Math 317 Week 04: Fourier Series

March 18, 2014

Table of contents

References

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"It was not accidental that the notion of function generally accepted now was first formulated in the celebrated memoir of Dirichlet (1837) dealing with the convergence of Fourier series; or that the definition of Riemann's integral in its general form appeared in Riemann's *Habilitationsschrift* devoted to trigonometric series; or that the theory of sets, one of the most important developments of nineteenth-century mathematics, was created by Cantor in his attempts to solve the problem of the sets of uniqueness for trigonometric series. In more recent times, the integral of Lebesgue was developed in close connexion with the theory of Fourier series, and the theory of generalized functions (distributions) with that of Fourier integrals."

—— Antonio Zygmund, 1958[1]

1. Introduction

1.1. Trigonometric series

DEFINITION 1. (TRIGONOMETRIC SERIES) *A trigonometric series is a special type of infinite series of functions*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\,\pi\,x}{L}+b_n\sin\frac{n\,\pi\,x}{L}\right\}. \tag{1}$$

DEFINITION 2. (PERIODIC FUNCTION) *A function $f:\mathbb{R}\mapsto\mathbb{R}$ is said to be periodic if there is $T>0$ such that $f(x+T)=f(x)$ for all $x\in\mathbb{R}$. Such T is called a period of f. If $L:=\min\{T|\,T$ is a period of $f(x)\}$ exists and is positive, we call L the fundamental period.*

Remark 3. When we say "the period of f is ..." we often mean fundamental period. For example, the period of $\sin x$ is $2\,\pi$. On the other hand, when we say $f(x)$ is a $2\,\pi$-periodic function, we often mean $2\,\pi$ is one period of f.

Exercise 1. What would happen if we allow $T=0$ in the definition above? (Hint:[2])

Exercise 2. What is the period of $\sin\left(\sqrt{3}\,x\right)$? (Hint:[3])

LEMMA 4. *If $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\,\pi\,x}{L}+b_n\sin\frac{n\,\pi\,x}{L}\right\}$, then $f(x)$ is periodic with $2\,L$ being a period.*

Proof. We have

$$\begin{aligned} f(x+2\,L) &= \frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\,\pi\,(x+2\,L)}{L}+b_n\sin\frac{n\,\pi\,(x+2\,L)}{T}\right\}\\ &= \frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\,\pi\,x}{L}+2\,n\,\pi\right)+b_n\sin\left(\frac{n\,\pi\,x}{T}+2\,n\,\pi\right)\right\}\\ &= \frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\,\pi\,x}{L}+b_n\sin\frac{n\,\pi\,x}{T}\right\}\\ &= f(x). \end{aligned} \tag{2}$$

1. In Preface to volume one of *Trigonometric Series*.
2. Every $f(x)$ is periodic since $f(x)=f(x+0)$ is always true.
3. $\frac{2\,\pi}{\sqrt{3}}$.

Thus $2L$ is a period of $f(x)$. □

Exercise 3. Must $2L$ be the fundamental period of $f(x)$? Justify your claim. (Hint:[4])

Note. In the following we will study the possibility of representing functions with trigonometric series. From the above lemma it is clear that such functions must be $2L$-periodic. Therefore in the following we will use the same symbol $f(x)$ to denote a function defined on $[-L, L)$ and $\tilde{f}(x)$, its periodic extension to $\mathbb{R}$:

$$\tilde{f}(x) = f(x - 2kL), \qquad x \in [2kL, 2(k+1)L). \tag{3}$$

1.2. d'Alembert + Euler vs Bernoulli

1.2.1. One-dimensional wave equation

Imagine a perfectly elastic, stretched string initially lying motionless along the x-axis, with two ends fixed at $x=0$ and $x=l$. Assume that the string has a uniform density ρ and a uniform tension T. Now imagine the string was pulled a bit in to become $y = f(x)$ at $t=0$ (keep the two ends fixed). Then the string would obviously oscillate when $t>0$. We denote its (time-dependent) position by $u(x,t)$, that is at time t_0 the string's position is given by the curve $y = u(x,t)$.

Since at time $t=0$, the string is at position $y = f(x)$, we have $u(x,0) = f(x)$. We further assume that the string is still at $t=0$, thus $\frac{\partial u}{\partial t}(x,0) = 0$. These two requirements are imposed at $t=0$, and is thus called "initial conditions".

On the other hand, assume we keep the two ends of the string fixed for all time, thus obtaining the boundary conditions:

$$\forall t>0, \qquad u(0,t) = u(l,t) = 0. \tag{4}$$

Next we derive the partial differential equation that governs the motion of the string. For simplicity we ignore gravity. We also assume that the string is pulled just a little bit so that the tension T does not change. We study the motion of a small segment, from x to $x + \Delta x$, of the string. Newton's second law gives

$$(\text{mass of the segment}) \cdot (\text{vertical acceleration of the segment}) = (\text{vertical force}) \tag{5}$$

which translates to

$$\rho\, \Delta x \frac{\partial^2 u}{\partial t^2} = T\, [\sin(\alpha(x+\Delta x)) - \sin(\alpha(x))]. \tag{6}$$

where $\alpha(x)$ is the angle between the string and the x-axis at the point $(x, u(x,t))$. By our assumption α is very small, therefore

$$T\, [\sin(\alpha(x+\Delta x)) - \sin(\alpha(x))] \approx T\, [\tan(\alpha(x+\Delta x)) - \tan(\alpha(x))]. \tag{7}$$

Divide both sides by Δx and taking limit $\Delta x \longrightarrow 0$, we have

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \cdot \left\{ \frac{\partial(\tan(\alpha(x)))}{\partial x} \right\}. \tag{8}$$

4. $\cos(2x)$.

but from calculus we know that

$$\tan(\alpha(x)) = \frac{\partial u}{\partial x}(x, t). \tag{9}$$

Thus we have

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \cdot \frac{\partial^2 u}{\partial x^2}. \tag{10}$$

Since $T, \rho > 0$ we can denote $c^2 := T/\rho$ and the equation becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{11}$$

This is called the one-dimensional wave equation.

Remark 5. The physical meaning of $c > 0$ is the propagation speed of the waves along the string.

Summarizing, the position of the string, $u(x, t)$, satisfies the following initial-boundary value system:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \forall x \in (0, l), \forall t > 0; \qquad \text{(equation)} \tag{12}$$

$$u(x, 0) = f(x); \qquad \frac{\partial u}{\partial t}(x, 0) = 0 \qquad \forall x \in [0, l]; \qquad \text{(initial conditions)} \tag{13}$$

$$u(0, t) = u(l, t) = 0 \qquad \forall t > 0; \qquad \text{(boundary conditions)} \tag{14}$$

1.2.2. d'Alembert's solution

In 1747[5], Jean Le Rond d'Alembert (1717 - 1783) solved (12 – 14) as follows.

First, through a change of variables, he obtained the following solution for (11) in

$$u(x, t) = \Phi(x + c\,t) + \Psi(x - c\,t) \tag{15}$$

where Φ, Ψ have continuous second-order derivatives. More specifically, he obained:

> If $u(x, t)$ is twice continuously differentiable in (x, t), then there are twice continuously differentiable functions $\Phi(x)$ and $\Psi(x)$ such that (15) holds.

Exercise 4. Prove the above claim. (Hint:[6])

Now the boundary and initial conditions become, for all $x \in [0, 1]$ and $t > 0$,

$$\Phi(x) + \Psi(x) = f(x), \;\; c\,\Phi'(x) - c\,\Psi'(x) = 0, \;\; \Phi(c\,t) + \Psi(-c\,t) = 0, \;\; \Phi(1 + c\,t) + \Psi(1 - c\,t) = 0. \tag{16}$$

After some algebra, d'Alembert obtained the solution as

$$u(x, t) = \frac{F(x + c\,t) + F(x - c\,t)}{2}. \tag{17}$$

5. J. le R. d'Alembert, *Recherches sur la courbe que forme une corde tendue mise en vibration*, Mémoires de l'Académie Royale de Berlin, 3 (1747: publ. 1749), 214- 219.

6. Change of variable $\xi := x + c\,t, \eta := x - c\,t$. Then (12) becomes $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$.

where $F(x)$ is obtained from $f(x)$ through first extending $f(x)$ to an odd function on $[-l, l]$, then extending this odd function to a periodic function with period $2l$.

$$F(x) := \begin{cases} f(x) & x \in [0, l] \\ -f(-x) & x \in [-l, 0] \end{cases} \text{ and } \forall x \in \mathbb{R}, f(x + 2l) = f(x). \tag{18}$$

Exercise 5. Obtain solution formula (17). (Hint:[7])

Naturally, d'Alembert required $f(x)$ to be twice continuously differentiable. In fact he required that $f(x)$ can be extended periodically to a function that is twice continuously differentiable on the whole real line $\mathbb{R}$. This is a very restrictive requirement: For example a plucked violin string does not satisfy it. Euler in 1748 tried to argue that the formula (17) still makes sense even when $f(x)$ fails to meet the requirement. d'Alembert disagreed.

1.2.3. Bernoulli's solution

In 1755, Daniel Bernoulli published a memoir[8] claiming that every motion of the string can be written as

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n \pi x}{l}\right) \cos\left(\frac{n \pi c t}{l}\right) \tag{19}$$

while c_n satisfy

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n \pi x}{l}\right). \tag{20}$$

Exercise 6. Let $u_1(x, t)$, $u_2(x, t)$ satisfy (12 – 14) and let $C_1, C_2 \in \mathbb{R}$. Prove that the linear combination $C_1 u_1 + C_2 u_2$ also satisfies (12 – 14).[9]

The idea is to write $u(x, t)$ as sum of infinitely many functions:

$$u(x, t) = \sum_{n=1}^{\infty} X_n(x) T_n(t). \tag{21}$$

and require each $X_n(x) T_n(t)$ to satisfy as much of (12 – 14) as possible.

> Complete solutions to Exercises 7 – 11 can be found in any introductory book on partial differential equations.

Exercise 7. Prove that if $X(x) T(t)$ satisfies (12), then there is $\lambda \in \mathbb{R}$ such that

$$X''(x) - \lambda X(x) = 0; \qquad T''(t) - c^2 \lambda T(t) = 0. \tag{22}$$

Exercise 8. Prove that if furthermore we require $X(x) T(t)$ to satisfy the boundary conditions (14), then the only possible values of λ are

$$\lambda_n = -\left(\frac{n \pi}{l}\right)^2 \tag{23}$$

and for each λ_n, the solution for the X equation is given by

$$C \sin\left(\frac{n \pi x}{l}\right) \tag{24}$$

7. Full derivation can be found in any introductory PDE book.

8. D. Bernoulli, *Réflexions et éclaircissemens sur les nouvelles vibrations des cordes exposées dans les mémoires de l'Académie de 1747 et 1748*, Mémoires de l'Académie Royale de Berlin, 9 (1753:publ. 1755), 147 - 172.

9. Equations with such property are called "linear". The understanding of linear partial differential equations have been more or less complete now.

where $C \in \mathbb{R}$ is arbitrary.

Exercise 9. Prove that for each λ_n, we have

$$X_n(x)\,T_n(t) = \sin\left(\frac{n\,\pi\,x}{l}\right)\left[A_n \cos\left(\frac{n\,c\,\pi\,t}{l}\right) + B_n \sin\left(\frac{n\,c\,\pi\,t}{l}\right)\right] \tag{25}$$

where $A_n, B_n \in \mathbb{R}$ are arbitrary.

Exercise 10. Prove that if $u(x,t)$ is given by (21), then

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\,\pi\,x}{l}\right); \tag{26}$$

$$0 = \sum_{n=1}^{\infty} B_n \frac{n\,c\,\pi}{l} \sin\left(\frac{n\,\pi\,x}{l}\right). \tag{27}$$

Argue that this leads to Bernoulli's solution (19 – 20).

Exercise 11. Spot as many theoretical issues in the above as you can. What kind of theorems are needed to settle all these issues?

Euler thought (19 – 20) are nonsensical, d'Alembert agreed, while Bernoulli wouldn't change his mind.

1.3. Fourier's claim

Jean Baptiste Joseph Fourier (21 March 1768 – 16 May 1830)[10], during his service under Napoleon as Prefect of the Department of Isère in Grenoble, devoted his spare time to the study of the propagation of heat.

Example 6. Consider a rod of length L lying along the x axis, with ends at 0 and L. Denote the temperature at position x and time t by $u(x,t)$. Let its temperature distribution at $t=0$ be $f(x)$. Assume that the temperature at both ends are kept at 0. Then $u(x,t)$ satisfies

$$\frac{\partial u}{\partial t}(x,t) = \kappa\,\frac{\partial^2 u}{\partial x^2}(x,t) \qquad \forall x \in (0,L), \forall t > 0; \tag{28}$$

$$u(x,0) = f(x) \qquad \forall x \in (0,L); \tag{29}$$

$$u(0,t) = u(L,t) = 0 \qquad \forall t > 0, \tag{30}$$

where $\kappa > 0$.

Exercise 12. Derive (28 – 30).

Exercise 13. Solve (28 – 30) using Bernoulli's method.

In 1807 Fourier finished a paper on the theory of heat, in which he sided with Bernoulli and claimed that any function can be written as a trignometric series. His theory met much criticism and had to wait till 1822 to be formally published in the form of the book *Théorie analytique de la chaleur*.

The justification and generalization of his claim would become the foundation of modern analysis.

10. For more about Fourier's colorful life serving the Jacobins, Napoleon, and Louis XVIII, as well as his involvement in the discovery of the Rosetta Stone, see Chapter 1 of (ELENA:EVOLUTION) .

2. Calculation of Fourier Expansions

2.1. Formulas for expansion coefficients

Assume that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\,\pi\,x}{L} + b_n \sin \frac{n\,\pi\,x}{L} \right\}. \tag{31}$$

We would like to calculate a_n and b_n.

The key property that makes calculation of Fourier expansion possible are the following orthogonality relations:

$$\int_{-L}^{L} \cos\left(\frac{n\,\pi\,x}{L}\right) \mathrm{d}x = \int_{-L}^{L} \sin\left(\frac{n\,\pi\,x}{L}\right) \mathrm{d}x = 0; \tag{32}$$

$$\int_{-L}^{L} \cos\frac{n\,\pi\,x}{L} \cos\frac{m\,\pi\,x}{L}\, \mathrm{d}x = \begin{cases} L & n = m \\ 0 & n \neq m \end{cases}; \tag{33}$$

$$\int_{-L}^{L} \sin\frac{n\,\pi\,x}{L} \cos\frac{m\,\pi\,x}{L}\, \mathrm{d}x = 0; \tag{34}$$

$$\int_{-L}^{L} \sin\frac{n\,\pi\,x}{L} \sin\frac{m\,\pi\,x}{L}\, \mathrm{d}x = \begin{cases} L & n = m \\ 0 & n \neq m \end{cases}. \tag{35}$$

Exercise 14. Prove (33 – 35). (Hint:[11])

From these properties it is easy to calculate:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\,\pi\,x}{L}\, \mathrm{d}x. \qquad n = 0, 1, 2, \ldots \tag{37}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\frac{n\,\pi\,x}{L}\, \mathrm{d}x. \qquad n = 1, 2, \ldots \tag{38}$$

Exercise 15. Certain assumptions have to be made to obtaining (37-38). Identify these assumptions. (Hint:[12])

Remark 7. These formulas were first derived by Leonhard Euler in 1777 (published in 1793).[13]

Exercise 16. (An ultimate exercise for constant-coefficient ODE) Formally "solve" the equation

$$y(x + 2\,\pi) - y(x) = f(x) \tag{39}$$

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11. You need to know the following formulas:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B; \qquad \sin(A \pm B) = \sin A \cos B \pm \sin B \cos A. \tag{36}$$

12. Termwise integration. A sufficient condition is uniform convergence of the series in (31).

13. Remember that he was against Daniel Bernoulli's idea of representing functions by trigonometric series!

to obtain (37-38) through the following argument due to Euler:

- Taylor expansion:

$$(2\,\pi)\,y'(x)+\frac{(2\,\pi)^2}{2!}\,y''+\frac{(2\,\pi)^3}{3!}\,y'''+\cdots=f(x). \tag{40}$$

  This is a linear non-homogeneous equation of infinite order.

- Solve the homogeneous part

$$(2\,\pi)\,y'(x)+\frac{(2\,\pi)^2}{2!}\,y''+\frac{(2\,\pi)^3}{3!}\,y'''+\cdots=0 \tag{41}$$

  which is linear and constant-coefficient to obtain general solution.

- Use undetermined coefficients to solve the original equation.

- Try to use variation of parameters to solve the equation.

Definition 8. *For any $f(x)$ and any $L>0$, we can calculate a_n, b_n using (37-38) and obtain a Fourier series*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\,\pi\,x}{L}+b_n\sin\frac{n\,\pi\,x}{L}\right\}. \tag{42}$$

This is called the Fourier expansion of $f(x)$ on $[-L,L]$.

Remark 9. It is important to realize that the Fourier expansion of $f(x)$ may or may not converge to $f(x)$. To emphasize this point, we avoid using equality and write

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\,\pi\,x}{L}+b_n\sin\frac{n\,\pi\,x}{L}\right\}. \tag{43}$$

Remark 10. There is a subtle difference, first noticed and emphasized by Bernhard Riemann, between "Fourier series" and "trigonometric series". A Fourier series is automatically a trigonometric series. However there are trigonometric series that cannot be obtained from and function $f(x)$ and the formulas (37-42), and thus are not Fourier series.

2.2. Examples

Example 11. Compute the Fourier series for

$$f(x)=x, \qquad -\pi\leqslant x<\pi. \tag{44}$$

Solution. We have $L=\pi$. Now we compute the coefficients.

- First

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x\,\mathrm{d}x=0; \tag{45}$$

- Next

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(n\,x)\,\mathrm{d}x \\ &= \frac{1}{n\,\pi}\int_{-\pi}^{\pi} x\,\mathrm{d}\sin(n\,x) \\ &= \frac{1}{n\,\pi}\left[x\sin(n\,x)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\sin(n\,x)\,\mathrm{d}x\right] \\ &= 0. \end{aligned} \tag{46}$$

- Finally

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(n\,x)\,\mathrm{d}x \\ &= -\frac{1}{n\,\pi}\int_{-\pi}^{\pi} x\,\mathrm{d}\cos(n\,x) \\ &= -\frac{1}{n\,\pi}\left[x\cos(n\,x)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\cos(n\,x)\,\mathrm{d}x\right] \\ &= -\frac{1}{n\,\pi}[\pi(-1)^n - (-\pi)(-1)^n] \\ &= -\frac{2(-1)^n}{n} \\ &= \frac{2(-1)^{n+1}}{n}. \end{aligned} \tag{47}$$

Summarizing, the Fourier expansion for x is

$$x \sim \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(n\,x). \tag{48}$$

Exercise 17. Find $x_0 \in \mathbb{R}$ such that

$$x_0 \neq \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(n\,x_0). \tag{49}$$

(Hint:[14])

Example 12. Compute the Fourier series for

$$f(x) = \begin{cases} 1 & -2 \leqslant x < 0 \\ x & 0 < x < 2 \end{cases}. \tag{50}$$

Solution. Clearly $L = 2$ and $\frac{n\,\pi\,x}{L}$ becomes $\frac{n\,\pi\,x}{2}$. We compute

- first

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,\mathrm{d}x = \frac{1}{2}\int_{-2}^{0}\mathrm{d}x + \frac{1}{2}\int_{0}^{2} x\,\mathrm{d}x = 2 \Longrightarrow \frac{a_0}{2} = 1; \tag{51}$$

14. $x_0 = \pm\pi$.

- next

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\,\pi\,x}{2}\right)\mathrm{d}x \\ &= \frac{1}{2}\int_{-2}^{0}\cos\left(\frac{n\,\pi\,x}{2}\right)\mathrm{d}x + \frac{1}{2}\int_{0}^{2} x\cos\left(\frac{n\,\pi\,x}{2}\right)\mathrm{d}x \\ &= \frac{1}{n\,\pi}\sin\left(\frac{n\,\pi\,x}{2}\right)\Big|_{-2}^{0} + \frac{1}{n\,\pi}\int_{0}^{2} x\,\mathrm{d}\sin\left(\frac{n\,\pi\,x}{2}\right) \\ &= 0 + \frac{1}{n\,\pi}\left[x\sin\left(\frac{n\,\pi\,x}{2}\right)\Big|_{0}^{2} - \int_{0}^{2}\sin\left(\frac{n\,\pi\,x}{2}\right)\mathrm{d}x\right] \\ &= \frac{1}{n\,\pi}\frac{2}{n\,\pi}\cos\left(\frac{n\,\pi\,x}{2}\right)\Big|_{0}^{2} \\ &= \frac{2}{(n\,\pi)^2}\,[(-1)^n - 1]; \end{aligned} \tag{52}$$

- and finally

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\,\pi\,x}{2}\right)\mathrm{d}x \\ &= \frac{1}{2}\int_{-2}^{0}\sin\left(\frac{n\,\pi\,x}{2}\right)\mathrm{d}x + \frac{1}{2}\int_{0}^{2} x\sin\left(\frac{n\,\pi\,x}{2}\right)\mathrm{d}x \\ &= -\frac{1}{n\,\pi}\cos\left(\frac{n\,\pi\,x}{2}\right)\Big|_{-2}^{0} - \frac{1}{n\,\pi}\int_{0}^{2} x\,\mathrm{d}\cos\left(\frac{n\,\pi\,x}{2}\right) \\ &= -\frac{1}{n\,\pi}\,[1-\cos(-n\,\pi)] - \frac{1}{n\,\pi}\left[x\cos\left(\frac{n\,\pi\,x}{2}\right)\Big|_{0}^{2} - \int_{0}^{2}\cos\left(\frac{n\,\pi\,x}{2}\right)\mathrm{d}x\right] \\ &= \frac{1}{n\,\pi}\,[(-1)^n - 1] - \frac{1}{n\,\pi}\left[2\,(-1)^n - \frac{2}{n\,\pi}\sin\left(\frac{n\,\pi\,x}{2}\right)\Big|_{0}^{2}\right] \\ &= -\frac{1}{n\,\pi}\,[(-1)^n + 1]. \end{aligned} \tag{53}$$

Summarizing, we have

$$f(x) = \begin{cases} 1 & -2 < x < 0 \\ x & 0 < x < 2 \end{cases} \sim 1 + \sum_{n=1}^{\infty} \frac{2}{(n\,\pi)^2}\,[(-1)^n - 1]\cos\left(\frac{n\,\pi\,x}{2}\right) - \frac{1}{n\,\pi}\,[(-1)^n + 1]\sin\left(\frac{n\,\pi\,x}{2}\right). \tag{54}$$

Exercise 18. Calculate the Fourier expansion on $[-\pi, \pi]$ for

$$f(x) = \begin{cases} -\pi & -\pi \leqslant x < 0 \\ x & 0 \leqslant x < \pi \end{cases}. \tag{55}$$

(Ans: [15].)

Exercise 19. Prove the following: Let $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\,\pi\,x}{L} + b_n\sin\frac{n\,\pi\,x}{L}\right\}$ on $[-L, L]$.

- If $f(x)$ is odd, then all $a_n = 0$;
- If $f(x)$ is even, then all $b_n = 0$.

(Hint:[16])

15. $-\frac{\pi}{4} - \sum_{n=1}^{\infty}\left\{\frac{2}{(2\,n-1)^2\,\pi}\cos(2\,n-1)\,x + \frac{1-2\,(-1)^n}{n}\sin n\,x\right\}$.
16. Say $f(x)$ is odd. Write $\int_{-L}^{L} = \int_{0}^{L} + \int_{-L}^{0}$. Make a change of variable $x = -u$ in the second integral.

3. Pointwise Convergence

Note. For simplicity of presentation, from now on we will restrict ourselves to the case $L=\pi$. Every result proved in this case can be generalized straightforwardly to the general case.

Let $f(x)$ be integrable on $[-\pi,\pi]$ and let $\frac{a_0}{2}+\sum_{n=1}^{\infty}\{a_n\cos(n\,x)+b_n\sin(n\,x)\}$ be its Fourier expansion on $[-\pi,\pi]$, with

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos n\,x\,\mathrm{d}x,\qquad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin n\,x\,\mathrm{d}x. \tag{56}$$

In the following discussion we always identify $f(x)$ with its 2π-periodic extension.

3.1. Partial sum

As we are interested not only in the convergence of $\frac{a_0}{2}+\sum_{n=1}^{\infty}\{a_n\cos(n\,x)+b_n\sin(n\,x)\}$ but also whether its limit is $f(x)$, we naturally try to study $\lim_{n\to\infty}|f(x)-S_n(x)|$ where

$$S_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}\{a_k\cos(k\,x)+b_k\sin(k\,x)\}. \tag{57}$$

Substituting (56) into (57) we have

$$\begin{aligned}
S_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,\mathrm{d}x\\
&\quad+\sum_{k=1}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi}\{f(x)\,[\cos(k\,u)\cos(k\,x)+\sin(k\,u)\sin(k\,x)]\}\,\mathrm{d}u\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left\{f(u)\left[\frac{1}{2}+\sum_{k=1}^{n}\cos(k\,(x-u))\right]\right\}\mathrm{d}u\\
&= \int_{-\pi}^{\pi}f(u)\,D_n(x-u)\,\mathrm{d}u.
\end{aligned} \tag{58}$$

Here

$$D_n(t):=\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{n}\cos(k\,t)=\frac{\sin[(n+1/2)\,t]}{2\pi\sin(t/2)},\qquad n=0,1,2,3,\ldots. \tag{59}$$

are called the n-th *Dirichlet kernel*.

What is a "kernel"?

The field of analysis, from late 19th to mid-20th century, has been dominated by the study of linear operators, that is linear mappings from one space of functions to another space of functions.

Intuitively, an operator can be seen as a "blackbox" whose input and output are both functions Input $f \xrightarrow{\text{Operator}}$ Output g, or more formally

$$A(f) = g. \tag{60}$$

Exercise 20. Argue that differentiation is an operator.

The study of general operators turns out to be very difficult. As a consequence the focus before mid-20th century has been linear operators, that is operators L such that

$$L(a\, f + b\, g) = a\, L(f) + b\, L(g) \tag{61}$$

for function f, g and numbers a, b.

Exercise 21. Prove that

$$L(f)(x) := \int_0^x f(t)\, \mathrm{d}t \tag{62}$$

is a linear operator.

Exercise 22. Prove that Taylor expansion to degree 3 at $x_0 = 0$

$$f(x) \mapsto T(f)(x) := f(0) + f'(0)\, x + \frac{f''(0)}{2}\, x^2 + \frac{f'''(0)}{6}\, x^3 \tag{63}$$

is a linear operator.

Exercise 23. Let $f(x)$ has Fourier expansion $\frac{a_0}{2} + \sum_{k=1}^{\infty} \{a_k \cos(k\, x) + b_k \sin(k\, x)\}$ on $[-\pi, \pi]$. Prove that for any $n \in \mathbb{N}$ the mapping from f to the partial sum

$$f(x) \mapsto S_n(x) := \frac{a_0}{2} + \sum_{k=1}^{n} \{a_k \cos(k\, x) + b_k \sin(k\, x)\} \tag{64}$$

is a linear operator. To emphasize this we can denote the partial sums by $S_n(f)(x)$.

Many linear operators allow an "integral" representation: Say L is a linear operator defined on $f: A \mapsto \mathbb{R}$. Then under certain conditions, there is $K(x, y): A \times A \mapsto \mathbb{R}$ such that

$$L(f)(x) = \int_A K(x, y)\, f(y)\, \mathrm{d}y. \tag{65}$$

This function $K(x,\, y)$ is called the "kernel" of the operator L. Clearly the properties of the abstract object L can be understood through the study of the more concrete – thus easier to study – object $K(x, y)$.

Exercise 24. Find the kernel for the operator defined in (62).

In some cases the kernel $K(x, y)$ is of the special form $k(x - y)$ where $k: A \mapsto \mathbb{R}$. This property would significantly reduce the difficulty of understanding the operator as now we only need to understand the function $k(\xi)$.

Exercise 25. Prove

$$\frac{1}{2\,\pi} + \frac{1}{\pi}\sum_{k=1}^{n} \cos(k\,t) = \frac{\sin[(n+1/2)\,t]}{2\sin(t/2)}. \tag{66}$$

(Hint:[17])

Exercise 26. Prove that

$$\int_{-\pi}^{\pi} D_n(t)\,\mathrm{d}t = 1. \tag{67}$$

(Hint:[18])

Exercise 27. Prove that

$$\lim_{n\to\infty}\left[\lim_{t\to 0} D_n(t)\right] = \infty. \tag{68}$$

(Hint:[19])

Exercise 28. Let f be periodic with period $2\,\pi$. Prove that

$$\int_{-\pi}^{\pi} f(u)\,D_n(x-u)\,\mathrm{d}u = \int_{-\pi}^{\pi} D_n(t)\,f(x-t)\,\mathrm{d}t = \int_{-\pi}^{\pi} D_n(t)\,f(x+t)\,\mathrm{d}t. \tag{69}$$

(Hint:[20])

Exercise 29. Prove that

$$S_n(x) = \int_0^{\pi} D_n(t)\,[f(x+t) + f(x-t)]\,\mathrm{d}t. \tag{70}$$

(Hint:[21])

3.2. Convergence

The key observation is that, for any $s \in \mathbb{R}$,

$$S_n(x) - s = \int_0^{\pi} D_n(t)\,[f(x+t) + f(x-t) - 2\,s]\,\mathrm{d}t. \tag{71}$$

Exercise 30. Prove (71). (Hint:[22])

THEOREM 13. *Let $x_0 \in (-\pi, \pi)$. Let $f(x)$ satisfy at x_0*

$$\exists \delta, \alpha, M > 0, \qquad \forall |x - x_0| < \delta, \qquad |f(x) - f(x_0)| < M\,|x - x_0|^{\alpha}, \tag{72}$$

then $\lim_{n\to\infty} S_n(x_0) \longrightarrow f(x_0)$.

Remark 14. Note that mere continuity of $f(x)$ at x_0 is not enough to guarantee convergence. On the other hand, from the above theorem we see that convergence is guaranteed if $f(x)$ is only slightly better than just being continuous (the hypothesis in the theorem is called “Hoelder continuity”). This fact makes it difficult to explicit construct a continuous function whose Fourier expansion does not converge to itself. We will discuss a bit more in §4.2.

17. $2\sin(t/2)\cos(k\,t) = \sin(k\,t + t/2) - \sin(k\,t - t/2)$;
18. Use (59).
19. L'Hospital for the inner limit.
20. Change of variable $t = x - u$, then notice that $D_n(t)$ is even.
21. Split the RHS of (69) to $\int_0^{\pi} + \int_{-\pi}^{0}$ then change variable in the latter integral.
22. $D_n(t)$ is even + Exercises 26, 29.

Proof. Without loss of generality we take $x_0 = 0$. Since $D_n(t)$ is an even function all we need to prove is

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\, D_n(x)\, \mathrm{d}x = f(0). \tag{73}$$

To do this we first notice that

$$f(0) = \left[\int_{-\pi}^{\pi} D_n(x)\right] f(0) = \int_{-\pi}^{\pi} f(0)\, D_n(x)\, \mathrm{d}x \tag{74}$$

which means we should study

$$\left|\int_{-\pi}^{\pi} [f(x) - f(0)]\, D_n(x)\, \mathrm{d}x\right|. \tag{75}$$

Note that the difficulty here is that $\int_{-\pi}^{\pi} |D_n(x)| \to \infty$ therefore we could not put the absolute value inside the integral.

We claim:

For any $\delta > 0$,

$$\lim_{n\to\infty} \left|\int_{\delta}^{\pi} [f(x) - f(0)]\, D_n(x)\, \mathrm{d}x + \int_{-\pi}^{-\delta} [f(x) - f(0)]\, D_n(x)\, \mathrm{d}x\right| = 0. \tag{76}$$

This claim is a consequence of the Riemann-Lebesgue Lemma, which claims

Let $f(x)$ be Riemann integrable on $[a, b]$. Then

$$\lim_{t\to\infty} \int_a^b f(x) \cos(t\, x)\, \mathrm{d}x = \lim_{t\to\infty} \int_a^b f(x) \sin(t\, x)\, \mathrm{d}x = 0. \tag{77}$$

(see Problem 3).

Now for any $\varepsilon > 0$, we take δ such that for all $x \in (-\delta, \delta)$,

$$|f(x) - f(0)| < M\, |x|^{\alpha}; \qquad \frac{|x|^{\alpha}}{|2 \sin(x/2)|} < 2\, |x|^{\alpha - 1}; \qquad \frac{4\, M}{\pi\, \alpha}\, \delta^{\alpha} < \frac{\varepsilon}{2}. \tag{78}$$

Then we have

$$\begin{aligned}
\left|\int_{-\delta}^{\delta} [f(x) - f(0)]\, D_n(x)\, \mathrm{d}x\right| &\leqslant \int_{-\delta}^{\delta} |f(x) - f(0)|\, |D_n(x)|\, \mathrm{d}x \\
&\leqslant M \int_{-\delta}^{\delta} |x|^{\alpha} \left|\frac{\sin[(n + 1/2)\, x]}{2\, \pi \sin(x/2)}\right| \mathrm{d}x \\
&\leqslant \frac{2\, M}{\pi} \int_{-\delta}^{\delta} |x|^{\alpha - 1}\, \mathrm{d}x \\
&= \frac{4\, M}{\pi\, \alpha}\, \delta^{\alpha} < \frac{\varepsilon}{2}.
\end{aligned} \tag{79}$$

Now thanks to (76), there is $N \in \mathbb{N}$ such that

$$\forall n > N, \quad \left|\int_{\delta}^{\pi} [f(x) - f(0)]\, D_n(x)\, \mathrm{d}x + \int_{-\pi}^{-\delta} [f(x) - f(0)]\, D_n(x)\, \mathrm{d}x\right| < \frac{\varepsilon}{2}. \tag{80}$$

Putting things together, we see that for all $n > N$,

$$|f(0) - S_n(0)| = \left| \int_{-\pi}^{\pi} [f(x) - f(0)]\, D_n(x)\, \mathrm{d}x \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{81}$$

Thus ends the proof. □

Exercise 31. Explain why we could take $x_0 = 0$ "without loss of generality". (Hint:[23])

Exercise 32. Prove (76) using Riemann-Lebesgue Lemma.

Problem 1. Assume that at x_0 the following is satisfied:

- $A := \lim_{x \to x_0+} f(x)$, $B := \lim_{x \to x_0-} f(x)$ exist;
- There is $\delta, \alpha, M > 0$ such that for all $x \in (x_0 - \delta, x_0)$, $|f(x) - A| < M\,|x - x_0|^{\alpha}$, and for all $x \in (x_0, x_0 + \delta)$, $|f(x) - B| < M\,|x - x_0|^{\alpha}$. Then

$$\lim_{n \to \infty} S_n(x_0) = \frac{A + B}{2} = \frac{1}{2}\Big[\lim_{x \to x_0+} f(x) + \lim_{x \to x_0-} f(x)\Big]. \tag{82}$$

Exercise 33. How does the Fourier series converge at end points $\pm\pi$? (Hint:[24])

Example 15. Consider the Fourier series for $f(x)$ with period 2π and $f(x) = x$ for $-\pi < x < \pi$. f is continuous for $-\pi < x < \pi$ but $f(\pi-) = \pi \neq -\pi = f(-\pi+)$. Clearly the hypotheses of Theorem 13 (or more precisely Problem 1) are satisfied at every x. For example at any $x_0 \in (-\pi, \pi)$ we have

$$|f(x) - f(x_0)| \leqslant |x - x_0| \tag{83}$$

for all other x.

Therefore its Fourier series will converge to $\tilde{f}$ with period 2π and

$$\tilde{f}(x) = \begin{cases} x & -\pi < x < \pi \\ 0 & x = \pm\pi \end{cases} \tag{84}$$

Example 16. Consider the Fourier series for $f(x)$ with period 4 and

$$f(x) = \begin{cases} 1 & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}. \tag{85}$$

We see that f is continuous for $-2 < x < 0$ and $0 < x < 2$, while has jump discontinuity at $-2, 0, 2$. As a consequence its Fourier series converges to $\tilde{f}$ with period 4 and

$$\tilde{f}(x) = \begin{cases} 1 & -2 < x < 0 \\ 1/2 & x = 0 \\ x & 0 < x < 2 \\ 3/2 & x = \pm 2 \end{cases}. \tag{86}$$

23. Change of variable $x - x_0 \longrightarrow t$ and then use periodicity.
24. Remember that f is periodic. So at π, we have $f(\pi-) = f(\pi)$, $f(\pi+) = f(-\pi)$. So the limit is $\frac{1}{2}(f(\pi) + f(-\pi))$.

3.3. Differentiation and integration

If the constants δ, M, α can be taken to be uniform in x on an interval $[a, b]$, that is

$$\forall x \in [a,b], \quad \forall |y-x| < \delta, \qquad |f(x) - f(y)| < M\,|x-y|^{\alpha}, \tag{87}$$

then the convergence in Theorem 13 is uniform. Thus we can state differentiation and integration theorems based on condition (87). However we will not state them as for most practical situation, the following weaker result suffices.

DEFINITION 17. *A function is called piecewise continuous on $[a, b]$ if and only if there are finitely many points $a = a_0 < a_1 < \ldots < a_m = b$, such that $f(x)$ is continuous on each open interval (a_i, a_{i+1}) and furthermore the one-sided limits of $f(x)$ exist and are finite at every a_i.*

THEOREM 18. (UNIFORM CONVERGENCE OF FOURIER SERIES) *Let $f(x)$ be a continuous, $2\,\pi$-periodic functions on $\mathbb{R}$. If f' is piecewise continuous, then the Fourier series for f converges uniformly to f on $\mathbb{R}$.*

Exercise 34. Prove that if f' is piecewise continuous, then (87) is satisfied at every x with $\alpha = 1$. (Hint:[25])

Exercise 35. Check that the functions in Examples 15 and 16 satisfies the hypotheses of Theorem 18.

Example 19. Consider the $2\,\pi$-periodic function defined by $f(x) = x$ for $x \in (-\pi, \pi]$. We have $f'(x)$ exists everywhere except at $(2\,k+1)\,\pi$, $k \in \mathbb{Z}$. Furthermore $f'(x) = 1$ on $(-\pi, \pi)$. Therefore f is continuous and f' piecewise continuous and we can apply Theorem 18 to obtain

$$\frac{f(x+) + f(x-)}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\,x). \tag{88}$$

for all $x \in \mathbb{R}$. In particular, at $x = \frac{\pi}{2}$ we obtain

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{89}$$

From Theorem 18 the following is immediate.

THEOREM 20. (DIFFERENTIATION AND INTEGRATION)

- *Let $f(x)$ be continuous, $2\,\pi$-periodic. Let $f'(x)$ and $f''(x)$ be piecewise continuous. Then the Fourier series of $f'(x)$ converges to $\frac{f'(x+) + f'(x-)}{2}$ at every x.*

- *Let $f(x)$ be piecewise continuous. Then we can integrate its Fourier series termwise, that is*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b \frac{a_0}{2}\,\mathrm{d}x + \sum_{n=1}^{\infty} \int_a^b [a_n \cos(n\,x) + b_n \sin(n\,x)]\,\mathrm{d}x. \tag{90}$$

25. If f' is piecewise continuous, then it is bounded. Apply MVT.

Exercise 36. Let $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,x) + b_n \sin(n\,x)$. Prove that

$$f' \sim \sum_{n=1}^{\infty} [a_n \cos(n\,x) + b_n \sin(n\,x)]' = \sum_{n=1}^{\infty} (n\,b_n) \cos(n\,x) + (-n\,a_n) \sin(n\,x). \tag{91}$$

Exercise 37. Prove Theorem 20. (Hint:[26])

Remark 21. A surprising fact is that Fourier series can always be integrated termwise, be the convergence uniform or not. See Problem 8.

Example 22. Find the functions represented by the series obtained by the termwise integration of the following series

$$\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2\,n+1)\,x}{(2\,n+1)}. \tag{92}$$

which is Fourier series for

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}, \tag{93}$$

from $-\pi$ to x.

Solution. By Theorem 20 we know that the integrated series represents

$$\int_{-\pi}^{x} f(x)\,\mathrm{d}x = \begin{cases} \int_{-\pi}^{x} -1\,\mathrm{d}x & -\pi < x < 0 \\ \int_{-\pi}^{0} -1\,\mathrm{d}x + \int_{0}^{x} \mathrm{d}x & 0 < x < \pi \end{cases} = \begin{cases} -(x+\pi) & -\pi < x < 0 \\ -\pi + x & 0 < x < \pi \end{cases} = |x| - \pi. \tag{94}$$

Example 23. Consider $f(x)$ be periodic with period $2\,\pi$ and satisfying $f(x) = x^2$ on $[-\pi, \pi]$. Then $f(x)$ is continuous and with piecewise continuous derivatives f', f''. If we are given

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(n\,x), \tag{95}$$

then we can conclude from Theorems 18 and 20 that

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(n\,x); \tag{96}$$

$$2\,x = (x^2)' = \sum_{n=1}^{\infty} \frac{4}{n} (-1)^{n+1} \sin(n\,x), \qquad x \in (-\pi, \pi) \tag{97}$$

which gives

$$x = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin(n\,x), \qquad x \in (-\pi, \pi). \tag{98}$$

26. First notice that the trigonometric series obtained from termwise differentiation is the Fourier series of f'. Now if f'' is piecewise continuous then the convergence of this series is uniform. For the integraion part, consider the Fourier expansion of the function $F(x) - \frac{a_0}{2}\,x$ where $F(x) = \int_0^x f(t)\,\mathrm{d}t$. Note that $-\frac{a_0}{2}\,x$ is necessary as the term $\frac{a_0}{2}\,x$ cannot be part of a Fourier series.

4. FINER PROPERTIES AND GENERALIZATIONS

4.1. The Gibbs phenomenon

In a short letter to *Nature* in 1898, Yale mathematical physicist J. W. Gibbs (1839 - 1903) mentioned that, as the number of terms in the partial sum of a Fourier series for a discontinuous periodic function is increased, the amplitude of the wiggles will decrease everywhere except near the discontinuity. Furthermore, at the discontinuity, the maximum overshoot and undershoot will not decrease to zero. In fact the ratio between this maximum and the size of the jump approaches:

$$\frac{1}{\pi}\int_0^\pi \frac{\sin x}{x}\,\mathrm{d}x - 1 \approx 0.089. \tag{99}$$

Remark 24. Gibbs mentioned (99) without any justification. It turns out that, as early as 1848, a 22-year old student at Trinity College, Cambridge, named Henry Wilbraham (1825 - 1883) already discovered this phenomenon through hand-plotting partial sums and proved (99). Unfortunately he left academia in his early 30s and the paper was forgotten.

Example 25. We calculate the Fourier series on $[-\pi, \pi]$ for

$$f(x) = \begin{cases} -1 & -\pi \leqslant x < 0 \\ 1 & 0 \leqslant x < \pi \end{cases}. \tag{100}$$

Since $f(x)$ is odd, $a_n = 0$. We have

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n\,x\,\mathrm{d}x \\ &= \frac{1}{n\,\pi}\left[\cos n\,x|_{-\pi}^{0} - \cos n\,x|_0^\pi\right] \\ &= \frac{2\,(1-\cos n\,\pi)}{n\,\pi} \\ &= \begin{cases} \frac{4}{n\,\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}. \end{aligned} \tag{101}$$

Therefore

$$f(x) \sim \frac{4}{\pi}\left(\sin x + \frac{\sin 3\,x}{3} + \frac{\sin 5\,x}{5} + \cdots\right). \tag{102}$$

Exercise 38. Write a short program to illustrate this phenomenon.[27]

To prove Gibbs phenomenon for $f(x)$, we need to find the maximum of the partial sum. This can be done easily through calculating $S_{2n+1}'(x)$ and set it to 0.

Exercise 39. Prove that

$$S_{2n+1}'(x) = \frac{1}{\pi}\,\frac{\sin(2\,(n+1)\,x)}{\sin x}. \tag{103}$$

27. If you don't have time to do this, please visit http://www.sosmath.com/fourier/fourier3/gibbs.html to see an animation.

(Hint:[28])

Thus we consider $f(x^*)$ for $x^* = \pi/[2(n+1)]$. We have

$$\lim_{n\to\infty} S_{2n+1}(x^*) = \frac{1}{\pi}\int_0^\pi \frac{\sin x}{x}\,\mathrm{d}x \tag{104}$$

and the conclusion immediately follows.

Exercise 40. Prove (104). (Hint:[29])

Remark 26. The "official" proof of Gibbs phenomenon for general discontinuous functions, together with the coinage of "Gibbs phenomenon", was by Maxime Bocher (1861 - 1918). Today one of the most important prizes in Analysis is named after him.

4.2. More on pointwise convergence

Recall that $S_n(x) - s = \int_0^\pi D_n(t)\,[f(x+t) + f(x-t) - 2\,s]\,\mathrm{d}t$. We fix x and denote

$$\phi(t) := f(x+t) + f(x-t) - 2\,s. \tag{105}$$

Thus

$$S_n(x) - s = \int_0^\pi D_n(t)\,\phi(t)\,\mathrm{d}t. \tag{106}$$

THEOREM 27. (DINI) *If there is $\delta > 0$ such that*

$$\int_0^\delta \frac{|\varphi(t)|}{t}\,\mathrm{d}t < \infty \tag{107}$$

then

$$\lim_{n\to\infty} S_n(x) = s. \tag{108}$$

Show that Theorem 13 follows from the above result.

Proof. Exercise. □

The main stream study of convergence of Fourier series in the 19th century focused on functions with

finitely many maxima and minima and finitely many discontinuities.

The rationale behind such assumption is very hard to understand until one learns the following definition.

DEFINITION 28. (BOUNDED VARIATION) *A function $f(x)$ is said to have bounded variation on $[a,b]$ if and only if there is $M > 0$ such that for every partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$,*

$$\sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| < M. \tag{109}$$

28. Termwise differentiation (note that this is a finite sum), and notice $2\sin x\cos k\,x = \sin(k\,x + x) - \sin(k\,x - x)$.

29. $\frac{\sin x}{x}$ is continuous on $[0,\pi]$ and therefore is uniformly continuous on $[0,\pi]$.

We denote the total variation of f on $[a, b]$ as:

$$\mathrm{BV}(f, [a, b]) := \sup_P \left\{ \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \right\}. \tag{110}$$

Exercise 41. Prove that any bounded function with "finitely many maxima and minima and finitely many discontinuities" has bounded variation.

Problem 2. Let f have bounded variation on $[a, b]$.

a) Prove that $f = g - h$ where $g(x), h(x)$ are increasing. (Hint:[30])

b) Prove that at every $x \in (a, b)$, $f(x+)$ and $f(x-)$ exist.

c) Prove that f is Riemann integrable.

d) Prove that if f' exists and $|f'|$ is Riemann integrable, then

$$\mathrm{BV}(f, [a, b]) = \int_a^b |f'(x)| \, \mathrm{d}x. \tag{111}$$

Theorem 29. (Jordan) *If there is $\delta > 0$ such that $\phi(t)$ has bounded variation on $[0, \delta]$, then*

$$\lim_{n \to \infty} S_n(x) = s. \tag{112}$$

Proof. First it suffices to prove the claim for $\phi(t)$ increasing, satisfying $\phi(0) = 0$. Second it suffices to prove

$$\lim_{n \to \infty} \int_0^\delta \frac{\sin (n + 1/2) \, t}{t} \phi(t) \, \mathrm{d}t = 0. \tag{113}$$

For any $\varepsilon > 0$, take $\eta > 0$ such that $\phi(\eta) < \varepsilon / (2 A)$ where

$$A := \sup_{a < b} \left| \int_a^b \frac{\sin x}{x} \, \mathrm{d}x \right|. \tag{114}$$

Then we have

$$\lim_{n \to \infty} \int_\eta^\delta \frac{\sin (n + 1/2) \, t}{t} \phi(t) \, \mathrm{d}t = 0 \tag{115}$$

by Riemann-Lebesgue Lemma.

Finally we have

$$\int_0^\eta \frac{\sin (n + 1/2) \, t}{t} \phi(t) \, \mathrm{d}t = \phi(\eta) \int_\xi^\eta \frac{\sin (n + 1/2) \, t}{t} \, \mathrm{d}t = \phi(\eta) \int_{\left(n + \frac{1}{2}\right) \xi}^{\left(n + \frac{1}{2}\right) \eta} \frac{\sin x}{x} \, \mathrm{d}x \tag{116}$$

which is bounded by $A \cdot (\varepsilon / (2 A)) = \varepsilon / 2$. Here the first equality is the second intermediate value theorem for integrals. $\square$

Exercise 42. Prove that there is $A > 0$ such that for any $a < b$,

$$\left| \int_a^b \frac{\sin x}{x} \, \mathrm{d}x \right| < A. \tag{117}$$

30. Take $g(x) = \mathrm{BV}(f, [a, x])$.

(Hint: Integration by parts)

Exercise 43. Let $f(x), g(x)$ be continuous on $[a, b]$. Furthermore assume $g(a) = 0$ and $g(x)$ is increasing. Then there is $c \in [a, b]$ such that

$$\int_a^b f(x)\, g(x)\, \mathrm{d}x = g(b) \int_c^b f(x)\, \mathrm{d}x. \tag{118}$$

(Hint:[31])

Exercise 44. Fill in all the details of the above proof.

COROLLARY 30. *Let $f(x)$ be 2π-periodic and have bounded variation on $[a, b] \subseteq \mathbb{R}$. Then its Fourier series converges uniformly on $[a, b]$ to $\frac{f(x+) + f(x-)}{2}$.*

THEOREM 31. (DE LA VALLÉE-POUSSIN) *Let $\psi(t) = t^{-1} \int_0^t \phi(u)\, \mathrm{d}u$ and s be such that $\psi(0+) = 0$. Assume $\psi(t)$ has bounded variation. Then*

$$\lim_{n \to \infty} S_n(x) = s. \tag{119}$$

Proof. Notice that $\phi(t) = \psi(t) + t\, \psi'(t)$. The conclusion follows from application of Jordan's test to the first term and Dini's test to the second. □

Exercise 45. Consider $f(x) = (\log(1/x))^{-1}$. Show that Jordan's condition is satisfied but Dini's is not.

Exercise 46. Consider $f(x) = x^{1/2} \sin(1/x)$. Show that Dini's condition is satisfied by Jordan's is not.

Remark 32. It turns out that, in the context of approximating continuous functions using trigonometric series, Fourier series is not the best choice.

The best choice is given by Fejer:

$$F_N(x) := \frac{S_0(x) + \cdots + S_{N-1}(x)}{N}. \tag{120}$$

It can be proved that, as long as both $f(x+), f(x-)$ exist, then

$$\lim_{N \to \infty} F_N(x) = f(x). \tag{121}$$

See Chapter 13 of (TITCHMARSH) for proofs and discussions.

A direct corollary of this is the following Weierstrass' Second Approximation Theorem:

> Let $f(x)$ be continuous on $[-\pi, \pi]$. For any $\varepsilon > 0$, there is a trigonometric polynomial $T(x)$ such that $|f(x) - T(x)| < \varepsilon$ for all $x \in [-\pi, \pi]$.

4.3. L^2 convergence

> "In all expositions of Fourier's series which have come to my notice, it is expressly stated that the series can represent a discontinuous function. The idea that a real discontinuity can replace a sum of continuous curves is so utterly at variance with the physicists' notions of quantity, that it seems to me to be worth while giving a very elementary statement of the problem in such simple form that the mathematicians can at once point to the inconsistency if any there be."

31. Set $F(x) := \int_a^x f(t)\, \mathrm{d}t$, integrate by parts and apply intermediate value theorem.

——— Albert Michelson (1852 - 1931)[32]

Definition 33. (L^2 functions) *Let $A \subseteq \mathbb{R}^N$. We say $f(x) \in L^2(A)$ if and only if $f^2(x)$ is integrable*[33] *on A. We further denote*

$$\|f(x)\|_{L^2(A)}^2 := \int_A f(x)^2 \,\mathrm{d}x. \tag{122}$$

Exercise 47. Let $p \geqslant 1$ be real, $f(x) \geqslant 0$. Then $f(x)$ is Riemann integrable on $[0, 1]$ if and only if $f(x)^p$ is Riemann integrable on $[0,1]$. (Hint:[34])

Theorem 34. *Let $f(x) \in L^2([-L, L])$. Let $\alpha_0, \ldots, \alpha_N, \beta_1, \ldots, \beta_N \in \mathbb{R}$ and define*

$$T_N(x) := \frac{\alpha_0}{2} + \sum_1^N \left\{\alpha_n \cos \frac{n\,\pi\,x}{L} + \beta_n \sin \frac{n\,\pi\,x}{L}\right\}. \tag{123}$$

Then

$$\|f(x) - T_N(x)\|_{L^2([-L,L])}^2 = \|f(x) - S_N(x)\|_{L^2([-L,L])}^2 + \|S_N(x) - T_N(x)\|_{L^2([-L,L])}^2. \tag{124}$$

Here $S_N(x) = \frac{a_0}{2} + \sum_1^N \left\{a_n \cos \frac{n\,\pi\,x}{L} + b_n \sin \frac{n\,\pi\,x}{L}\right\}$ is the partial sum of the Fourier expansion of f on $[-T, T]$. Furthermore we have

$$\|f(x) - S_N(x)\|_{L^2([-L,L])}^2 = \|f(x)\|_{L^2([-L,L])}^2 - \left[\frac{a_0^2}{2} + \sum_{n=1}^N (a_n^2 + b_n^2)\right] L. \tag{125}$$

$$\|S_N(x) - T_N(x)\|_{L^2([-L,L])} = \left[\frac{(a_0 - \alpha_0)^2}{2} + \sum_{n=1}^\infty [(a_n - \alpha_n)^2 + (b_n - \beta_n)^2]\right] L. \tag{126}$$

Proof. Direct calculation using the orthogonality relations (33 – 35). Left as exercise. ☐

Corollary 35. *Let $f(x) \in L^2([-L, L])$. Then the solution to*

$$\min_{\alpha_0, \ldots, \alpha_N, \beta_1, \ldots, \beta_N} \|f(x) - T_N(x)\|_{L^2([-T,T])} \tag{127}$$

is $T_N(x) = S_N(x)$.

Corollary 36. (Bessel's inequality) *Let $f(x) \in L^2([-L, L])$, let $\frac{a_0}{2} + \sum_{n=1}^\infty \left\{a_n \cos \frac{n\,\pi\,x}{L} + b_n \sin \frac{n\,\pi\,x}{L}\right\}$ be its Fourier expansion. Then*

$$L\left[\frac{a_0^2}{2} + \sum_{n=1}^\infty (a_n^2 + b_n)^2\right] \leqslant \int_{-L}^L f(x)^2 \,\mathrm{d}x. \tag{128}$$

Exercise 48. Prove $\lim_{n\to\infty} a_n = 0$ using Bessel's inequality.

32. Of "Michelson-Morley experiment" fame. In a 1898 letter to the journal *Nature*. This letter led to several more letters on *Nature* in the same year by Cambridge mathematician A. E. H. Love, the French mathematician Henri Poincaré, and Gibbs. One of Gibbs' letter is the starting of the study of the Gibbs' pehnomenon. Interestingly, Poincaré was on Michelson's side. For more on the story of Gibbs' phenomenon, see (Nahin:Euler) .

33. For the theory to be truly useful the "integrability" here has to be Lebesgue's integrability. However for now we can safely ignore this subtle point.

34. Cauchy-Schwarz.

Remark 37. (PARSEVAL'S EQUALITY) It turns out that equality holds in (128). One way to prove this without invoking functional analysis is as follows.

Exercise 49. Prove that, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\int_{-L}^{L} (f(x) - S_N(x))^2 \, \mathrm{d}x < \varepsilon. \tag{129}$$

Then conclude that equality holds in (128). (Hint: Apply Weierstrass' approximation theorem together with the fact that the Fourier partial sum is optimal in the L^2 sense).

Exercise 50. Assume

$$\lim_{n \longrightarrow \infty} \|f_n(x) - f(x)\|_{L^2([-L,L])} = 0. \tag{130}$$

Does it follow that $\lim_{n \to \infty} f_n(x) = f(x)$?

Remark 38. As we have seen, to prove pointwise convergence, $f(x)$ needs to more regular than being only continuous. The weakening of this assumption turned out to be very difficult. In 1966 Lennart Carleson finally proved that for any $f(x) \in L^2([-L, L])$, we have $\lim_{N \to \infty} S_N(x) = f(x)$ almost everywhere. The subtlety of this problem can be clearly seen from a related result by A. N. Kolmogorov in 1926: There is $f(x) \in L^1$, that is $|f(x)|$ is integrable, whose Fourier expansion diverges everywhere.

Exercise 51. Prove that if $f(x) \in L^1([-\pi, \pi])$, then its Fourier expansion is well-defined.

4.4. Sturm-Liouville theory

4.4.1. Separation of variables for partial differential equations

We use

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} + P(x,t), \qquad a < x < b; \qquad u(x,0) = f(x), \qquad + \text{boundary conditions} \tag{131}$$

to illustrate the method.

1. Require $X(x)\,T(t)$ to solve the homogeneous equation

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} \tag{132}$$

which leads to eigenvalue problem for X:

$$X'' - K\,X = 0 + \text{boundary conditions}. \tag{133}$$

The only solution to this problem is $X = 0$ unless K equals one of countably many values $K_1, K_2, \ldots$. When $K = K_n$, there is a solution X_n such that all the solutions can be written as $X = C\,X_n$ for some $C \in \mathbb{R}$.

2. Expand

$$f(x) = \sum_n f_n\,X_n. \tag{134}$$

Expand

$$P(x,t) = \sum_n p_n(t)\,X_n. \tag{135}$$

3. Solve
$$T_n' - \beta K T_n = p_n(t), \qquad T_n(0) = f_n \tag{136}$$
to obtain T_n.

4. Write down the solution
$$u(x,t) = \sum_n T_n(t) X_n(x). \tag{137}$$

Question. For arbitrary $f(x)$ and $P(x,t)$, is it always possible to write $f(x) = \sum_n f_n X_n$ and $P(x,t) = \sum_n p_n(t) X_n$ with X_n's the eigenfunctions obtained in Step 1? If so, how?

4.4.2. Sturm-Liouville theory

Jacques Charles François Sturm (1803 – 1855) and Joseph Liouville (1809 – 1882) studied the following problem: Given an general eigenvalue problem
$$-(p(x) X')' + q(x) X = \lambda w(x) X, \qquad a < x < b \tag{138}$$
with boundary conditions
$$\alpha_1 X(a) + \beta_1 X'(a) = 0; \qquad \alpha_2 X(b) + \beta_2 X(b) = 0. \tag{139}$$
What can we say about the eigenvalues/eigenfunctions?

The following is an informal version of their results.

THEOREM 39. (STURM-LIOUVILLE, WOOLLY VERSION) *The following hold true:*

1. *The eigenvalues are countable, and can be ordered by their sizes.*
2. *For each eigenvalue λ_n, the eigenfunction can be written as $C X_n$, where C is an arbitrary constant.*
3. *The X_n's are "orthogonal" in the following sense:*
$$\int_a^b X_m(x) X_n(x) w(x) \, \mathrm{d}x = 0 \text{ whenever } m \neq n. \tag{140}$$
4. *The X_n's are "complete" in the following sense: Any reasonable $f(x)$ (for example, bounded) has exactly one representation as linear combination of X_n's:*
$$f(x) = \sum_n f_n X_n. \tag{141}$$
The "=" here means
$$\lim_{N \to \infty} \int \left| f(x) - \sum_{n<N} f_n X_n \right| \mathrm{d}x = 0. \tag{142}$$

Remark 40.

- As far as I know the proof has to involve the theory of compact linear operators;
- Many "special functions", such as Bessel functions, Legendre functions, are such eigenfunctions.
- We see that the properties of these "eigenfunction expansions" are very similar to the Fourier expansion. However the study of their pointwise convergence is much more complicated.
- Study of such eigenfunctions is a dominating topic for analysts in the early 1900s.

Example 41. Consider the eigenvalue problem

$$X'' - K X = 0; \qquad X(0) = X(L) = 0; \tag{143}$$

We know that the eigenfunctions are

$$X_n = \sin\left(\frac{n \pi x}{L}\right), \qquad n = 1, 2, 3, \ldots \tag{144}$$

Then from the above theorem we know that any $f(x)$ can be written as

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n \pi x}{L}\right). \tag{145}$$

We will see later that this expansion has a name: Fourier Sine Series.

Example 42. Consider the eigenvalue problem

$$X'' - K X = 0; \qquad X'(0) = X'(L) = 0. \tag{146}$$

We know that the eigenfunctions are

$$X_n = \cos\left(\frac{n \pi x}{L}\right), \qquad n = 0, 1, 2, 3, \ldots \tag{147}$$

So the above theorem tells us any $f(x)$ can be written as

$$f(x) = \sum_{n=0}^{\infty} f_n \cos\left(\frac{n \pi x}{L}\right). \tag{148}$$

Such expansion is called: Fourier Cosine Series.

4.4.3. Eigenfunction expansions

To put the theorem into practice, we need to know how to compute the coefficients.

- Problem: Determine f_n's in

$$f(x) = \sum_n f_n X_n. \tag{149}$$

- Idea: Use "orthogonality":

$$\int_a^b X_m(x) X_n(x) w(x) \, \mathrm{d}x = 0 \text{ when } m \neq n. \tag{150}$$

- Let's set a particular n_0 and try to find out f_{n_0}. As we try to use the above orthogonality, naturally we multiply both sides of

$$f(x) = \sum_n f_n X_n. \tag{151}$$

by $X_{n_0}(x) w(x)$, and then integrate from a to b. We have

$$\begin{aligned} \int_a^b f(x) X_{n_0}(x) w(x) \, \mathrm{d}x &= \int_a^b \left[\sum_n f_n X_n\right] X_{n_0}(x) w(x) \, \mathrm{d}x \\ &= \sum_n f_n \int_a^b X_n(x) X_{n_0}(x) w(x) \, \mathrm{d}x. \end{aligned} \tag{152}$$

As

$$\int_a^b X_n(x) X_{n_0}(x) w(x)\,\mathrm{d}x = 0 \text{ for all } n \neq n_0 \tag{153}$$

we see that the right hand side in fact has exactly one nonzero term:

$$\int_a^b X_{n_0}(x)^2\, w(x) \mathrm{d}x. \tag{154}$$

Thus we reach

$$\int_a^b f(x)\, X_{n_0}(x)\, w(x)\,\mathrm{d}x = f_{n_0} \int_a^b X_{n_0}(x)^2\, w(x) \mathrm{d}x. \tag{155}$$

and consequently

$$f_{n_0} = \frac{\int_a^b f(x)\, X_{n_0}(x)\, w(x)\,\mathrm{d}x}{\int_a^b X_{n_0}(x)^2\, w(x) \mathrm{d}x}. \tag{156}$$

Example 43.

- Fourier Cosine Series.

  In this case

$$X_n = \cos\left(\frac{n\,\pi\,x}{L}\right). \qquad n = 0, 1, 2, 3, \ldots \tag{157}$$

  We have

$$\int_0^L \left[\cos\left(\frac{n\,\pi\,x}{L}\right)\right]^2 \mathrm{d}x \;=\; \int_0^L \frac{\cos\left(\frac{2\,n\,\pi\,x}{L}\right) + 1}{2}\,\mathrm{d}x = \frac{L}{2}. \tag{158}$$

  Note that the above calculation is wrong when $n = 0$. We have to calculate the $n = 0$ case separately:

$$\int_0^L 1^2\,\mathrm{d}x = L. \tag{159}$$

  So the f_n's in the Fourier Cosine expansion

$$f(x) = \sum_{n=0}^{\infty} f_n \cos\left(\frac{n\,\pi\,x}{L}\right) \tag{160}$$

  are given by

$$f_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x; \qquad f_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\,\pi\,x}{L}\right) \mathrm{d}x, \text{ for } n = 1, 2, 3, \ldots \tag{161}$$

  A more popular way of writing it is setting $a_0 = 2\, f_0$, and $a_n = f_n$ to get a universal formula

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\,\pi\,x}{L}\right) \mathrm{d}x, \qquad n = 0, 1, 2, 3, \ldots \tag{162}$$

  The Fourier cosine series then reads

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\,\pi\,x}{L}\right). \tag{163}$$

- Fourier Sine Series.

  In this case

$$X_n = \sin\left(\frac{n\,\pi\,x}{L}\right), \qquad n = 1, 2, 3, \ldots \tag{164}$$

Similar calculation as in the previous case gives

$$f(x)=\sum_{n=1}^{\infty} f_n \sin\left(\frac{n\,\pi\,x}{L}\right) \Longrightarrow f_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\,\pi\,x}{L}\right)\mathrm{d}x, \qquad n=1,2,3,\ldots \tag{165}$$

- Often, to emphasize the relation between Fourier Cosine/Sine series and Fourier series, the following notation is used:
  - Fourier Cosine:

    $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\,\pi\,x}{L}\right),\ a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\,\pi\,x}{L}\right)\mathrm{d}x,\quad n=0,1,2,3,\ldots \tag{166}$$

  - Fourier Sine:

    $$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\,\pi\,x}{L}\right), b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\,\pi\,x}{L}\right)\mathrm{d}x,\ n=1,2,3,\ldots \tag{167}$$

Example 44. Compute the Fourier cosine series for

$$f(x)=e^x, \qquad 0<x<1. \tag{168}$$

Solution. We have $L=1$. First

$$a_0=\frac{2}{1}\int_0^1 e^x\,\mathrm{d}x=2\,(e-1). \tag{169}$$

next

$$\begin{aligned}
a_n &= 2\int_0^1 e^x \cos(n\,\pi\,x)\,\mathrm{d}x \\
&= 2\int_0^1 \cos(n\,\pi\,x)\,\mathrm{d}e^x \\
&= 2\left[\cos(n\,\pi\,x)\,e^x|_0^1 + n\,\pi\int_0^1 e^x\sin(n\,\pi\,x)\,\mathrm{d}x\right] \\
&= 2\,[e\,(-1)^n-1]+2\,n\,\pi\int_0^1 \sin(n\,\pi\,x)\,\mathrm{d}e^x \\
&= 2\,[e\,(-1)^n-1]+2\,n\,\pi\left[e^x\sin(n\,\pi\,x)|_0^1 - n\,\pi\int_0^1 e^x\cos(n\,\pi\,x)\,\mathrm{d}x\right] \\
&= 2\,[e\,(-1)^n-1]-2\,(n\,\pi)^2\int_0^1 e^x\cos(n\,\pi\,x)\,\mathrm{d}x \\
&= 2\,[e\,(-1)^n-1]-(n\,\pi)^2\,a_n.
\end{aligned} \tag{170}$$

Therefore

$$a_n=\frac{2\,[e\,(-1)^n-1]}{1+(n\,\pi)^2}. \tag{171}$$

So the Fourier cosine series is given by

$$e^x=e-1+\sum_{n=1}^{\infty}\frac{2\,[e\,(-1)^n-1]}{1+(n\,\pi)^2}\cos(n\,\pi\,x). \tag{172}$$

5. More Exercises and Problems

5.1. Basic exercises

We will not further group the exercises since most of them involve both calculation of expansion and justification of convergence.

Exercise 52. Let $f(x)=|x|$. Prove that its Fourier series on $[-\pi,\pi]$ is given by

$$|x|=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos(3x)}{3^2}+\cdots\right]. \tag{173}$$

Then prove

$$\frac{1}{1}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}. \tag{174}$$

Justify every step of your argument.

Exercise 53. In 1744 Euler wrote in a letter

$$\frac{\pi-t}{2}=\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}. \tag{175}$$

Derive this expansion and explain in what sense it holds. Then derive

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}=\frac{\pi^2}{6}-\frac{\pi}{2}x+\frac{x^2}{4}. \tag{176}$$

Setting $x=\pi/2$ to obtain

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}. \tag{177}$$

Integrate again to obtain

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{\pi^3}{32}. \tag{178}$$

Remark. Note that $1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots$ is still unknown.

(Hint:[35])

Exercise 54. Consider the function

$$f(x)=\begin{cases}1 & |x|<\pi/2\\ -1 & \pi/2<|x|<\pi\end{cases}. \tag{179}$$

Prove that its Fourier expansion on $(-\pi,\pi)$ is

$$f(x)\sim\frac{4}{\pi}\left[\cos x-\frac{\cos(3x)}{3}+\frac{\cos(5x)}{5}-\cdots\right]. \tag{180}$$

Now use the fact that $|f(x)|^2=1$ to conclude

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8} \tag{181}$$

through term-wise integration. Finally from this result prove that

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots=\frac{\pi^2}{6}. \tag{182}$$

Justify every step of your argument.

Exercise 55. Let $\alpha\in\mathbb{R}$ be non-integer. Show that the Fourier expansion of $\cos(\alpha x)$ on $[-\pi,\pi]$ is

$$\cos(\alpha x)=\frac{\sin(\alpha\pi)}{\alpha\pi}+\frac{\alpha\sin(\alpha\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2-n^2}2\cos(nx). \tag{183}$$

Use this to prove that

$$\pi=2+4\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2}. \tag{184}$$

35. Termwise integration.

Justify every step of your argument.

Exercise 56. Prove

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots=\frac{\pi^2}{6}. \tag{185}$$

through Fourier expansion of $\left(\frac{\pi-x}{4}\right)^2$ on $[0,2\pi]$.

Exercise 57. (FOLLAND) Let $b\in\mathbb{R}$. Find the sums

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+b^2} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{1}{n^2+b^2} \tag{186}$$

by considering the Fourier expansion of $f(x)=e^{|b|x}$ on $(-\pi,\pi)$.

Exercise 58. Let f be continuous and periodic with period 2π. Let $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos n\,x+b_n\sin n\,x$ be its Fourier expansion on $[-\pi,\pi]$. Further assume that $\sum_{n=1}^{\infty}|a_n|$ and $\sum_{n=}^{\infty}|b_n|$ are convergent series. Prove that the Fourier series converges to f uniformly. (Hint:[36])

5.2. More exercises

Exercise 59. Is $\cos(x)+\cos\left(\sqrt{3}\,x\right)$ periodic? (Hint:[37])

Exercise 60. Let $f(x)$ be periodic with fundamental period $L>0$. Let T be a period of f. Prove that there is $n\in\mathbb{N}$ such that $T=n\,L$. (Hint:[38])

Exercise 61. Find f_1, f_2 with fundamental periods $L_1, L_2>0$, such that the fundamental period L of the function $f+g$ satisfies $L<\min\{L_1,L_2\}$. (Hint:[39])

Exercise 62. Is the Dirichlet function $D(x)=\begin{cases}1 & x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}\end{cases}$ periodic? If so, what is its fundamental period? (Hint:[40])

Remark. The Dirichlet function $D(x)$ is Dirichlet's response to the debate of "what is a function": a rule, a graph, or an arbitrary assignment. This debate was ignited by the study of trigonometric series, and was finally settled when most people agreed with the last statement.

Exercise 63. Find f_1, f_2, both with positive fundamental periods, such that $f_1(x)+f_2(x)=D(x)$ the Dirichlet function. (Hint:[41])

Exercise 64. Let $D_n(t)$ be Dirichlet kernel. Prove

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}|D_n(t)|\,\mathrm{d}t=\infty. \tag{188}$$

(Math 317, 2014, HW4, Q5).

Exercise 65. Prove

$$\sum_{n=-\infty}^{\infty}\frac{1}{1+n^2}=\pi\,\frac{1+e^{-2\pi}}{1-e^{-2\pi}}. \tag{189}$$

through expansion of $f(x)=e^{-x}$ over $(0,2\pi)$ and Parseval's relation.

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36. by Weierstrass's M-test the Fourier series converges uniformly. Let $F(x)$ be the sum. Then $F(x)$ is continuous and $F(x)-f(x)$ has Fourier expansion $0+\sum 0\cos n\,x+0\sin n\,x$. But $F-f$ is continuous, therefore $F-f=0$. (Here we applied the result from a problem in Homework 4)

37. No. Assume otherwise, there is $T>0$ such that

$$\cos(x+T)+\cos\left(\sqrt{3}\,(x+T)\right)=\cos(x)+\cos\left(\sqrt{3}\,x\right) \tag{187}$$

for all x. Take $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ of both sides and derive $\cos(x+T)=\cos(x)$, $\cos\left(\sqrt{3}\,(x+T)\right)=\cos\left(\sqrt{3}\,x\right)$.

38. Assume otherwise. Then there is n such that $T\in(n\,L,(n+1)\,L)$. Prove that $T-n\,L$ is also a period which leads to contradiction.

39. $f_1=1$ only for $x=n$ odd, $f_2=1$ only for $x=n$ even.

40. Any $r\in\mathbb{Q}^+$ is a period. There is no fundamental period.

41. Take $f_1=1$ only for $x\in\mathbb{N}$. $f_2=D(x)-f_1$. Then clearly f_1 has fundamental period 1. On the other hand, clearly 1 is also a period of f_2. Prove that for any $T<1$ $f_2(x)=f_2(x+T)$ cannot hold through discussing $T\in\mathbb{Q}$ and $T\notin\mathbb{Q}$ separately.

Exercise 66. Be convinced by the Poisson summation formula:

$$\sum_{k=-\infty}^{\infty} \delta(t-k) = 1 + 2 \sum_{k=1}^{\infty} \cos(2\pi k t) \tag{190}$$

through plotting partial sums of the right hand side.

5.3. Problems

Problem 3. (RIEMANN-LEBESGUE LEMMA) Let $f(x)$ be Riemann integrable on $[a, b]$. Then

$$\lim_{t\to\infty} \int_a^b f(x)\cos(t\,x)\,\mathrm{d}x = \lim_{t\to\infty} \int_a^b f(x)\sin(t\,x)\,\mathrm{d}x = 0. \tag{191}$$

(Hint: [42])

Problem 4. (RIEMANN-LEBESGUE LEMMA FOR IMPROPER INTEGRALS) Let $|f(x)|$ be improperly Riemann integrable on $[a, b]$. Then

$$\lim_{t\to\infty} \int_a^b f(x)\cos(t\,x)\,\mathrm{d}x = \lim_{t\to\infty} \int_a^b f(x)\sin(t\,x)\,\mathrm{d}x = 0. \tag{192}$$

Discuss:

a) Is it enough to assume the improper integrability of $f(x)$?

b) Can we take the interval to be infinite?

Problem 5. (SS) Let D_n be the Dirichlet kernel. Prove that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(x)|\,\mathrm{d}x = \frac{4}{\pi^2} \ln n + O(1). \tag{193}$$

Then show that for any $n \in \mathbb{N}$, there is a continuous function f_n such that $\forall x \in \mathbb{R}$, $|f_n(x)| \leqslant 1$, but

$$S_n(f)(0) \geqslant c \ln n \tag{194}$$

for some constant c independent of n.

Problem 6. (WIRTINGER'S INEQUALITY) Let $f(x) \in C^1([0, 2\pi])$ satisfy

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = 0. \tag{195}$$

Prove

$$\int_0^{2\pi} [f'(x)]^2\,\mathrm{d}x \geqslant \int_0^{2\pi} [f(x)]^2\,\mathrm{d}x. \tag{196}$$

Problem 7. Prove the general Gibbs phenomenon.

Problem 8. Prove that Fourier series can always be integrated termwise.

42. Consider first piecewise constant functions.