represent the tangent vector using $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.

5.2. More exercises

Exercise 61. Let a curve be given in spherical coordinates. Derive the formula for its arc length.

Exercise 62. Let L be a C^1 curve. Let $\boldsymbol{x}_0, \boldsymbol{x} \in L$. Denote by $|\boldsymbol{x}_0 \boldsymbol{x}\,|$ the arc length of the part of the curve between the two points. Prove

$$\lim_{\boldsymbol{x} \longrightarrow \boldsymbol{x}_0} \frac{|\boldsymbol{x}_0 \boldsymbol{x}\,|}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} = 1. \tag{168}$$

Exercise 63. Treat the interval $[a, b] \subseteq \mathbb{R}$ as a curve in $\mathbb{R}^2$. Then the usual integral

$$\int_a^b f(x)\,\mathrm{d}x \tag{169}$$

in single variable calculus, according to our definitions of line integrals, is a line integral of a scalar function, or a vector function?

Exercise 64. (DO CARMO) Assume that all normals of a parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a circle. (Note that we do not have any restriction on the dimension N)

Exercise 65. (BRAND) Let L be a curve in $\mathbb{R}^2$ from $\boldsymbol{x}_A$ to $\boldsymbol{x}_B$. Prove that

$$A = \frac{1}{2} \int_L x\,\mathrm{d}y - y\,\mathrm{d}x \tag{170}$$

where A is the area of the region D enclosed by the line segments connecting the origin to $\boldsymbol{x}_A, \boldsymbol{x}_B$ together with L. (Hint:[16])

5.3. Problems

Problem 3. Let L be rectifiable. Let L' be the straight line connecting the two ends of L. Then $l(L) \geqslant l(L')$.

Problem 4. Let $f: \mathbb{R}^N \mapsto \mathbb{R}$ be continuous. Let L be a C^1 curve in $\mathbb{R}^N$. Let $P = \{\boldsymbol{x}_0, \boldsymbol{x}_1, ..., \boldsymbol{x}_m\}$ be a partition of L. Denote by L_P the union of line segments: $\overline{\boldsymbol{x}_0 \boldsymbol{x}_1} \cup \overline{\boldsymbol{x}_1 \boldsymbol{x}_2} \cup ... \cup \overline{\boldsymbol{x}_{m-1} \boldsymbol{x}_m}$, oriented $\boldsymbol{x}_0 \rightarrow \boldsymbol{x}_1 \rightarrow .... \rightarrow \boldsymbol{x}_m$. Prove that

$$\lim_{d(P) \longrightarrow 0} \left| \int_{L_P} f\,\mathrm{d}s - \int_L f\,\mathrm{d}s \right| = 0. \tag{171}$$

Problem 5. Let $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}: [a, b] \mapsto \mathbb{R}^3$ satisfy the Frenet formulas:

$$\boldsymbol{t}' = \kappa \boldsymbol{n}; \tag{172}$$

$$\boldsymbol{n}' = -\kappa \boldsymbol{t} - \tau \boldsymbol{n}; \tag{173}$$

$$\boldsymbol{b}' = \tau \boldsymbol{n}. \tag{174}$$

Further assume that $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ form an orthonormal basis at $s = a$. Prove that $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ form an orthonormal basis for all $s \in [a, b]$.

Problem 6. Let $\boldsymbol{x}: [a, b] \mapsto \mathbb{R}^2$ be a plane curve. Denote by $\theta(s)$ the angle between the x-axis and the tangent $\boldsymbol{t}(s)$ (counter-clockwise, starting from positive x-axis). Prove that

$$\boldsymbol{t}'(s) = (\theta'(s))\,\boldsymbol{t}^{\perp} \tag{175}$$

where $\boldsymbol{t}^{\perp} := \begin{pmatrix} -t_2 \\ t_1 \end{pmatrix}$. What is the relation between θ' and the curvature?

Problem 7. (DO CARMO) Let a plane curve be given through polar coordinates: $\rho = \mathrm{P}(\theta)$. Prove that the curvature is

$$\kappa(\theta) = \frac{2\,(\mathrm{P}'(\theta))^2 - \mathrm{P}(\theta)\,\mathrm{P}''(\theta) + \mathrm{P}(\theta)^2}{[\mathrm{P}'(\theta)^2 + \mathrm{P}(\theta)^2]^{3/2}}. \tag{176}$$

Problem 8. (DO CARMO) Show that the knowledge of the binormal $\boldsymbol{b}(s)$ of a curve with nonzero torsion everywhere, determines the curvature and the absolute value of the torsion.

Problem 9. (DO CARMO) Show that the knowledge of the normal $\boldsymbol{n}(s)$ of a curve with nonzero torsion everywhere, determines the curvature and the torsion.

16. Along the two straight lines the integrals are 0.

Problem 10. Let $\boldsymbol{x}: [0, l] \mapsto \mathbb{R}^3$ be parametrized by arc length and satisfy $\boldsymbol{x}'(s) \neq 0, \boldsymbol{x}''(s) \neq 0$ at every s. Prove that

$$\boldsymbol{x}(s) = \boldsymbol{x}(0) + \left(s - \frac{\kappa(0)^2 s^3}{3!}\right) \boldsymbol{t}(0) + \left(\frac{s^2 \kappa(0)}{2} + \frac{s^3 \kappa'(0)}{3!}\right) \boldsymbol{n}(0) - \frac{s^3}{3!} \kappa(0) \tau(0) \boldsymbol{b}(0) + R(s) \tag{177}$$

where $\lim_{s \to 0} R(s)/s^3 = 0$.

Problem 11. (DO CARMO) Let $\boldsymbol{x}: [0, l] \mapsto \mathbb{R}^3$ be a smooth curve with arc length parametrization. Let $s_0 \in (0, l)$. Assume that a plane P passing s_0 satisfy the following:

- P contains the tangent line at s_0;
- Given any $(a, b) \ni s_0$, there exist points of $\boldsymbol{x}([0, l])$ in both sides of P.

Prove that P is the osculating plane.

Problem 12. (DO CARMO) Show that the curvature $\kappa(t) \neq 0$ of a regular parametrized curve in $\mathbb{R}^3$ is the curvature at the same point of the projection of the curve to its osculating plane at this point.

Problem 13. (DO CARMO) Let $\boldsymbol{x}$ be a simple closed C^1 plane curve with length l. Further assume that the curvature satisfies $0 \leqslant \kappa \leqslant c$ everywhere along the curve. Prove

$$l \geqslant \frac{2\pi}{c}. \tag{178}$$

(Hint:[17])

Problem 14. (DO CARMO) Let $\boldsymbol{x}(t): t \in [0, l]$ be a closed convex C^1 plane curve positively oriented. We define a new curve

$$\boldsymbol{y}(t): t \in [0, l], \qquad \boldsymbol{y}(t) := \boldsymbol{x}(t) - r\, \boldsymbol{n}(t) \tag{180}$$

where $r > 0$ and $\boldsymbol{n}(t)$ is the normal vector.

a) Let $l(\boldsymbol{y}), l(\boldsymbol{x})$ denote the lengths of the two curves respectively. Prove that

$$l(\boldsymbol{y}) = l(\boldsymbol{x}) + 2\pi r; \tag{181}$$

b) Let $A(\boldsymbol{y}), A(\boldsymbol{x})$ denote the areas enclosed by the two curves respectively. Prove that (Hint:[18])

$$A(\boldsymbol{y}) = A(\boldsymbol{x}) + r\, l + \pi r^2; \tag{182}$$

c) What would happen if we drop the "convexity" hypothesis?

(You can use the fact that $\int_L \kappa\, \mathrm{d}s = 2\pi$ where κ is the (signed) curvature of the curve).

17. Let A denote the area enclosed. Green's formula:

$$2A = \int \boldsymbol{x} \cdot (-\boldsymbol{n})\, \mathrm{d}s = \int \kappa^{-1} \boldsymbol{x} \cdot \boldsymbol{t}'\, \mathrm{d}s. \tag{179}$$

Integrate by parts then use isoperimetric inequality.

18. Similar idea as the previous problem.