Math 317 Week 05: Cardinal Arithmetics

March 18, 2014

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References

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"The infinite! No other question has ever moved so profoundly the spirit of man."
— David Hilbert (1862 - 1943)

"The concept of the infinite has intrigued and perplexed people for thousands of years. Over the centuries, authorities like Aristotle (384 - 322 B.C.) and Thomas Aquinas (1225 - 1274) argued that (with the exception of God) nothing was actually infinite, but some things could be potentially infinite, ... They thought of an induction proof ... as a potentially infinite process. The work of Georg Cantor (1845 – 1918) transformed this situation. Cantor's investigations of the foundations of the real numbers led him to work with actual infinities."
— Thomas Q. Sibley. *The Foundations of Mathematics*, John Wiley & Sons, Inc., 2009. Chapter 5.

1. Counting Infinite Sets

1.1. Cardinality

Definition 1. *Two sets A and B are said to have the same cardinality (or "power"), denoted $A \sim B$, if and only if there is a bijection*[1] $f: A \mapsto B$.

Example 2. $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$, $C = \{1, 2, 3, 4\}$. Then

$$A \sim B, \qquad A \nsim C. \tag{1}$$

Remark 3. This is clearly a generalization of everyday counting, where we set up a bijection between the set and a finite subset of $\mathbb{N}$. Such idea of comparing the sizes of sets can be traced back to Aristotle, who puzzled over the obvious contradiction of the following two facts about two circles of different size: They have the same number of points; They have different circumference.

What did Aristotle miss?

He didn't realize that, the size of a point set has several different measures: its cardinality (how many points are there), its measure (how long is the circumference), its density (whether the points "fill" a continuum or not). This point was only fully realized near the end of the 19th century.

Theorem 4. $\sim$ *is an equivalence relation:*

i. $A \sim A$;

ii. $A \sim B \Longrightarrow B \sim A$;

iii. $A \sim B$ *and* $B \sim C$ *then* $A \sim C$.

Exercise 1. prove the above theorem.

Example 5. In his *Dialogue Concerning the Two Chief World Systems*, published in 1632, Galileo Galilei claimed that

> "There are as many squares as there are natural numbers because they are just as numerous as their roots."

Let's prove that this is indeed the case.[2]

1. Recall that a bijection $A \mapsto B$ is a function $A \mapsto B$ that is one-to-one and onto, more specifically, one-to-one means $f(x) = f(y) \Longrightarrow x = y$; Onto means $f(A) = B$.

2. In fact Galileo wrote in his *Dialogues Concerning the Two New Sciences*, First day, "If I should ask further how many squares there are, one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root has its own square, while no square has more than one root and no root more than one square."

Proof. Denote by B the set of all squares. Consider the function $f(x) = x^2$. We prove that it is one-to-one and onto from $\mathbb{N}$ to B.

- one-to-one.

  We need to show that $n_1 \neq n_2 \Longrightarrow f(n_1) \neq f(n_2)$. Assume the contrary. Then we have

$$n_1^2 = n_2^2 \Longrightarrow (n_1 - n_2)(n_1 + n_2) = 0 \Longrightarrow n_1 - n_2 = 0 \tag{2}$$

  since $n_1, n_2 \in \mathbb{N} \Longrightarrow n_1 + n_2 \in \mathbb{N}$ and therefore $n_1 + n_2 \neq 0$. Thus we have reached contradiction.

- onto.

  For any $m \in B$, by definition of B we have $m = n^2 = f(n)$ for some $n \in \mathbb{N}$. □

Exercise 2. Prove that the cardinality of the set of even numbers is the same as that of the odd numbers.

Example 6. $(-1, 1) \sim \mathbb{R}$.

Solution. We define $f: (-1, 1) \mapsto \mathbb{R}$ as

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} - 1 & x > 0 \\ \frac{1}{x} + 1 & x < 0 \end{cases}. \tag{3}$$

Clearly f is one-to-one. To see that it is onto, take any $y \in \mathbb{R}$.

- If $y = 0$, then $y = f(0)$;
- If $y > 0$, then $y + 1 > 1 \Longrightarrow \frac{1}{y+1} \in (0, 1) \Longrightarrow y \in f((-1, 1))$ since $f\left(\frac{1}{y+1}\right) = y$.
- If $y < 0$, then $1 - y > 1 \Longrightarrow \frac{1}{y-1} \in (-1, 0) \Longrightarrow y \in f((-1, 1))$.

Therefore f is a bijection and the conclusion follows.

DEFINITION 7. (FINITE SET) *Let A be a set. We say A is finite if and only if there is $n \in \mathbb{N}$ such that $A \sim \{1, 2, \ldots, n\}$. Otherwise we say A is infinite. In the first case we denote $|A| = n$.*

Exercise 3. Let $|A| = m$, $|B| = n$ with $m, n \in \mathbb{N}$ and $m \neq n$. Prove that $A \nsim B$. (Hint:[3])

PROPOSITION 8. *Let A be finite and B infinite. Then $A \nsim B$.*

Proof. Assume otherwise. Then there is a bijection $g: A \mapsto B$. Now as A is finite, there is $n \in \mathbb{N}$ and a bijection $f: \{1, 2, \ldots, n\} \mapsto A$. Let $h := g \circ f: \{1, 2, \ldots, n\} \mapsto B$. Then h is a bijection and B is finite. Contradiction. □

Exercise 4. (SIBLEY: FOUNDATION) Critique the following "proof" that $\mathbb{N}$ is finite.

Proof. Let $P(n)$ be the statement that the set $\{1, \ldots, n\}$ is finite. It is easy to prove through induction that all $P(n)$ are true. Therefore $\mathbb{N}$ is finite. □

(Hint:[4])

Exercise 5. Prove the following through explicit construction of the bijection:

a) $[0, 1) \sim [0, \infty)$;

b) $[0, 1] \sim [2, 4]$;

c) $\mathbb{N} \sim \{n \in \mathbb{N} | n \text{ even}\}$;

3. Obviously the conclusion holds for $m = 1$. Assume the contrary. Let $m_0 > 1$ be the smallest natural number such that there is $|A| = m_0$ and $|B| = n_0 > m_0$ such that $A \sim B$. Argue that this contradicts the minimality of m_0.

4. Induction can prove $P(n)$ is true for every $n \in \mathbb{N}$ but this does not mean $P(\infty)$ is true.

d) $\mathbb{N} \sim \{p \in \mathbb{N} | p \text{ prime}\}$;

e) $\mathbb{N} \sim \mathbb{Z}$.

(Hint:[5])

DEFINITION 9. *Let A, B be sets. We say the cardinality of A is smaller than that of B, denoted $A \lesssim B$, if and only if there is a one-to-one function $f: A \mapsto B$. If $A \lesssim B$ and $A \nsim B$, then we say the cardinality of A is strictly smaller than that of B.*

Exercise 6. For any sets A, B, if $A \subseteq B$ then $A \lesssim B$. (Hint:[6])

1.2. The Schröder-Bernstein Theorem

Intuitively, two sets having equal cardinality have the same "number" of elements. Thus for each set A, we would like to assign a (possibly infinite) "cardinal number" $|A|$ to it to denote its cardinality. When the set is finite, the task is easy, our "cardinal number"s are simply natural numbers. On the other hand, for infinite sets the situation gets misty. We can of course play the game of just dictating:

Call the cardinality of A a cardinal number, denote it by $|A|$.

But the real question is, suppose we do that, then what are the properties of such new, transfinite number? In particular, if we naturally denote $|A| \leqslant |B| \Longleftrightarrow A \lesssim B$, do we have

$$|A| \leqslant |B|, |B| \leqslant |A| \Longrightarrow |A| = |B|? \tag{4}$$

That this indeed holds is guaranteed by the following theorem, which may be the most useful theorem in proving cardinal relations.

THEOREM 10. (SCHRÖDER-BERNSTEIN) [7]*Let A, B be two sets. If $A \lesssim B$ and $B \lesssim A$ then $A \sim B$.*

Proof. (FROM ROTMAN-KNEEBONE)

We first claim a lemma:

LEMMA 11. *Let A be any set, and let $f: \mathcal{P}(A) \mapsto \mathcal{P}(A)$ be such that $X \subseteq Y \Longrightarrow f(X) \subseteq f(Y)$. Then there is $T \subseteq A$ such that $f(T) = T$.*

Proof. Set

$$S := \{X | X \subseteq A, X \subseteq f(X)\}. \tag{5}$$

Then

$$T := \cup_{X \in S} X. \tag{6}$$

We will prove that $f(T) = T$.

- $T \subseteq f(T)$. Let $X \in S$ be arbitrary. We have $X \subseteq f(X) \subseteq f(T)$. Therefore $T \subseteq f(T)$.
- $f(T) \subseteq T$. Since $T \subseteq f(T)$ we have $f(T) \subseteq f(f(T))$, that is $f(T) \in S$. Consequently by definition of T, $f(T) \subseteq T$. □

Nowe we prove the theorem.

5. $\frac{1}{1-x} - 1$; $2(x+1)$; $n \mapsto 2n$; $n \mapsto p_n$; $2, 4, 6, \ldots$ maps to $\mathbb{N}$, $3, 5, 7, \ldots$ maps to $-\mathbb{N}$, 1 maps to 0.

6. $f(x) = x$.

7. Felix Bernstein (1878 - 1956) was a student in Cantor's seminar.

Siner $A \lesssim B$ there is $f: A \mapsto B$ and similarly there is $g: B \mapsto A$. Now we define $F: \mathcal{P}(A) \mapsto \mathcal{P}(A)$ through

$$F(X) := A - g(B - f(X)). \tag{7}$$

Then for any $X \subseteq Y$ we have $F(X) \subseteq F(Y)$ and it follows from the Lemma 11 that there is T satisfying $F(T) = T$, that is $T = A - g(B - f(T))$, or

$$g(B - f(T)) = A - T. \tag{8}$$

Thus the bijection is given by

$$x \in A \mapsto \begin{cases} f(x) & x \in T \\ g^{-1}(x) & x \in A - T \end{cases} \tag{9}$$

and the proof ends. □

Remark 12. Note that the difficulty in proving this theorem is "intrinsic", in the sense that it is very hard to reduce it to a highly related while essentially finite problem. One is forced to think infinitely.

Remark 13. The basic idea of this proof can be understood as follows. Note that

$$F(\varnothing) = A - g(B) \supseteq \varnothing; \qquad F(A) = A - g(B - f(A)) \subseteq A. \tag{10}$$

Therefore when X "expands" from $\varnothing$ to A, $F(X)$ "expands" from some set "larger than" $\varnothing$ to some other set "smaller than" A. Thus intuitively, at some point during this "expansion", X "catches up" with $F(X)$. In other words there should be a T satisfying $F(T) = T$.

Exercise 7. Let $f: [0, 1] \mapsto [0, 1]$ be continuous. Assume $f(0) \geqslant 0$, $f(1) \leqslant 1$, then there is $t \in [0, 1]$ such that $f(t) = t$. (Hint:[8])

Remark 14. The following proof from (BREUER: INTRODUCTION) may be easier to understand.

> It is clear that there is $C \subsetneq A$ such that $A \sim C$. On the other hand there is $F \subsetneq A$ such that $F \sim B$. Further we have $C \subsetneq F \subsetneq A$. Now write $A = C \cup D \cup E$ where $F = C \cup D$. Then we have
>
> $$C = C_1 \cup D_1 \cup E_1 \tag{11}$$
>
> such that $C \sim C_1, D \sim D_1, E \sim E_1$. Continuing we have a chain. Now denote $G := \cap C_n$. We have
>
> $$A = G \cup D \cup E \cup D_1 \cup E_1 \cup \cdots \tag{12}$$
>
> On the other hand
>
> $$F = G \cup D \cup E_1 \cup D_1 \cup \cdots \tag{13}$$
>
> and we have found our one-to-one correspondence!

Exercise 8. Represent the fixed point T in Lemma 11 using G, D_n, E_n. (Hint:[9])

Exercise 9. Let $A = B = [0, 1]$ and let $f(x) = g(x) = x/2$. Obtain the bijection explicitly.

Exercise 10. Assume $|A| < |B| < |C|$. Does it follow that $|A| < |C|$? Justify your claim.

NOTATION 15. *We will follow the tradition and use German letters* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}, \ldots$ *to denote cardinal numbers.*

8. Intermediate value Theorem applied to $f(t) - t$.
9. $G \cup D_1 \cup D_2 \cup \cdots$

2. Countable Sets

2.1. Countable sets

DEFINITION 16. (DENUMERABLE; COUNTABLE) *A set is A is denumerable if and only if $A \sim \mathbb{N}$. A set is countable if and only if it is either finite or denumerable.*

NOTATION 17. *In this section we will denote $\mathfrak{n} := |\mathbb{N}|$ (traditionally German letters are used).*

Note that it is implied in this definition that $\mathbb{N}$ is the "smallest" infinite set. Therefore we need the following theorem to guarantee the consistency of the definition.

THEOREM 18. *Let A be denumerable. Let $B \subseteq A$. Then B is either finite or denumerable.*

Proof. Assume B is infinite, we prove that B is denumerable. Since $B \subseteq A \sim \mathbb{N}$, we have $B \lesssim \mathbb{N}$. The conclusion would follow from Schröder-Bernstein once we prove the existence of a one-to-one mapping $\mathbb{N} \mapsto B$.

Since $A \sim \mathbb{N}$, there is a bijection $g: A \mapsto \mathbb{N}$. Thus $g(B) \subseteq \mathbb{N}$ could be written as an increasing sequence $b_1 < b_2 < \cdots$. Now we define

$$f: \mathbb{N} \mapsto B \qquad f(n) := g^{-1}(b_n). \tag{14}$$

Clearly this mapping is one-to-one and we have $\mathbb{N} \lesssim B$. By Schröder-Bernstein $\mathbb{N} \sim B$. □

From the above theorem we see that any infinite set A satisfies $A \gtrsim \mathbb{N}$. Thus we will also denote $\aleph_0 := \mathfrak{n}$. Here $\aleph$ is a Hebrew letter and pronounced "aleph".

2.2. Operations on countable sets

THEOREM 19. $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$. *Furthermore, $\mathbb{N} \sim \mathbb{N} \times \cdots \times \mathbb{N}$ (n times).*

Proof. We order elements in $\mathbb{N} \times \mathbb{N}$ as

$$(1,1),(1,2),(2,1),(1,3),(2,2),(3,1),\ldots \tag{15}$$

and construct a bijection.

On the other hand, we can construct two one-to-one functions:

$$f: \mathbb{N} \mapsto \mathbb{N} \times \mathbb{N}, \qquad f(n) := (n,1) \tag{16}$$

and

$$g: \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}, \qquad g(m,n) := 2^m 3^n. \tag{17}$$

Now the conclusion follows from Schröder-Bernstein. □

Exercise 11. Prove $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$ directly though $f(m,n) := 2^{m-1}(2n-1)$.

Exercise 12. Prove $\mathbb{N} \sim \mathbb{N} \times \mathbb{Q}$. (Hint:[10])

Exercise 13. Prove that $\mathbb{Q} \times \mathbb{Q} \sim \mathbb{N}$. (Hint:[11])

10. Order $\mathbb{N} \times \mathbb{Q}$ as $(1,r_1),(1,r_2),(2,r_1),\ldots$
11. Order $(r_1,r_1),(r_1,r_2),(r_2,r_1),\ldots$

THEOREM 20. *Let $A_1, \ldots, A_n$ be countable. Define $A := \cup_{k=1}^{n} A_k$. Then A is countable.*

Proof. Define $B_1 := A_1, B_2 := A_2 - A_1, B_3 := A_3 - (A_1 \cup A_2), \ldots, B_n := A_n - (A_1 \cup \cdots \cup A_{n-1})$. Then clearly $B_1, \ldots, B_n$ are disjoint and still countable.

Without loss of generality, we assume $B_1, \ldots, B_n$ are all denumerable that is none of them is finite. Thus we have bijections $f_1, \ldots, f_n$ satisfying $f_k: B_k \mapsto \mathbb{N}$. Now we construct

$$g: A \mapsto \mathbb{N} \times \mathbb{N}, \qquad g(x) := (k, f_k(x)) \text{ for } x \in B_k. \tag{18}$$

Clearly g is one-to-one and therefore $A \lesssim \mathbb{N} \times \mathbb{N} \sim \mathbb{N}$. On the other hand it is clear that $\mathbb{N} \lesssim A$. Thus by Schröder-Bernstein theorem we conclude $A \sim \mathbb{N}$ and is thus countable. □

Exercise 14. Prove the above theorem without assuming $B_1, \ldots, B_n$ are all denumerable. (Hint:[12])

Exercise 15. (SIBLEY:FOUNDATION) What is wrong with the following proof of "all infinite sets have the same cardinality"?

Let $A = \{a_1, a_2, \ldots\}, B = \{b_1, b_2, \ldots\}$. Then $f(a_n) = b_n$ is the desired bijection.

(Hint:[13])

2.3. A first encounter with Axiom of Choice

"At first it seems obvious, but the more you think about it, the stranger the deductions from this axiom seem to become; in the end you cease to understand what is meant by it."
—– Bertrand Russell commenting on Axiom of Choice.

Axiom of Choice. Given any family of nonempty sets $\{A_i: i \in I\}$ where I is any set, there is a function $f: I \mapsto \cup A_i$ such that $f(i) \in A_i$.

When is Axiom of Choice used?

Whenever we pick exactly one element from every set of a collection of sets, but do not give an explicit rule of how to choose this "representative" element for every set, we are applying Axiom of Choice. Note that this "rule" must enable us to pick the elements all at the same time. In particular, when we are saying

"take r_1 so that ..., then take r_2 so that ..., then take r_3 so that ..., and so on"

we are using Axiom of Choice, as the elements are chosen one by one, not at the same time. Put it bluntly, most of our proofs involving infinitely many objects relies on Axiom of Choice.

Exercise 16. (SIBLEY:FOUNDATION) For each family of sets, give, if possible, an explicit rule to pick one element from each set. (If you can give such a rule, you don't need the Axiom of Choice.)

a) $\{[a, b]: a, b \in \mathbb{R}, a < b\}$;

b) $\{(a, b): a, b \in \mathbb{R}, a < b\}$;

c) All non-empty subsets of $\mathbb{R}$.

d) All finite non-empty subsets of $\mathbb{R}$.

e) All countable nonempty subsets of $\mathbb{R}$.

f) All nonempty subsets of $\mathbb{Z}$.

g) $\{P_n: n \in \mathbb{N}\}$ where P_n is the set of all polynomials of degree n.

12. Thus some of the B_k's are finite.
13. To list A, B as such, one need to first assume that $A \sim \mathbb{N}, B \sim \mathbb{N}$.

h) $\{F_{r,s}: r, s \in \mathbb{R}\}$ where $F_{r,s}$ is the set of all functions $f: \mathbb{R} \mapsto \mathbb{R}$ such that $f(r) = s$.

(Hint:[14])

THEOREM 21. *Let A_n, $n \in \mathbb{N}$ be countable sets. Then $\cup_{n=1}^{\infty} A_n$ is countable.*

Proof. Naïvely we would do the following:

- First since each A_n is countable, we can list its elements in a row $a_{n1}, a_{n2}, \ldots$ and obtain the following infinite matrix:

$$\begin{matrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{matrix} \tag{19}$$

- Now simply delete all the duplicate entries. It is clear that $\cup_{n=1}^{\infty} A_n \lesssim \mathbb{N} \times \mathbb{N}$ and the conclusion follows from Schröder-Bernstein.

In the above naïve argument we in fact applied Axiom of Choice twice. In the first step we picked infinitely many bijections simultaneously, while in the second step we picked simultaneously one element from each set of duplicate entries. □

Exercise 17. Assume $I \sim \mathbb{N}$. Let A_i be defined for each $i \in I$. Then $\cup_{i \in I} A_i$ is countable. (Hint:[15])

Exercise 18. Do we really need Axiom of Choice in the second step? (Hint: [16])

Example 22. A number is called "algebraic" if and only if it is a solution to an algebraic equation

$$a_n x^n + \cdots + a_1 x + a_0 = 0 \tag{20}$$

with $a_n, \ldots, a_0 \in \mathbb{Z}$ and $a_n \neq 0$. Let A be the set of all algebraic numbers, then $A \sim \mathbb{N}$.

Proof. For any equation

$$a_n x^n + \cdots + a_1 x + a_0 = 0 \tag{21}$$

with $a_n, \ldots, a_0 \in \mathbb{Z}$ and $a_n \neq 0$, define its "height" to be

$$h := n + |a_n| + \cdots + |a_1| + |a_0|. \tag{22}$$

Now define

$$A_k := \{r \in \mathbb{C} | \, r \text{ solves an equation with height } k\}. \tag{23}$$

Clearly each A_k is finite and thus countable. The conclusion now follows. □

Exercise 19. Prove that a number is algebraic if and only if it solves

$$\alpha_n x^n + \cdots + \alpha_1 x + \alpha_0 = 0 \tag{24}$$

with $\alpha_n, \ldots, \alpha_0 \in \mathbb{Q}$ and $\alpha_n \neq 0$. (Hint:[17])

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14. a, $\frac{a+b}{2}$, not possible, first one, not possible, not possible, first one according to lexicographic ordering, not possible. (Note that these are just my opinion)

15. Let $r: I \mapsto \mathbb{N}$ be a bijection.

16. Seems to me the 2nd application of AoC is not necessary. Since the whole thing can be well-ordered, we could simply pick the first one.

17. Write each $\alpha_k = \frac{p_k}{q_k}$. Multiply the equation by $\prod_{k=0}^{n} q_k$.

3. Uncountable Sets

3.1. The continuum

DEFINITION 23. *A set A is uncountable if and only if it is not countable, that is if and only if $|A| > \mathfrak{n} = \aleph_0$.*

The existence of uncountable set is a highly non-trivial fact and nobody truly realized it before Cantor.

THEOREM 24. $[0,1]$ *is uncountable.*

Proof. Assume the contrary. Then we have

$$[0,1] = \{x_1, x_2, \ldots\} \tag{25}$$

with $i \neq j \Longrightarrow x_i \neq x_j$.

Now we define a sequence of nested intervals as follows. First divide [0, 1] to [0, 1/3] ∪ [1/3, 2/3] ∪ [2/3, 1]. At least one of the three intervals does not contain x_1. Call it I_1. Then we divide I_1 into three equal-size compact intervals. At least one of the three does not contain x_2. Call it I_2. And so on.

Thus we have a sequence of nested intervals $I_1 \supset I_2 \supset I_3 \supset \cdots$ satisfying

$$|I_n| = 3^{-n}; \qquad x_n \notin I_n. \tag{26}$$

By the Nested Intervas Theorem, there is $c \in [0,1]$ such that

$$c \in \cap_{n=1}^{\infty} I_n. \tag{27}$$

By assumption there is $n_0 \in \mathbb{N}$ such that $c = x_{n_0}$. But then $c \notin I_{n_0} \Longrightarrow c \notin \cap_{n=1}^{\infty} I_n$. Contradiction. □

Exercise 20. Prove that $\mathbb{R}$ is uncountable. (Hint:[18])

Exercise 21. Prove that there exists a real number that cannot be described by words or formulas. (Hint:[19])

NOTATION 25. *We will denote $\mathfrak{c} := |\mathbb{R}|$. We have just proved that $\aleph_0 = \mathfrak{n} < \mathfrak{c}$.*

Remark 26. The Continuum Hypothesis (CH) states that there is no set A satisfying $\mathfrak{n} < |A| < \mathfrak{c}$. Thanks to works by Kurt Gödel and Paul Cohen, it has been known since the 1960s that, roughly speaking, CH is independent of other parts of mathematics – it cannot be proved, and including it would not cause any new contradiction. More specifically, Gödel prove the latter through elaborate recursive construction, while Cohen solved the whole problem by inventing a new, powerful technique called “forcing”.

Remark 27. An alternative proof is as follows. Instead of $[0,1]$, we prove $(0,1]$ is uncountable.

Exercise 22. Prove that the uncountability of $(0,1]$ implies that of $[0,1]$.

Proof. Every $0 < x \leqslant 1$ can be uniquely written as a non-terminating decimal $0.x_1 x_2 \cdots$. Assume $(0,1]$ is countable. Then its elements can be listed as

$$\begin{array}{l} 0.a_{11}a_{12}a_{13}\cdots \\ 0.a_{21}a_{22}a_{23}\cdots \\ 0.a_{31}a_{32}a_{33}\cdots \\ \vdots \end{array} \tag{28}$$

Now define

$$b = 0.b_1 b_2 \cdots \tag{29}$$

18. $\mathbb{R} \gtrsim [0,1]$.

19. Possible words and formulas a countable.

where

$$b_n = \begin{cases} 1 & a_{nn} \neq 1 \\ 2 & a_{nn} = 1 \end{cases}. \tag{30}$$

We see that b cannot be in the list. Contradiction. □

Exercise 23. What can go wrong if we define $b_n = (a_{nn} + 1) \mod 10$? (Hint:[20])

Example 28. $\mathbb{R} - \mathbb{N} \sim \mathbb{R}$.

Proof. Clearly $\mathbb{R} - \mathbb{N} \lesssim \mathbb{R}$. Now all we need is a one-to-one mapping from $\mathbb{R}$ to $\mathbb{R} - \mathbb{N}$. Once this is done the conclusion follows from Schröder-Bernstein.

Denote $A := \{n\sqrt{2} | n \in \mathbb{N}\}$. We define

$$f: \mathbb{N} \mapsto A \qquad f(n) = (2n+1)\sqrt{2}; \tag{31}$$

$$g: A \mapsto A \qquad g(n\sqrt{2}) = 2n\sqrt{2}; \tag{32}$$

Then we define $h: \mathbb{R} \mapsto \mathbb{R} - \mathbb{N}$ as

$$h(x) := \begin{cases} f(x) & x \in \mathbb{N} \\ g(x) & x \in A \\ x & \text{otherwise} \end{cases}. \tag{33}$$

Clearly h is one-to-one. □

Exercise 24. Prove that $\mathbb{R} - \mathbb{Q} \sim \mathbb{R}$. (Hint:[21])

More generally, we have

THEOREM 29. *Let A be such that $|A| > \mathfrak{n}$. Let $B \subseteq A$ be countable. Then $A - B \sim A$.*

Proof. First assume B is denumerable, that is $B \sim \mathbb{N}$. Since $|A| > \mathfrak{n}$, $A - B$ is infinite. Therefore there is $C \subseteq A - B$ such that $C \sim \mathbb{N}$.

Now let f, g be two bijections from $\mathbb{N}$ to B, C respectively. We define

$$h: A \mapsto A - B \qquad h(a) := \begin{cases} a & a \in A - B - C \\ g(2n) & a = g(n) \in C \\ g(2n+1) & a = f(n) \in B \end{cases}. \tag{34}$$

Thus h is one-to-one and $A \lesssim A - B$. Since $A - B \lesssim A$ is obvious, we have $A - B \sim A$ by Schröder-Bernstein. □

Exercise 25. Let A be infinite and B be countable. Then $A \cup B \sim A$. Explain why this is not equivalent to the above theorem. (Hint:[22])

3.2. Cardinality of $\mathbb{R}^N$

"I see it, but I don't believe it."

—– Georg Cantor, 1874, in a letter to Richard Dedekind, on discovering spaces of different dimension may in fact have the same number of points and thus be indistinguishable as sets. [23]

20. We may get a new sequence of digits but it may not be a new real number, as decimal representation is not unique.

21. $A = \mathbb{Q}\sqrt{2} := \{r\sqrt{2} | r \in \mathbb{Q}\}$.

22. $|A| > \mathfrak{n}$ may not hold.

23. (VILENKIN: STORY) "Cantor searched for three years (from 1871 to 1874) for a proof that it was impossible to set up a one-to-one correspondence between the points of the segment and the points of the square."

THEOREM 30. $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$.

Proof. It suffices to prove $(0,1) \times (0,1) \sim (0,1)$.

We will construct a one-to-one mapping from $(0,1) \times (0,1)$ to $(0,1)$. For any $(x,y) \in (0,1) \times (0, 1)$, we write their unique infinite decimal expansions (that is we choose 0……99999…. instead of 0……0000….) :

$$x = 0.x_1 x_2 \ldots; \qquad y = 0.y_1 y_2 \cdots. \tag{35}$$

Now we map

$$(x,y) \mapsto 0.x_1 y_1 x_2 y_2 \ldots\ldots \tag{36}$$

It is clear that this is one-to-one. Now the conclusion follows immediately from Schröder-Bernstein. □

Exercise 26. Prove $(0,1) \sim \mathbb{R}$ and justify the above strategy of proving $(0,1) \times (0,1) \sim (0,1)$ (instead of proving directly $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$). (Hint:[24])

Remark 31. A bit conter-intuitively, the above mapping is not a bijection. In fact this was exactly what Cantor did in his first "proof" of $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$ that he communicated to Richard Dedekind in 1877. Later in June 1877 Dedekind noticed a flaw in the proof caused by the ambiguity of decimal representation of numbers. For example, if we require that no number should end in 00…., then there is no pair $(0,1) \times (0,1)$ that is mapped to the number

$$0.13101010\ldots. \in (0,1). \tag{37}$$

This flaw turned out to be fixable.

The fix is as follows. We divide each decimal into "blocks": Each non-zero digit that is not preceded by zero is to form a block by itself, but any ruin of consecutive zeros is to form a single block with the non-zero digit that comes immediately after it. Thus

$$x = 0.X_1 X_2 \ldots; \qquad y = 0.Y_1 Y_2 \ldots \tag{38}$$

where each X_i (or Y_i) is a "block". Finally we define

$$(x,y) \mapsto 0.X_1 Y_1 X_2 Y_2 \ldots.. \tag{39}$$

and this gives the desired bijection.

Remark 32. The claim also follows from Peano's construction of a "square-filling curve", a function $f(x): [0,1] \mapsto [0,1] \times [0,1]$ that is continuous and onto, about which Henri Poincaré commented "How was it possible that intuition could so deceive us?".

Exercise 27. Find any description of the construction of "square-filling curves" and try to prove that the resulting function is continuous.

Exercise 28. Assuming the existence of Peano's curve. Prove that $[0,1] \sim [0,1] \times [0,1]$. Does Axiom of Choice play a role in the proof? (Hint:[25])

Note that $f(x)$ is not one-to-one. It turned out that $f: [0,1] \mapsto [0,1] \times [0,1]$ that is at the same time continuous and bijective does not exist. Thus the intuition of dimension is not totally deceptive after all.

3.3. Power sets and hierarchy of cardinals

DEFINITION 33. (POWER SETS) *Let A be a set. Then its power set is defined as*

$$\mathcal{P}(A) := \{B \mid B \subseteq A\}. \tag{40}$$

24. $(0,1) \sim (0,2) \sim (-1,1) \sim \mathbb{R}$.
25. Yes.

Example 34. Let $A=\{1,2,3\}$. Then

$$\mathcal{P}(A)=\{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}. \tag{41}$$

Exercise 29. Let A be finite. Then $\mathcal{P}(A)$ is also finite. Furthermore $|\mathcal{P}(A)|=2^{|A|}$.

Exercise 30. Let A be finite. Consider the set of all functions from A to $\{0,1\}$:

$$B:=\{f: A\mapsto\{0,1\}\}. \tag{42}$$

Then $B\sim\mathcal{P}(A)$. (Hint:[26])

Theorem 35. *$A\sim B\Longrightarrow\mathcal{P}(A)\sim\mathcal{P}(B)$.*

Proof. Exercise. □

Exercise 31. Prove that $A\lesssim B\Longrightarrow\mathcal{P}(A)\lesssim\mathcal{P}(B)$. (Hint:[27])

Theorem 36. *$\mathcal{P}(\mathbb{N})\sim\mathbb{R}$.*

Proof.

- $\mathcal{P}(\mathbb{N})\lesssim\mathbb{R}$. We define $f:\mathcal{P}(\mathbb{N})\mapsto\mathbb{R}$ as follows.

  $$f(\varnothing)=0; \tag{43}$$

  $$f(\{n_1,\ldots,n_k,\ldots\})=0.a_1a_2a_3\cdots \tag{44}$$

  where

  $$a_n=\begin{cases}1 & \exists k\in\mathbb{N},\quad n=n_k\\ 0 & \text{otherwise}\end{cases}. \tag{45}$$

  It is clear that f is one-to-one and the conclusion follows.

- $\mathbb{R}\lesssim\mathcal{P}(\mathbb{N})$.

  It suffice to prove $\mathbb{R}^+-\mathbb{Q}^+\sim\mathcal{P}(\mathbb{Q}^+)$. Recalling the construction of $\mathbb{R}$ through Dedekind's cuts, we see that there is a one-to-one mapping from $\mathbb{R}^+$ to $\mathcal{P}(\mathbb{Q}^+)$.

Now the conclusion follows from Schröder-Bernstein. □

Theorem 37. *There is no largest cardinal.*

Proof. For any set A we define its "power set" $\mathcal{P}(A):=\{B|\,B\subseteq A\}$. We prove that $|\mathcal{P}(A)|>|A|$. The best way to present this may be the following. Clearly

$$\mathcal{P}(A)\sim C:=\{\text{characteristic functions on } A\}. \tag{46}$$

Now assume that there is a bijection $A\mapsto C: a\mapsto f_a$. Define

$$\varphi: A\mapsto\{0,1\} \tag{47}$$

through

$$\varphi(a):=1-f_a(a). \tag{48}$$

Then there is $b\in A$ such that $\varphi=f_b$. But then $\varphi(b)=f_b(b)=1-f_b(b)\Longrightarrow f_b(b)=1/2$. Contradiction! □

26. $f\in B$ maps to $f^{-1}(1)\subseteq A$.

27. $A\lesssim B$ then there is one-to-one $f: A\mapsto B$. Thus $\mathcal{P}(A)\sim\mathcal{P}(f(A))$. But there is a one-to-one mapping $\mathcal{P}(f(A))\mapsto\mathcal{P}(B)$: $F(C)=C$.

4. Advanced Topics, Notes, and Comments

4.1. Cardinal Arithmetics

4.1.1. Arithmetic operations

DEFINITION 38. (ADDITION) *Let A, B be disjoint. Let the cardinal numbers $\mathfrak{m} := |A|$, $\mathfrak{n} := |B|$. Then we define*

$$\mathfrak{m} + \mathfrak{n} := |A \cup B|. \tag{49}$$

Exercise 32. Prove that the above definition does not depend on the choices of the specific sets A, B.

Exercise 33. Let $\mathfrak{m}, \mathfrak{n}$ be arbitrary cardinal numbers. Prove that $\mathfrak{m} + \mathfrak{n} \geqslant \mathfrak{m}$.

Example 39. $\mathfrak{c} + \mathfrak{n} = \mathfrak{c}$.

Proof. Take $A = [0, 1)$, $B = \{2 - n^{-1} | n \in \mathbb{N}\}$. Then we have

$$\mathfrak{c} + \mathfrak{n} = |A \cup B| \leqslant |\mathbb{R}| = \mathfrak{c}. \tag{50}$$

Thus ends the proof as the other direction follows from the above exercise. □

DEFINITION 40. (SUBTRACTION) *Let $B \subseteq A$. Let the cardinal numbers $\mathfrak{m} := |A|$, $\mathfrak{n} := |B|$. Then if $\mathfrak{m} > \mathfrak{n}$ we can define*

$$\mathfrak{m} - \mathfrak{n} := |A - B|. \tag{51}$$

Exercise 34. $\mathfrak{c} - \mathfrak{n} = \mathfrak{c}$.

Exercise 35. Let $m \in \mathbb{N}$. Then $\mathfrak{n} - m = \mathfrak{n}$.

Exercise 36. Explain why the assumption $\mathfrak{m} > \mathfrak{n}$ is necessary.

DEFINITION 41. (MULTIPLICATION) *Let the cardinal numbers $\mathfrak{m} := |A|$, $\mathfrak{n} := |B|$. Then we define*

$$\mathfrak{m} \cdot \mathfrak{n} := |A \times B|. \tag{52}$$

Exercise 37. Show that this definition naturally generalizes to the situation of finite product and powers.

Exercise 38. Prove that when $\mathfrak{m}, \mathfrak{n}$ are finite the definition reduces to that of usual multiplication.

Exercise 39. Prove that $\mathfrak{c}^2 = \mathfrak{c}$; $\mathfrak{n}^4 = \mathfrak{n}$.

Exercise 40. Prove that $\mathfrak{c} \cdot \mathfrak{n} = \mathfrak{c}$.

Exercise 41. Let $\mathfrak{m}$ be a cardinal number and $n \in \mathbb{N}$. Prove that $\mathfrak{m} \cdot n = \mathfrak{m} + \cdots + \mathfrak{m}$ (n terms).

DEFINITION 42. (EXPONENTIAL) *Let the cardinal numbers $\mathfrak{m} := |A|$, $\mathfrak{n} := |B|$. Define C to be the set of all functions from B to A:*

$$C := \{f: B \mapsto A\}. \tag{53}$$

Then define

$$\mathfrak{m}^{\mathfrak{n}} := |C|. \tag{54}$$

Exercise 42. Notice that it is $B \mapsto A$ instead of $A \mapsto B$. Work on finite sets to make sure you understand why.

Exercise 43. Prove that $2^{\mathfrak{n}} = \mathfrak{c}$.

Exercise 44. Prove that $2^{\mathfrak{c}} > \mathfrak{c}$.

Exercise 45. Prove that $\mathfrak{n}^{\mathfrak{n}} = \mathfrak{c}$.

Exercise 46. Prove that the set of all real sequences has cardinality $\mathfrak{c}$, that is $\mathfrak{c}^{\mathfrak{n}} = \mathfrak{c}$. Then prove that the set of all real functions has cardinality $2^{\mathfrak{c}}$.

Exercise 47. What is the cardinality of the set of all real continuous functions?

Exercise 48. What is the cardinality of the set of all complex analytic functions?

4.1.2. Cardinal arithmetics

The following rules are satisfied by cardinal numbers. The proofs are left as exercises.

- $\mathfrak{m} \cdot (\mathfrak{n} + \mathfrak{p}) = \mathfrak{m} \cdot \mathfrak{n} + \mathfrak{m} \cdot \mathfrak{p}$;
- $\mathfrak{m}^{(\mathfrak{n}+\mathfrak{p})} = \mathfrak{m}^{\mathfrak{n}} \cdot \mathfrak{m}^{\mathfrak{p}}$;
- $(\mathfrak{m} \cdot \mathfrak{n})^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{p}} \cdot \mathfrak{n}^{\mathfrak{p}}$;
- $(\mathfrak{m}^{\mathfrak{n}})^{\mathfrak{p}} = \mathfrak{m}^{(\mathfrak{n}\mathfrak{p})}$;

Exercise 49. Let $\mathfrak{m}, \mathfrak{n}, \mathfrak{p}$ be cardinal numbers. Assume $\mathfrak{m} \leqslant \mathfrak{n}$. Prove

$$\mathfrak{m} + \mathfrak{p} \leqslant \mathfrak{n} + \mathfrak{p}; \quad \mathfrak{m} \cdot \mathfrak{p} \leqslant \mathfrak{n} \cdot \mathfrak{p}; \quad \mathfrak{m}^{\mathfrak{p}} \leqslant \mathfrak{n}^{\mathfrak{p}}; \quad \mathfrak{p}^{\mathfrak{m}} \leqslant \mathfrak{p}^{\mathfrak{n}}. \tag{55}$$

THEOREM 43. (CANTOR) *Let* $\mathfrak{m}$ *be any cardinal number. Then* $2^{\mathfrak{m}} > \mathfrak{m}$.

Proof. We have done that already. Exercise. □

COROLLARY 44. *Let* W *be the collection of all sets. Then* W *is not a set.*

Proof. Assume otherwise. Denote $\mathfrak{w} := |W|$. Then $\mathcal{P}(W)$ has cardinality $2^{\mathfrak{w}} > \mathfrak{w}$. One the other hand, each member of $\mathcal{P}(W)$, being a set, is also a member of W, which includes all possible sets. This means $\mathcal{P}(W) \subseteq W$ and consequently $2^{\mathfrak{w}} \leqslant \mathfrak{w}$. Contradiction. □

Example 45. $2^{\mathfrak{c}} = \mathfrak{c}^{\mathfrak{c}}$.

Proof. Obviously $2^{\mathfrak{c}} \leqslant \mathfrak{c}^{\mathfrak{c}}$. For the other direction, we have

$$\mathfrak{c}^{\mathfrak{c}} = |\{\text{all functions on } \mathbb{R}\}| \leqslant |\mathcal{P}(\mathbb{R} \times \mathbb{R})| = 2^{|\mathbb{R} \times \mathbb{R}|} = 2^{\mathfrak{c} \cdot \mathfrak{c}} = 2^{\mathfrak{c}}. \tag{56}$$

Thus the proof after application of Schröder-Bernstein. □

5. More Exercises and Problems

5.1. Basic exercises

5.1.1. Comparing sizes of sets

Exercise 50. Let A, B be sets and let $|\cdot|$ denote cardinality. Prove $|A \times B| = |B \times A|$. (Hint:[28])

Exercise 51. Let A, B, A_1, B_1, C be sets. Assume $A \sim A_1, B \sim B_1$, then

$$A \times B \sim C \Longrightarrow A_1 \times B_1 \sim C. \tag{57}$$

(Hint:[29])

Exercise 52. Let A, B, C be sets. Prove that

a) $A \lesssim A$;

b) $(A \lesssim B)$ and $(B \lesssim C) \Longrightarrow A \lesssim C$.

Do we have $(A \lesssim B, A \nsim B)$ and $(B \lesssim C) \Longrightarrow (A \lesssim C, A \nsim C)$? Justify. (Hint:[30])

Exercise 53. (Sibley:Foundation) Critique the following "proof" of $\mathbb{N} \sim (0, 1)$.

> Clearly $\mathbb{N} \lesssim (0, 1)$. For the other direction, list all numbers of $(0, 1)$ as follows. First $0.1 - 0.9$, then $0.01 - 0.99$, then $0.001 - 0.999$, and so on. Clearly this gives a one-to-one function $f: (0, 1) \mapsto \mathbb{N}$ and consequently $(0, 1) \lesssim \mathbb{N}$. By Schroeder-Bernstein we conclude that $\mathbb{N} \sim (0, 1)$.

(Hint:[31])

Exercise 54. (Sibley:Foundation) Prove that $\{(x, y) | x^2 + y^2 \leqslant 1\} \sim [-1, 1]^2$. (Hint:[32])

5.1.2. Countable sets

Exercise 55. Prove the following:

a) $\mathbb{N} \sim \mathbb{N} \cup \{0\}$;

b) Let X be any infinite set and $y \notin X$. Prove $X \sim X \cup \{y\}$.

c) Let X be any infinite set that does not contain any natural number. Prove $X \sim X \cup \mathbb{N}$.

(Hint:[33])

Exercise 56. Let $E \subseteq \mathbb{R}$ be open. Prove that E is the union of countably many open intervals. (Hint:[34])

Exercise 57. (Sibley: Foundation) Let S_n be the set of all subsets of $\mathbb{N}$ that have size n. For instance, $\{3, 7\} \in S_2, \{1, 4, 9\} \in S_3$.

a) Prove that S_2 is countable;

b) Prove for all $n \in \mathbb{N}$ that S_n is countable;

c) Prove that the set of all finite subsets of $\mathbb{N}$ is countable.

d) What about the set of all subsets of $\mathbb{N}$? Can you prove that it is countable?

(Hint:[35])

Exercise 58. Let A be the set of all finite rational sequences. Prove that $A \sim \mathbb{N}$.

Exercise 59. (Sibley:Foundation) What is wrong with the following proof of $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$?

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28. $(a, b) \mapsto (b, a)$ is bijection.
29. $(a, b) \mapsto c$ becomes $(f(a), g(b)) \mapsto c$.
30. Yes. Schröder-Bernstein.
31. This only gives 1-1 mapping from all rationals ending with 000.... to $\mathbb{N}$.
32. For $\gtrsim$, use polar coordinate for say $\{1/4 \leqslant x^2 + y^2 \leqslant 1\}$.
33. Show that any infinite set has a subset $\sim \mathbb{N}$.
34. Consider all $(a_n, b_n) \subseteq E$ where a_n, b_n are rational.
35. $S_2 \lesssim \mathbb{N} \times \mathbb{N}, S_3 \lesssim \mathbb{N} \times \mathbb{N} \times \mathbb{N}$...

Clearly $\mathbb{N} \lesssim \mathbb{N} \times \mathbb{N}$. Now for $(a_1...a_n, b_1...b_n) \in \mathbb{N} \times \mathbb{N}$, we define

$$f(a_1...a_n, b_1...b_n) = a_1 b_1 \cdots a_n b_n \tag{58}$$

For example $f(314,\ 676) = 361746$. Clearly f is one-to-one. Now the conclusion follows from Schröder-Bernstein. (Hint:[36])

5.1.3. Uncountable sets

Exercise 60. What is the cardinality of the set of transendental numbers? Justify. (MATH317 2014 HW5)

Exercise 61. What is the cardinality of $\mathbb{Q}^{\mathbb{R}}$? Justify. (Hint:[37])

Exercise 62. (SIBLEY:FOUNDATION) Let $D = \{x \in (0,1) |$ the decimal expansion of x contains only odd digits$\}$. Find the cardinality of D and justify your answer. (Hint:[38])

Exercise 63. Let S be the set of all bijections from $\mathbb{N}$ to $\mathbb{N}$. Prove that $|B| = \mathfrak{c}$, that is $B \sim \mathbb{R}$. (Hint:[39])

Exercise 64. Let A be the set of all characteristic functions on $\mathbb{R}$; Let B be the set of all continuous functions on $\mathbb{R}$. Prove that $|A| = 2^{\mathfrak{c}}$, $|B| = \mathfrak{c}$. (Hint:[40])

Exercise 65. What is the cardinality of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$? (Hint:[41])

5.2. More exercises

Exercise 66. Let S be the set of all bijections from $\mathbb{R}$ to $\mathbb{R}$. What is the cardinality of S? Justify. (Hint:[42])

Exercise 67. (BABY RUDIN) $E \subseteq \mathbb{R}^N$ is called "perfect" if and only if

i. E is closed;

ii. Every point in E is a limit point of E.

Prove that a perfect set must be uncountable. Then show that the uncountability of $[0,1]$ as well as the Cantor set follows from this. (Hint:[43]) (MATH317 2014 HW4)

Exercise 68. Let the equivalence relation $\sim$ be defined as

$$x \sim y \Longleftrightarrow x - y \in \mathbb{Q}. \tag{61}$$

---

36. What about 37 and 2456?

37. $\mathbb{Q}^{\mathbb{R}} \gtrsim \mathcal{P}(\mathbb{R})$ is obvious. On the other hand $\mathbb{Q}^{\mathbb{R}} \lesssim$ the set of all functions $\mathbb{R} \mapsto \mathbb{R} \lesssim$ all subsets of $\mathbb{R} \times \mathbb{R} \sim \mathcal{P}(\mathbb{R} \times \mathbb{R})$. But $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$.

38. Note that two decimals containing only odd digits must represent two different numbers. Thus the answer is $5^{\mathbb{N}}$ which satisfies $\mathbb{R} \sim 2^{\mathbb{N}} \lesssim 5^{\mathbb{N}} \lesssim \mathcal{P}(\mathbb{N} \times \mathbb{N}) \sim \mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

39. For every 0-1 sequence $\{a_n\}$ (that is a_n takes only 0 or 1), we can define a bijection f as follows: Divide $\mathbb{N}$ into infinitely many blocks $\{0,1\}, \{2,3\}, \{4,5\}, ...$ Then for every $n \in \mathbb{N}$,

- If $a_n = 1$, then $f(2n) = 2n+1$, $f(2n+1) = f(2n)$;
- If $a_n = 0$, then $f(2n) = 2n$, $f(2n+1) = f(2n+1)$.

Clearly f is a bijection. Therefore $B \gtrsim \{0,1\}^{\mathbb{N}} \sim \mathbb{R}$.
On the other hand, identifying each function with its graph, we have $B \subset \mathcal{P}(\mathbb{N} \times \mathbb{N}) \Longrightarrow B \lesssim \mathcal{P}(\mathbb{N} \times \mathbb{N}) \sim \mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

40. $A \sim \mathcal{P}(\mathbb{R})$; $B \sim \mathbb{R}^{\mathbb{Q}} \sim \mathbb{R}^{\mathbb{N}} \sim \mathbb{R}$.

41. Obviously it is no more than the number of continuous functions. On the other hand, considering

$$f_x(t) := \begin{cases} e^{-1/(x-t)^2} & x \neq t \\ 0 & x = t \end{cases} \tag{59}$$

we see that it is at least as many as $\mathbb{R}$.

42. On one hand it is $\lesssim \mathcal{P}(\mathbb{R} \times \mathbb{R}) \sim \mathcal{P}(\mathbb{R})$; On the other hand for every $A \subseteq \mathbb{R}^+$, we construct bijection

$$f(x) = \begin{cases} x & x \text{ or } -x \text{ belongs to } A \\ -x & x \text{ or } -x \text{ belongs to } \mathbb{R}^+ - A \\ 0 & x = 0 \end{cases}. \tag{60}$$

Thus it is no less than $\mathcal{P}(\mathbb{R}^+) \sim \mathcal{P}(\mathbb{R})$.

43. Assume the contrary. Try to construct nested intervals excluding all points.

Then what is the cardinality of the quotient set $\mathbb{R}/\sim$?

Exercise 69. Let $\mathfrak{a}, \mathfrak{b}$ be infinite cardinals with $\mathfrak{a} \leqslant \mathfrak{b}$. Find $\mathfrak{a}+\mathfrak{b}, \mathfrak{a}\cdot\mathfrak{b}, \mathfrak{a}^{\mathfrak{b}}, \mathfrak{b}^{\mathfrak{a}}$.

Exercise 70. (SIBLEY:FOUNDATION) Let A, B be sets with $A \subseteq B$. Let $|A| = \mathfrak{a}, |B| = \mathfrak{b}$ with $\mathfrak{a} < \mathfrak{b}$. Define

$$\mathfrak{b} - \mathfrak{a} := |B - A|. \tag{62}$$

You need to justify all your claims.

a) Prove that the definition is consistent.

b) Show that the definition is not consistent anymore if $\mathfrak{a} < \mathfrak{b}$ is relaxed to $\mathfrak{a} \leqslant \mathfrak{b}$.

c) Let $n \in \mathbb{N}$. Find $\aleph_0 - n$.

d) Find $\mathfrak{c} - \aleph_0$.

Exercise 71. Let $\mathfrak{m}$ be any transfinite cardinal number. Prove that

$$\mathfrak{m} + \mathfrak{m} = 2 \cdot \mathfrak{m}, \qquad \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}^2. \tag{63}$$

Does it hold that $\mathfrak{m} + \mathfrak{m} = \mathfrak{m} \cdot 2$?

Exercise 72. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be cardinals. Prove

$$\mathfrak{a} \cdot b = \mathfrak{b} \cdot \mathfrak{a}; \qquad (\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathfrak{d} = \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{d}); \qquad \mathfrak{a} \cdot (\mathfrak{b} + \mathfrak{d}) = \mathfrak{a} \cdot \mathfrak{b} + \mathfrak{a} \cdot \mathfrak{d}. \tag{64}$$

5.3. Problems

Problem 1. (SIBLEY:FOUNDATION) Let $P_0 := \mathbb{N}$ and define P_n recursively by $P_{n+1} := \mathcal{P}(P_n)$.

a) Show that the cardinality of $S_0 := \cup_{n \in \mathbb{N}} P_n$ is larger than any $|P_n|$.

b) What if we apply the same construction to S_0?

Problem 2. Is it true that $\mathcal{P}(A) \sim \mathcal{P}(B) \Longrightarrow A \sim B$?

Problem 3. (BABY RUDIN) A point $\boldsymbol{x} \in \mathbb{R}^N$ is called a condensation point of a set E if and only if for every $\delta > 0$, $E \cap B(\boldsymbol{x}, \delta)$ is uncountable.

Let $E \subseteq \mathbb{R}^N$ be uncountable. Let P be the set of all condensation points of E. Prove

a) P is perfect;

b) $P^c \cap E$ is countable.

(Hint: [44])

Problem 4. Richard Dedekind (1831 - 1916) defined "infinite set" as follows.

> A is infinite if and only if there is $B \subsetneq A$, such that $A \sim B$.

Prove that this definition is equivalent to Definition 7. Then use this definition to prove that $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ are all infinite.

Problem 5. The following is due to Ramsey.

> Let D be denumerable (that is countable and infinite). Let $E := \{\{a, b\} | a, b \in D\}$ be the set of **un-ordered** pairs of D. Let $A \cup B$ be a partition of E, that is $E = A \cup B, A \cap B = \varnothing$. Then for any partition $A \cup B$ there is a denumerable subset $D' \subseteq D$ such that
>
> $$E' := \{\{c, d\} | c, d \in D'\} \tag{65}$$
>
> is either a subset of A, or a subset of B.

44. Consider all open balls with rational centers and radii. Then consider those balls sharing with E only countably many points.