Math 317 Week 02: Infinite Series of Functions

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1. Limits of Sequence and Series of Functions

1.1. Questions to be answered

Consider a domain $A \subseteq \mathbb{R}^N$ and $f_n(x): A \mapsto \mathbb{R}$. Assume that for every $x \in A$, the limit $\lim_{n\rightarrow\infty} f(x)$ exists, then we can define a new function

$$f(x): A \mapsto \mathbb{R} \tag{1}$$

through

$$\forall x \in A, \qquad f(x) := \lim_{n\rightarrow\infty} f_n(x). \tag{2}$$

Now it is natural to ask: What properties of f can be inferred from those of f_n? In particular,

- If f_n are all continuous, is f continuous?
- If f_n are all differentiable, is f differentiable?
- If f_n are all differentiable and f is also differentiable, what is the relation between derivatives of f and those of f_n?
- If f_n are all integrable, is f integrable?
- If f_n are all integrable and f is also integrable, what is the relation between $\int_A f(x)\,\mathrm{d}x$ and $\int_A f_n(x)\,\mathrm{d}x$?

Similarly, we can consider a function series $\sum_{n=1}^{\infty} u_n(x)$. If for each $x \in A$, $\sum_{n=1}^{\infty} u_n(x)$ converges as a real infinite series, then $\sum_{n=1}^{\infty} u_n(x)$ defines a function $A \mapsto \mathbb{R}$. Now we can ask the same questions about the relations between properties of $u_n(x)$ and those of $u(x)$.

1.2. Examples

Example 1. Let $A := [0, 1]$ and $f_n(x) = x^n$. Then each f_n is differentiable (in fact infinitely differentiable) but

$$f(x) := \lim_{n\rightarrow\infty} f_n(x) = \begin{cases} 0 & 0 \leqslant x < 1 \\ 1 & x = 1 \end{cases}. \tag{3}$$

It is not continuous.

Exercise 1. Let $A := [0, 1]$, $f_n(x) = e^{-n^2 x^2}$. Find $\lim_{n\rightarrow\infty} f_n(x)$.

Example 2. Let $f_n(x) = \frac{\sin n x}{\sqrt{n}}$. Then

$$f(x) = \lim_{n\rightarrow\infty} f_n(x) = 0. \tag{4}$$

However $f_n'(x) = \sqrt{n} \cos n\, x$ whose limit is not $f'(x)$.

Example 3. Let $A := [0, 1]$. Let $f_n(x) := \lim_{m \to \infty} \cos((n!)\, \pi\, x)^{2m} = \begin{cases} 1 & (n!)\, x \in \mathbb{Z} \\ 0 & \text{elsewhere} \end{cases}$. Then each $f_n(x)$ is integrable on $[0, 1]$ but

$$f(x) := \lim_{n \to \infty} f_n(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \tag{5}$$

is not integrable.

Example 4. Let $A := [0, 1]$ and $f_n(x) = n\, x\, (1 - x^2)^n$. Then we have

$$f(x) = \lim_{n \to \infty} f_n(x) = 0. \tag{6}$$

But

$$\lim_{n \to \infty} \int_0^1 f_n(x)\, \mathrm{d}x = \frac{1}{2} \neq 0 = \int_0^1 f(x)\, \mathrm{d}x. \tag{7}$$

Exercise 2. Construct similar examples for $\sum_{n=1}^{\infty} u_n(x)$. (Hint:[1])

Exercise 3. Can you find $f_n(x, y): \mathbb{R}^2 \mapsto \mathbb{R}$ such that

$$\lim_{n \to \infty} f_n(x, y) = f(x, y), \qquad \lim_{n \to \infty} \frac{\partial f_n(x, y)}{\partial x} = \frac{\partial f(x, y)}{\partial x} \tag{8}$$

but $\lim_{n \to \infty} \frac{\partial f_n(x, y)}{\partial y} \neq \frac{\partial f(x, y)}{\partial y}$? Justify. (Hint:[2])

Note. From the above examples we see that even when A is a compact interval the situation is already complicated enough. Therefore in the following, we will take A to be a closed interval $[a, b]$ (or an open interval (a, b)) so that we can focus on the limiting process of f_n. Most results can be generalized to $A \subseteq \mathbb{R}^N$ more or less straightforwardly, at least for A reasonably nice.

1. Set $u_n = f_n - f_{n-1}$.
2. Take f_n to be independent of x.

2. Uniform Convergence

Note that in the following all functions are defined on a closed interval $[a, b]$, unless otherwise stated.

2.1. Uniform convergence for sequences

Definition 5. (Uniform convergence) *Let $\{f_n(x)\}$ be a infinite sequence of functions defined on $A \subseteq \mathbb{R}^N$. Say $f_n(x) \longrightarrow f(x)$ uniformly, if and only if*

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N}, \ \forall x \in A, \ \forall n > N, \qquad |f(x) - f_n(x)| < \varepsilon. \tag{9}$$

Exercise 4. Prove that none of the sequences in last section's examples converge uniformly.

Exercise 5. Let $f_n(x)$ converge uniformly. Is it true that $\{f_n(x)\}$ must be uniformly bounded ($\exists M > 0$, $\forall n \in \mathbb{N}$, $\forall x \in [a, b]$, $|f_n(x)| < M$)? (Hint:[3])

Exercise 6. Prove that the convergence $1 + x + x^2 + \cdots = 1/(1 - x)$ is not uniform on $(-1, 1)$. (Hint:[4])

Convergence vs Uniform convergence.

We compare:

Convergence.

$$\forall x \in A, \ \forall \varepsilon > 0, \ \exists N \in \mathbb{N}, \ \forall n > N, \qquad |f(x) - f_n(x)| < \varepsilon; \tag{10}$$

Uniform convergence.

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N}, \ \forall x \in A, \ \forall n > N, \qquad |f(x) - f_n(x)| < \varepsilon. \tag{11}$$

The only difference is the position of "$\forall x \in A$". More specifically, in "uniform convergence", the "speed" of convergence, quantified by N, is a function of ε only, while in "convergence", $N = N(\varepsilon, x)$ – it can change if x changes. Thus in "uniform convergence" the speed of convergence is "uniform" with respect to x.

2.1.1. Continuity

Theorem 6. *Let $f_n(x)$ be continuous on $[a, b]$ for every n. Assume $f_n(x) \to f(x)$ uniformly. Then $f(x)$ is continuous on $[a, b]$.*

Proof. Let $\varepsilon > 0$ and $x_0 \in [a, b]$ be arbitrary. Since $f_n(x) \longrightarrow f(x)$ uniformly, there is $N \in \mathbb{N}$ such that for all $n > N$, $\sup_{x \in E} |f_n(x) - f(x)| < \varepsilon/3$.

Now since $f_{N+1}(x)$ is continuous, there is $\delta > 0$ such that

$$\forall |x - x_0| < \delta \Longrightarrow |f_{N+1}(x) - f_{N+1}(x_0)| < \varepsilon/3. \tag{12}$$

Thus for the same δ, we have for every $|x - x_0| < \delta$,

$$\begin{aligned} |f(x) - f(x_0)| &\leqslant |f(x) - f_{N+1}(x)| \\ &\quad + |f_{N+1}(x) - f_{N+1}(x_0)| \\ &\quad + |f_{N+1}(x_0) - f(x_0)| \\ &< \varepsilon. \end{aligned} \tag{13}$$

3. No. Take $f_n(x) = \frac{1}{x} - \frac{1}{n}$.

4. Denote $f_n(x) = 1 + x + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}$. Thus $|f(x) - f_n(x)| = \frac{x^n}{1 - x}$. Now for any N, take $x = \left(\frac{1}{2}\right)^{1/n}$. Then we have $|f(x) - f_n(x)| > \frac{1}{2}$.

Thus ends the proof. □

Exercise 7. Let $f_n(x)$ be continuous at $x_0 \in [a, b]$. Assume $f_n(x) \longrightarrow f(x)$ on $[a, b]$. Give a sufficient condition for the continuity of $f(x)$ at x_0. Your condition should be weaker than "$f_n(x) \longrightarrow f(x)$ uniformly on $[a, b]$". Justify your claim. (Hint:[5])

Exercise 8. Show through an example that the above theorem is sufficient but not necessary. (Hint:[6])

2.1.2. Differentiability

THEOREM 7. *Let $f_n(x)$ be differentiable on $[a, b]$ and satisfies:*

i. There is $x_0 \in E$ such that $f_n(x_0)$ convergens;

ii. $f_n'(x)$ converges uniformly to some function $\varphi(x)$ on $[a, b]$;

Then

a) $f_n(x)$ converges uniformly to some function $f(x)$ on $[a, b]$;

b) $f'(x) = \varphi(x)$ on $[a, b]$.

Proof.

a) First we show that $\forall x \in [a, b]$, $f_n(x)$ converges. It suffices to show that the sequence $f_n(x)$ is Cauchy.

Let $\varepsilon > 0$ be arbitrary. Take $N_0 \in \mathbb{N}$ such that for all $n > N_0$,

$$\forall x \in [a, b], \qquad |f_n'(x) - \varphi(x)| < \frac{\varepsilon}{4\,(b - a)}. \tag{14}$$

On the other hand, since $f_n(x_0) \longrightarrow f(x_0)$, there is $N_1 \in \mathbb{N}$ such that for all $m, n > N_1$,

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}. \tag{15}$$

Now take $N = \max\{N_0, N_1\}$. We have, for any $m, n > N$,

$$\begin{aligned} |f_n(x) - f_m(x)| &\leqslant |f_n(x_0) - f_m(x_0)| \\ &\quad + |(f_n(x) - f_n(x_0)) - (f_m(x) - f_m(x_0))| \\ &< \frac{\varepsilon}{2} + |f_n'(\xi) - f_m'(\xi)|\,|b - a| < \varepsilon. \end{aligned} \tag{16}$$

Thus there is $f(x)$ defined on $[a, b]$ such that $f_n(x) \longrightarrow f(x)$. The proof of uniformity is left as exercise.

b) We consider

$$\frac{f_m(x) - f_m(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} = f_m'(\xi) - f_n'(\xi). \tag{17}$$

Thus we have

$$\frac{f_n(x) - f_n(x_0)}{x - x_0} \longrightarrow \frac{f(x) - f(x_0)}{x - x_0} \tag{18}$$

uniformly in x. Now define

$$F_n(x) := \begin{cases} \dfrac{f_n(x) - f_n(x_0)}{x - x_0} & x \neq x_0 \\ f_n'(x_0) & x = x_0 \end{cases}; \qquad F(x) := \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0} & x \neq x_0 \\ \varphi(x_0) & x = x_0 \end{cases} \tag{19}$$

We see that $F_n(x) \rightarrow F(x)$ uniformly. As every $F_n(x)$ is continuous, $F(x)$ is also continuous which means $f' = \varphi$. □

5. There is $\delta > 0$ such that $f_n(x) \longrightarrow f(x)$ uniformly on $[a, b] \cap (x_0 - \delta, x_0 + \delta)$.

6. $f_n(x) = x^n$ on $[0, 1)$.

Exercise 9. Finish the proof of a). (Hint:[7])

Exercise 10. Prove that

$$\frac{f_n(x) - f_n(x_0)}{x - x_0} \longrightarrow \frac{f(x) - f(x_0)}{x - x_0} \tag{20}$$

is uniform with respect to x. (Hint:[8])

Exercise 11. Show that it is not enough to assume only $f_n'(x) \longrightarrow \varphi(x)$ without uniformity. (Hint: [9])

Exercise 12. Show through example that the conditions in the above theorem is sufficient but not necessary. (Hint: [10]).

2.1.3. Integrability

Theorem 8. *Let $f_n(x)$ be Riemann integrable on $[a, b]$ for every n. Assume $f_n(x) \rightarrow f(x)$ uniformly. Then $f(x)$ is Riemann integrable on $[a, b]$ and*

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x. \tag{21}$$

Proof. Left as exercise. □

Exercise 13. Prove the theorem.

Exercise 14. Study $f_n(x) = 2\, n^2\, x\, e^{-n^2 x^2}$ with respect to the above theorems.

Exercise 15. Study $f_n(x) = \frac{n\, x}{1 + n^2\, x^2}$ with respect to the above theorems.

2.1.4. Checking uniform convergence

Theorem 9. *Let $A \subseteq \mathbb{R}^N$. Let $\{f_n(x)\}$ be a infinite sequence of functions defined on A. Then $f_n(x) \longrightarrow f(x)$ uniformly on E if and only if $\lim_{n \rightarrow \infty} M_n = 0$ where*

$$M_n := \sup_{x \in A} |f(x) - f_n(x)|. \tag{22}$$

Exercise 16. Prove the above theorem.

Example 10. Prove that $f_n(x) := \frac{x}{1 + n^2\, x}$ converges to 0 on $\mathbb{R}$ uniformly.

Proof. We have

$$M_n := \sup_{x \in \mathbb{R}} \left| \frac{x}{1 + n^2\, x} \right| < \frac{1}{n^2} \rightarrow 0 \tag{23}$$

and the conclusion follows. □

Theorem 11. (Cauchy criterion) *Let $E \subseteq \mathbb{R}^N$. Let $\{f_n(x)\}$ be a infinite sequence of functions defined on E. Then $f_n(x) \longrightarrow f(x)$ uniformly on E for some $f(x)$ defined on E if and only if*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall x \in E,\ \forall m, n > N, \qquad |f_m(x) - f_n(x)| < \varepsilon. \tag{24}$$

Exercise 17. Prove the above theorem. (Hint:[11])

7. Take $m \longrightarrow \infty$ in (15). Then use the fact that N is independent of x.
8. $\left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - \frac{f(x) - f(x_0)}{x - x_0} \right| = \left| \frac{[f_n(x) - f(x)] - [f_n(x_0) - f(x_0)]}{x - x_0} \right|$. Then apply MVT.
9. $f_n(x) = e^{-n^2 x^2}$.
10. $f_n(x) = \frac{1}{2\, n} \ln(1 + n^2\, x^2)$
11. First use the fact that $\{f_n(x)\}$ is Cauchy for every x to show the existence of $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Then take $m \rightarrow \infty$ in (24) to prove uniform convergence.

2.2. Uniform convergence for series

DEFINITION 12. *Let $A \subseteq \mathbb{R}^N$. Let $\sum_{n=1}^{\infty} u_n(x)$ be a infinite series of functions defined on A. Say $\sum_{n=1}^{\infty} u_n(x)$ converges to $f(x)$ uniformly, if and only if the partial sum $S_n(x) := \sum_{k=1}^{n} u_k(x)$ converges to $f(x)$ uniformly.*

THEOREM 13. *Let $A \subseteq \mathbb{R}^N$. Let $\sum_{n=1}^{\infty} u_n(x)$ be a infinite series of functions defined on A. Then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly if and only if $\lim_{n\to\infty} M_n = 0$ where*

$$M_n := \sup_{x\in A} |f(x) - S_n(x)| \tag{25}$$

with $S_n(x) := \sum_{k=1}^{n} u_k(x)$.

Exercise 18. Prove the above theorem.

Exercise 19. Let $\sum_{n=1}^{\infty} u_n(x)$ be uniformly convergent. Prove that $u_n(x) \longrightarrow 0$ uniformly.

Example 14. Prove that $1 + x + x^2 + \cdots$ does not converge to $\frac{1}{1-x}$ uniformly on $(-1, 1)$.

Proof. We have

$$M_n := \sup_{x\in(0,1)} \frac{x^n}{1-x} = \infty \tag{26}$$

and the conclusion follows. □

Exercise 20. Prove that $1 + x + x^2 + \cdots$ converges to $\frac{1}{1-x}$ uniformly on $(-a, a)$ for any $0 < a < 1$.

Exercise 21. Prove that $\sum_{n=1}^{\infty} \frac{x}{1+n^6 x}$ converges uniformly on $[0, \infty)$.

Exercise 22. Prove that $\sum_{n=1}^{\infty} n\, e^{-n x}$ convergens uniformly on $[\delta, \infty)$ for any $\delta > 0$, but does not converge uniformly on $(0, \infty)$.

THEOREM 15. (PROPERTIES OF UNIFORMLY CONVERGENT SERIES) *Let $\sum_{n=1}^{\infty} u_n(x)$ be a infinite series of functions. Assume . Then*

i. If $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $f(x)$ and each $u_n(x)$ is continuous, then $f(x)$ is continuous;

ii. If each $u_n(x)$ is differentiable and

1. $\sum_{n=1}^{\infty} u_n(x_0)$ converges for some x_0;

2. $\sum_{n=1}^{\infty} u_n'(x)$ converges to $\varphi(x)$ uniformly,

then

1. $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to some $f(x)$,

2. f is differentiable and $f'(x) = \varphi(x)$.

iii. If $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $f(x)$ and each $u_n(x)$ is integrable on $[a, b]$, then $f(x)$ is integrable on $[a, b]$ and furthermore

$$\sum_{n=1}^{\infty} \int_a^b u_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x. \tag{27}$$

Exercise 23. Prove the above theorem.

Example 16. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{3^n} \cos(n\,\pi\,x^2)$. Calculate $\lim_{x\to1} f(x)$.

By the above theorem $f(x)$ is continuous. Thus

$$\lim_{x\to1} f(x) = f(1) = \frac{3}{4}. \tag{28}$$

3. TESTS FOR UNIFORM CONVERGENCE FOR SERIES

3.1. The Weierstrass M-test

THEOREM 17. (WEIERSTRASS M-TEST) *Let $E \subseteq \mathbb{R}^N$ and let $\sum_{n=1}^{\infty} u_n(\boldsymbol{x})$ be a infinite series defined on E. Assume that there is a non-negative convergent series $\sum_{n=1}^{\infty} M_n$ such that*

$$|u_n(\boldsymbol{x})| \leqslant M_n \tag{29}$$

for all $\boldsymbol{x} \in E$. Then $\sum_{n=1}^{\infty} u_n(\boldsymbol{x})$ converges uniformly on E.

Exercise 24. Prove the theorem. (Hint:[12])

Exercise 25. Prove that "$\forall n \in \mathbb{N}$" can be replaced by "$\exists N \in \mathbb{N}, \forall n > N$".

Example 18. The series $\sum_{n=1}^{\infty} \frac{\cos(n^2 x)}{n^2}$ is uniformly convergent over any interval $[a, b]$.

Proof. Since

$$\left| \frac{\cos(n^2 x)}{n^2} \right| \leqslant \frac{1}{n^2} \tag{30}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is positive and convergent, application of the M-test gives the result. □

Exercise 26. Prove that $1 + x + x^2 + \cdots$ converges to $\frac{1}{1-x}$ uniformly on $(-a, a)$ for any $0 < a < 1$ using the M-test. (Hint:[13])

Exercise 27. Prove that $1 + x + x^2 + \cdots$ converges to $\frac{1}{1-x}$ uniformly on any (a, b) with $a, b \in (-1, 1)$. (Hint:[14])

Example 19. Prove that $\sum_{n=1}^{\infty} x^2 e^{-nx}$ converges uniformly on $(0, \infty)$.

Proof. Set $u_n(x) = x^2 e^{-nx}$. Then we have, for $x \in (0, \infty)$,

$$u_n'(x) = 0 \Longleftrightarrow 2\,x\,e^{-nx} - n\,x^2\,e^{-nx} = 0 \Longleftrightarrow x = \frac{2}{n}. \tag{31}$$

It is clear that $u_n'(x) > 0$ when $x < \frac{2}{n}$ and $u_n'(x) < 0$ when $x > \frac{2}{n}$. Therefore $u_n(x)$ reaches its maximum at $x = \frac{2}{n}$. Thus we have

$$0 < u_n(x) < \left(\frac{2}{n}\right)^2 \Longrightarrow |u_n(x)| < \frac{4}{n^2}. \tag{32}$$

Application of the M-test now gives the uniform convergence. □

Exercise 28. Let $1 < a < b$. Prove that the Dirichleet series $\sum_{n=1}^{\infty} n^{-s}$ is uniformly convergent for $s \in [a, b]$.

Exercise 29. Let $0 < b < 1$. The series $\sum_{n=1}^{\infty} r^n \cos(n\,\theta)$ converges uniformly for $r \in [0, b]$ and $\theta \in [0, 2\,\pi]$.

Problem 1. Consider $\sum_{n=1}^{\infty} u_n(x)$. Assume that it satisfies

$$\exists r \in (0, 1), \qquad \forall x \in [a, b], \qquad \left| \frac{u_{n+1}(x)}{u_n(x)} \right| \leqslant r. \tag{33}$$

Does it follow that $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly? Justify your answer. If your answer is "no", then furthermore remedy the situation through an extra assumption. (Hint:[15])

12. Any $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $\sum_{m=N}^{\infty} M_m < \varepsilon$.
13. $M_n = a^n$.
14. $M_n = \max\{|a|, |b|\}^n$.
15. We need $u_1(x)$ to be bounded. That is $\exists M > 0$ such that $|u_1(x)| < M$ for all x.

3.2. Dirichlet and Abel tests

THEOREM 20. (DIRICHLET'S TEST) *Consider* $\sum_{n=1}^{\infty} u_n(x)$ *with* $u_n(x) = a_n(x)\, b_n(x)$. *Assume:*

1. $\sum_{k=1}^{n} b_k(x)$ *is uniformly bounded, that is* $\exists M > 0$ *such that for all* $n \in \mathbb{N}$, $\left|\sum_{k=1}^{n} b_k(x)\right| < M$;
2. *For each* $x \in [a, b]$, $a_n(x)$ *is decreasing;*
3. $a_n(x) \longrightarrow 0$ *uniformly as* $n \longrightarrow \infty$.

Then $\sum_{n=1}^{\infty} a_n(x)\, b_n(x)$ *converges uniformly on* $[a, b]$.

Proof. Denote $S_n(x) := \sum_{k=1}^{n} b_k(x)$. Let $\varepsilon > 0$ be arbitrary.

Since $a_n(x) \longrightarrow 0$ uniformly as $n \longrightarrow \infty$, there is $N \in \mathbb{N}$ such that for all $n > N$, $|a_n(x)| < \frac{\varepsilon}{4\,M}$. Now for any $m > n > N$, we have

$$\begin{aligned} \left|\sum_{k=n+1}^{m} a_n(x)\, b_n(x)\right| &= \left|\sum_{k=n+1}^{m} a_k(x)\, [S_k(x) - S_{k-1}(x)]\right| \\ &= \left|\sum_{k=n+1}^{m-1} [a_k(x) - a_{k+1}(x)]\, S_k(x) + a_m(x)\, S_m(x) - a_{n+1}(x)\, S_n(x)\right| \\ &\leqslant \sum_{k=n+1}^{m-1} |a_k(x) - a_{k+1}(x)|\, |S_k(x)| + |a_m(x)|\, |S_m(x)| + |a_{n+1}(x)|\, |S_n(x)| \\ &\leqslant M \sum_{k=n+1}^{m-1} |a_k(x) - a_{k+1}(x)| + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= M \sum_{k=n+1}^{m-1} [a_k(x) - a_{k+1}(x)] + \frac{\varepsilon}{2} \\ &= M\, [a_{n+1}(x) - a_m(x)] + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned} \tag{34}$$

Thus ends the proof. □

Exercise 30. Let $a_n > 0$, decreasing with limit 0. Then $\sum_{n=1}^{\infty} a_n \sin(n\, x)$ is uniformly convergent in any closed interval not including a multiple of $2\,\pi$. (Hint:[16])

Exercise 31. State the above result for general intervals, not necessarily closed.

THEOREM 21. (ABEL'S TEST) *Assume* $\sum_{n=1}^{\infty} b_n(x)$ *converges uniformly. Assume* $\{a_n(x)\}$ *is monotone for each fixed* x *and is uniformly bounded. Then* $\sum_{n=1}^{\infty} a_n(x)\, b_n(x)$ *converges uniformly.*

Exercise 32. Prove the theorem.

Example 22. Prove $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ converges uniformly on $[0, 1]$.

Proof. We try to apply Abel's Test. Take $b_n(x) = \frac{(-1)^{n-1}}{n}$ and $a_n(x) = x^n$. Then $\sum_{n=1}^{\infty} b_n(x)$ converges uniformly. On the other hand, for any $x \in [0, 1]$, x^n monotonically decreasing with n. Furthermore we have $|x^n| \leqslant 1$ for all $x \in [0, 1]$ and all $n \in \mathbb{N}$. Thus we can apply Abel's Test and the conclusion follows. □

16. $\sin x + \cdots + \sin n\, x = \frac{1}{\sin(x/2)} \left[\sin\frac{x}{2} \sin x + \sin\frac{x}{2} \sin 2\, x + \cdots + \sin\frac{x}{2} \sin n\, x\right] = \frac{\cos\frac{x}{2} - \cos\left(n + \frac{1}{2}\right) x}{2 \sin(x/2)} = \frac{\sin\frac{n+1}{2}\, x \sin\frac{n\, x}{2}}{\sin(x/2)}$.

4. Advanced Topics, Notes, and Comments

4.1. Dini's Theorem and termwise integration

Theorem 23. (Dini) *Let $a, b \in \mathbb{R}$. Let $u_n(x) \geqslant 0$ and be continuous on $[a, b]$. Assume $S(x) = \sum_{n=1}^{\infty} u_n(x)$ is continuous on $[a, b]$. Then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly.*

Proof. Fix an arbitrary $\varepsilon > 0$. Take any $x \in [a, b]$. Since $\sum_{n=1}^{\infty} u_n(x) = S(x)$, there is $N(x) \in \mathbb{N}$ such that

$$\left| S(x) - \sum_{n=1}^{N(x)} u_n(x) \right| < \varepsilon/3. \tag{35}$$

Now as $\sum_{n=1}^{N(x)} u_n(x)$ is continuous, there is $\delta_1(x) > 0$ such that

$$\forall |y - x| < \delta_1(x), \qquad \left| \sum_{n=1}^{N(x)} u_n(y) - \sum_{n=1}^{N(x)} u_n(x) \right| < \varepsilon/3. \tag{36}$$

As $S(x)$ is continuous, there is $\delta_2(x) > 0$ such that

$$\forall |y - x| < \delta_2(x), \qquad |S(x) - S(y)| < \varepsilon/3. \tag{37}$$

Taking $\delta(x) := \min(\delta_1(x), \delta_2(x))$, we have

$$\forall y \in (x - \delta(x), x + \delta(x)), \qquad \left| S(y) - \sum_{n=1}^{N(x)} u_n(y) \right| < \varepsilon. \tag{38}$$

As $u_n(x) \geqslant 0$, we have for all $m > N(x)$,

$$\sum_{n=1}^{N(x)} u_n(y) \leqslant \sum_{n=1}^{m} u_n(y) \leqslant S(y) \Longrightarrow \forall y \in (x - \delta(x), x + \delta(x)), \left| S(y) - \sum_{n=1}^{m} u_n(y) \right| < \varepsilon. \tag{39}$$

Thus we have proved:

For every $x \in [a, b]$, there is $\delta(x) > 0$ and $N(x) \in \mathbb{N}$ such that

$$\forall n > N(x), \forall y \in (x - \delta(x), x + \delta(x)), \left| S(y) - \sum_{k=1}^{n} u_k(y) \right| < \varepsilon. \tag{40}$$

Since a, b are finite, $[a, b]$ is compact, thus

$$[a, b] \subseteq \cup_{x \in [a,b]} (x - \delta(x), x + \delta(x)) \Longrightarrow \exists x_1, \ldots, x_l, \quad [a, b] \subseteq \cup_{i=1}^{l} (x_i - \delta(x_i), x_i + \delta(x_i)). \tag{41}$$

Now take $N := \max\{N(x_1), N(x_2), \ldots, N(x_l)\}$. Take any $n > N$. For any $x \in [a, b]$, there is $i \in \{1, \ldots, l\}$ such that $x \in (x_i - \delta(x_i), x_i + \delta(x_i))$. Since $n > N \geqslant N(x_i)$, we have $|S(x) - \sum_{k=1}^{n} u_k(x)| < \varepsilon$ and uniform convergence is proved. $\square$

Exercise 33. Try to prove Dini's theorem using proof by contradiction.

Exercise 34. Prove Dini's theorem as follows. Let $\varepsilon > 0$ be arbitrary. Define

$$E_n := \{x \in [a,b] \mid |S(x) - [u_1(x) + \cdots + u_n(x)]| < \varepsilon\}. \tag{42}$$

Let $C_n := [a,b] - E_n$. Prove that there is $N \in \mathbb{N}$ such that $C_n = \varnothing$ for all $n > N$. (Hint:[17])

Exercise 35. Find counter-examples in the following situations.

a) The interval is not closed;

b) The interval is not bounded;

c) $u_n(x) \geqslant 0$ does not hold for all $n \in \mathbb{N}$.

THEOREM 24. *Let $u_n(x) \geqslant 0$ be Riemann integrable on $[a,b]$. Then*

$$\sum_{n=1}^{\infty} \int_a^b u_n(x)\,\mathrm{d}x = L\left(\sum_{n=1}^{\infty} u_n(x), [a,b]\right) \tag{43}$$

where $L(f,[a,b])$ denotes the lower integral of $f(x)$ on $[a,b]$.

Proof. Exercise. □

COROLLARY 25. *Let $u_n(x) \geqslant 0$ be Riemann integrable on $[a, b]$. Assume $S(x) = \sum_{n=1}^{\infty} u_n(x)$ is Riemann integrable on $[a,b]$. Then term-wise integration is OK:*

$$\sum_{n=1}^{\infty} \int_a^b u_n(x)\,\mathrm{d}x = \int_a^b \left(\sum_{n=1}^{\infty} u_n(x)\right) \mathrm{d}x. \tag{44}$$

Remark 26. We see that in some sense, the only obstacle for (44) to hold is the integrability of $S(x)$. This shows that Riemann integrability is too restrictive.

4.2. Product of infinite series

To study product and ratio of functions through infinite series, we need to first understand the product of infinite series.

Let $\sum_{n=1}^{\infty} a_n = A, \sum_{n=1}^{\infty} b_n = B$ be absolutely convergent. Then the numbers $a_m b_n$ in whatever order is also convergent, with limit $A\,B$.

THEOREM 27. *Let $\sum_{n=1}^{\infty} a_n = A$ be absolutely convergent and $\sum_{n=1}^{\infty} b_n = B$ be convergent (may be conditional). Define*

$$c_n = a_1 b_n + \cdots + a_n b_1. \tag{45}$$

Then $\sum_{n=1}^{\infty} c_n$ converges to $A\,B$.

Remark 28. $\sum_{n=1}^{\infty} c_n$ is called the *Cauchy product* of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.

17. Nested compact sets.

Proof. We try to prove that

$$\lim_{m\to\infty}\left|\sum_{n=1}^{m} c_n - \left(\sum_{n=1}^{m-1} a_n\right)\left(\sum_{n=1}^{m-1} b_n\right)\right| = 0. \tag{46}$$

Since $\sum a_n$ is absolutely convergent, we can denote $\sum_{n=1}^{\infty} |a_n| = M_1 \in \mathbb{R}$. Also since $\sum_{n=1}^{\infty} b_n$ is convergent, its partial sums are bounded. There is $M_2 \in \mathbb{R}$ such that $\left|\sum_{k=n+1}^{m} b_k\right| < M_2$ for all m, $n \in \mathbb{N}$.

Take any $\varepsilon > 0$. As $\sum_{n=1}^{\infty} b_n$ converges, there is $N_2 \in \mathbb{N}$ such that for any $m > n > N_2$,

$$\left|\sum_{k=n+1}^{m} b_n\right| < \frac{\varepsilon}{2\, M_1}. \tag{47}$$

On the other hand, there is $N_1 \in \mathbb{N}$ such that

$$\sum_{n>N_1} |a_n| < \frac{\varepsilon}{2M_2} \tag{48}$$

Now take $N = N_1 + N_2$. For any $m > N$, we have

$$\begin{aligned}
\left|\sum_{n=1}^{m} c_n - \left(\sum_{n=1}^{m-1} a_n\right)\left(\sum_{n=1}^{m-1} b_n\right)\right| &= \left|\sum_{k=2}^{N_1}\left(a_k\left(\sum_{l=m-k}^{m} b_l\right)\right)\right| \\
&\quad + \left|\sum_{k=N_1+1}^{m-1}\left[a_k\left(\sum_{l=m-k}^{m-k} b_l\right)\right]\right| \\
&\leqslant \sum_{k=2}^{N_1}\left[|a_k|\left|\sum_{l=m-k}^{m} b_l\right|\right] \\
&\quad + \sum_{k=N_1+1}^{m-1}\left[|a_k|\left|\sum_{l=m-k}^{m} b_l\right|\right] \\
&< \frac{\varepsilon}{2\, M_2}\left(\sum_{k=2}^{N_1} |a_k|\right) + M_1\left(\sum_{k=N_1+1}^{m-1} |a_k|\right) \\
&< \varepsilon.
\end{aligned} \tag{49}$$

Now clearly

$$\lim_{m\to\infty}\left(\sum_{n=1}^{m-1} a_n\right)\left(\sum_{n=1}^{m-1} b_n\right) = A\, B \tag{50}$$

and the conclusion follows. □

Example 29. Take $a_n = b_n = \frac{(-1)^{n+1}}{\sqrt{n+1}}$. Then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent but neither is absolutely convergent. However we can show that $|c_n| \geqslant 1$ and thus $\sum_{n=1}^{\infty} c_n$ is not convergent. On the other hand, if we take

$$a_1 = 3, \quad a_n = 3^{n-1}; \qquad b_1 = -2, \quad b_n = 2^{n-1}. \tag{51}$$

Then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge. However for $n > 2$,

$$c_n = a_1 b_n + \cdots + a_n b_1 = 0. \tag{52}$$

Thus $\sum_{n=1}^{\infty} c_n$ is absolutely convergent.

Exercise 36. Fill in the details of the argument in the example above.

COROLLARY 30. *It is easy to prove that $\sum_{n=1}^{\infty} d_n$ converges where*

$$d_n := a_1 b_n + a_2 b_n + \cdots + a_n b_n + a_n b_{n-1} + \cdots + a_n b_1. \tag{53}$$

4.3. Pathological functions

Example 31. (TENT FUNCTION OVER CANTOR SET) We define $f_n(x)$ as follows:

$$f_1(x) = \begin{cases} \frac{1}{6} u_0(3(x-1/2)+1/2) & x \in [1/3, 2/3] \\ 0 & \text{elsewhere} \end{cases}. \tag{54}$$

$$f_2(x) = f_1(x) + \frac{1}{3}\left[f_1(3(x-1/6)) + f_1(3(x-5/6))\right] \tag{55}$$

and so on.

It is easy to see that $f_n(x) \to f(x)$ uniformly which means $f(x)$ is continuous. On the other hand, $f(x)$ is not differentiable at infinitely many points.

Example 32. (USTC3) Define $u_0(x)$ through:

$$u_0(x) = |x| \text{ on } \left[-\frac{1}{2}, \frac{1}{2}\right], \qquad \forall x \in \mathbb{R}, u_0(x) = u_0(x+1). \tag{56}$$

Now define

$$u_k(x) := 4^{-k} u_0(4^k x). \tag{57}$$

It is easy to show that $\sum_{n=0}^{\infty} u_n(x)$ converges uniformly and therefore to a continuous function, denote it by $S(x)$, since each $u_n(x)$ is clearly continuous.

Now we prove that $S(x)$ is nowhere differentiable. Take any $x \in [0, 1)$. For each n we have a unique integer s_n such that

$$2 \cdot 4^n x \in [s_n, s_n + 1) \Longrightarrow x_n \in \left[\frac{s_n}{2 \cdot 4^n}, \frac{s_n + 1}{2 \cdot 4^n}\right). \tag{58}$$

Now take $x_n \in \left[\frac{s_n}{2 \cdot 4^n}, \frac{s_n + 1}{2 \cdot 4^n}\right]$ such that $|x_n - x_0| = 4^{-(n+1)}$. We will prove that

$$\lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \tag{59}$$

does not exist.

Consider $u_k(x_n) - u_k(x_0)$. When $k > n$, we have $u_k(x + 4^{-k}) = u_k(x)$ for every x which means

$$k > n \Longrightarrow u_k(x_n) - u_k(x_0) = 0. \tag{60}$$

Therefore

$$f(x_n) - f(x_0) = \sum_{k=0}^{n} [u_k(x_n) - u_k(x_0)]. \tag{61}$$

Consequently

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \sum_{k=0}^{n} \frac{u_k(x_n) - u_k(x_0)}{x_n - x_0} = \sum_{k=1}^{n} p_k \tag{62}$$

where each p_k is either 1 or -1.

Now observe that $\sum_{k=1}^{n} p_k$ is odd(even) $\Longleftrightarrow$ n is odd(even). Therefore the limit cannot exist.

Exercise 37. Does the proof still work if we replace 4 by some other numbers? Say 2 or 3?

Remark 33. The first such function was constructed by Karl Weierstrass (1815 - 1897) in 1872. The above example was proposed by van der Waerden. Weierstrass' original example is

$$\sum_{n=1}^{\infty} b^n \cos(a^n \pi x) \tag{63}$$

where $b \in (0,1)$ and a is an odd integer with $a\,b > 1 + \frac{3\pi}{2}$. Earlier in 1861, G. F. B. Riemann (1826 - 1866) proposed

$$\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2} \tag{64}$$

as a candidate for a continuous function that is nowhere differentiable. This turns out to be a highly nontrivial problem and was only settled in 1970.[18] Unfortunately Riemann was wrong, but not by much: his function is almost everywhere non-differentiable.

Exercise 38. Prove that both Riemann's and Weierstrass' functions are continuous.

Exercise 39. Plot the functions by van der Waerden, Weierstrass, and Riemann for different n, and imagine what the limiting function would look like.

Remark 34. We see from the above remark that, although the basic idea is very simple: Add wilder and wilder oscillations while at the same time make the series uniformly convergent (Riemann's proposal (64) is indeed the simplest function following this idea, as termwise differentiation leads to a divergent series), however to carry out this idea rigorously is very difficult.

Note. See Chapter 4 of (Vilenkin: Story) for more such pathological functions.

4.4. A theorem about trignometric series

Theorem 35. *Let $b_n > 0$ be decreasing. Then a necessary and sufficient condition for the uniform convergence of $\sum_{n=1}^{\infty} b_n \sin(n\,x)$ in any interval is that $\lim_{n\to\infty} n\,b_n = 0$.*

18. Joseph Gerver, "*The Differentiability of the Riemann Function at Certain Rational Multiples of π*,"American Journal of Mathematics, January 1970, pp. 33–55.

Proof.

- Necessity. For any large $m \in \mathbb{N}$, set $x = \frac{\pi}{2m}$ and let $n = \left[\frac{m}{2} + 1\right]$ where $[\cdot]$ denotes the integer part of the number. Now

$$\sum_{k=n}^{m} b_k \sin(k\,x) > b_m \left(\frac{m}{2} - 1\right) \sin\left(\frac{\pi}{4}\right) \tag{65}$$

  and the conclusion follows.

- First, through Abel's re-summation we can easily prove:

  Abel's lemma. If $b_1 \geqslant \cdots \geqslant b_n \geqslant 0$ and $m \leqslant a_1 + \cdots + a_n \leqslant M$ for all $n \in \mathbb{N}$, then

$$b_1\,m \leqslant a_1\,b_1 + \cdots + a_n\,b_n \leqslant b_1\,M. \tag{66}$$

  Now for our problem notice that it suffices to consider the convergence on $[0, \pi]$. Consider

$$s_{n,m} = \sum_{k=n}^{m} b_k \sin(k\,x). \tag{67}$$

  Let $\mu_n = \max_{k \geqslant n}(k\,b_k)$. Then we have $\mu_n \longrightarrow 0$ and is decreasing. Now consider

  - If $x \geqslant \pi/n$, Abel's lemma gives

$$|s_{n,m}| \leqslant b_n \frac{1}{\sin(x/2)} \leqslant \frac{b_n\,\pi}{x} \leqslant n\,b_n \leqslant \mu_n. \tag{68}$$

  - If $x \leqslant \pi/m$, we have

$$|s_{n,m}| \leqslant b_n\,n\,x + \cdots + b_m\,m\,x \leqslant (m - n + 1)\,\mu_n\,x \leqslant \pi\,\mu_n. \tag{69}$$

  - If $x \in (\pi/m, \pi/n)$, we split

$$s_{n,m} = s_{n,k} + s_{k+1,m} \tag{70}$$

    for some k to be decided later. Applying Abel's lemma to the second term and $\sin\theta \leqslant \theta$ to the first, we reach

$$|s_{n,m}| \leqslant k\,\mu_n\,x + \frac{b_{k+1}\,\pi}{x} \leqslant \mu_n \left[k\,x + \frac{\pi}{(k+1)\,x}\right]. \tag{71}$$

    Now take $k = \left[\frac{\pi}{x}\right]$ we reach

$$|s_{n,m}| \leqslant \mu_n\,(\pi + 1). \tag{72}$$

  In fact we can apply the split method from the very start and thus do not need to discuss the three cases. □

Remark 36. Such "splitting" is ubiquitous in proofs in harmonic analysis.

Exercise 40. Prove Abel's lemma.

Exercise 41. Prove $b_n \frac{1}{\sin(x/2)} \leqslant \frac{b_n\,\pi}{x}$.

5. MORE EXERCISES AND PROBLEMS

5.1. Basic exercises

5.1.1. Limits of functions and series

Exercise 42. (USTC3) For each of the following series, find all $x \in \mathbb{R}$ such that the series converges. Here $x, y > 0$.

$$\sum_{n=1}^{\infty} \frac{n-1}{n+1} \left(\frac{x}{3x+1} \right)^n; \qquad \sum_{n=1}^{\infty} \left(\frac{x(x+n)}{n} \right)^n; \qquad \sum_{n=1}^{\infty} n\, e^{-nx}; \qquad \sum_{n=1}^{\infty} \frac{x^n}{1+x^{2n}} \tag{73}$$

$$\sum_{n=1}^{\infty} \frac{(n+x)^n}{n^{n+x}}; \qquad \sum_{n=1}^{\infty} \frac{x^n y^n}{x^n + y^n}; \qquad \sum_{n=1}^{\infty} \frac{x^n}{n+y^n}; \qquad \sum_{n=1}^{\infty} \frac{\ln(1+x^n)}{n^p}. \tag{74}$$

Exercise 43. Find bounded functions $f_n(x): \mathbb{R} \mapsto \mathbb{R}$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{75}$$

for every $x \in \mathbb{R}$ but $f(x)$ is not bounded. (Hint:[19])

Exercise 44. Can you find $f_n(x)$: $\mathbb{R} \mapsto \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$, $\lim_{n\to\infty} f_n'(x) = f'(x)$, but $\lim_{n\to\infty} f_n''(x) \neq f''(x)$? Justify. (Hint:[20])

Exercise 45. Find $f_n(x)$, $f(x)$: $[0,1] \mapsto \mathbb{R}$ with $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0,1]$, and satisfy one of the following:

a) None of $f_n(x)$ is continuous, but $f(x)$ is continuous;

b) None of $f_n(x)$ is differentiable, but $f(x)$ is differentiable;

c) None of $f_n(x)$ is Riemann integrable on $[0,1]$, but $f(x)$ is Riemann integrable on $[0,1]$.

(Hint:[21])

5.1.2. Uniform convergence

Exercise 46. (USTC3) Consider

$$\sum_{n=1}^{\infty} (-1)^n x^n (1-x). \tag{76}$$

Prove:

a) It converges absolutely on $[0,1]$;

b) It converges uniformly on $[0,1]$;

c) The following series does not converge uniformly on $[0,1]$.

$$\sum_{n=1}^{\infty} |(-1)^n x^n (1-x)|. \tag{77}$$

(Hint:[22])

Exercise 47. (FOLLAND) Let $f_n(x) = x \arctan(n x)$.

a) Prove that $\lim_{n\to\infty} f_n(x) = \frac{\pi}{2} |x|$;

b) Prove that $\lim_{n\to\infty} f_n'(x)$ exists for every x, including $x = 0$, but the convergence is not uniform in any interval containing 0.

Exercise 48. Let $A_1, \ldots, A_k \subseteq \mathbb{R}$ be disjoint. Assume $f_n \longrightarrow f$ uniformly on each A_i. Prove that $f_n \to f$ uniformly on $A := A_1 \cup \cdots \cup A_k$. (Hint:[23])

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19. $f_n(x) = \min\{|x|^{-1}, n\}$.

20. $f_n(x) = \frac{\cos(n x)}{n \sqrt{n}}$.

21. $f_n = \frac{1}{n} D(x)$ where $D(x)$ is the Dirichlet function.

22. We have $\sum_{n=N}^{\infty} |(-1)^n x^n (1-x)| = (1-x) x^N \sum_{n=0}^{N} x^n = x^N$. Thus no matter what N is we can find x such that $x^N > 1/2$.

Exercise 49. (FOLLAND) Let f_n be continuous on $[a, b]$ and assume $f_n \longrightarrow f$ uniformly on (a, b). Prove that $f_n \longrightarrow f$ uniformly on $[a, b]$. (Hint:[24])

Exercise 50. (FOLLAND) Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2}$. Prove that f is continuous on $[0, \infty)$ and furthermore $\int_0^1 f(x)\, \mathrm{d}x = 1$.

5.1.3. Tests

Exercise 51. Let $p > 1$. Prove that $\sum_{n=1}^{\infty} n^{-p} \sin(n^2 x)$ converges uniformly on $\mathbb{R}$. (Hint:[25])

Exercise 52. (USTC3) Let each $u_n(x)$ be monotone on $[a, b]$. Assume the convergence of $\sum_{n=1}^{\infty} |u_n(a)|$ and $\sum_{n=1}^{\infty} |u_n(b)|$. Prove that $\sum_{n=1}^{\infty} |u_n(x)|$ converges uniformly. (Hint:[26])

Exercise 53. (USTC3) Consider

$$u_n(x) := \begin{cases} \frac{1}{n} & x = \frac{1}{n} \\ 0 & x \neq \frac{1}{n} \end{cases}. \tag{78}$$

Prove

a) $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[0, 1]$; (Hint:[27])

b) There is no convergent positive series $\sum_{n=1}^{\infty} a_n$ such that $|u_n(x)| \leqslant a_n$.

Exercise 54. (USTC3) Prove that

$$f(x) := \sum_{n=1}^{\infty} n\, e^{-nx} \tag{79}$$

is continuous on $(0, \infty)$. (Note that the convergence on $(0, \infty)$ is not uniform!) (Hint:[28])

Exercise 55. Calculate

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\cos n x}{n^2}. \tag{80}$$

Justify your answer.

Exercise 56. Calculate

$$\left(\frac{\sin(n^2 x)}{n^6} \right)''. \tag{81}$$

Justify your answer.

Exercise 57. (RIS) Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^n} = \int_0^1 x^x \, \mathrm{d}x. \tag{82}$$

(Hint: See footnote[29])

Exercise 58. Prove that the Riemann-ζ function

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x} \tag{83}$$

is continuous on $(1, \infty)$. Furthermore prove that it is infinitely continuously differentiable, that is all orders of derivatives exist and they are all continuous.

Exercise 59. Prove

$$\lim_{x \to 1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^x} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2. \tag{84}$$

(Hint: [30])

23. Take $N = \max\{N_1, \ldots., N_k\}$.
24. $|f_n(a) - f(a)| = \lim_{x \to a} |f_n(x) - f(x)|$. Apply Comparison theorem.
25. M-test.
26. Prove $|u_n(x)| \leqslant \max\{|u_n(a)|, |u_n(b)|\}$.
27. Note that at any x there is at most one $u_n(x) \neq 0$.
28. For any $x > 0$, take $\delta < x$ and prove uniform convergence on $[\delta, \infty)$.
29. Expand $e^{x \log x}$ and integrate term by term.
30. Dirichlet.

5.2. More exercises

Exercise 60. (USTC3) Study the uniform convergence of the following series.

$$\sum_{n=1}^{\infty} \frac{1}{(x+n)(x+n+1)} \text{ on } (0,\infty); \tag{85}$$

$$\sum_{n=1}^{\infty} \frac{n\,x}{(1+x)(1+2\,x)\cdots(1+n\,x)} \text{ on } (0,\delta), (\delta,\infty); \tag{86}$$

$$\sum_{n=1}^{\infty} \frac{n\,x}{1+n^2\,x^2} \text{ on } (-\infty,\infty); \tag{87}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n!}}\,(x^n+x^{-n}) \text{ on } \left(\frac{1}{2},2\right); \tag{88}$$

$$\sum_{n=2}^{\infty} \ln\left(1+\frac{x}{n\,(\ln n)^2}\right) \text{ on } (-a,a); \tag{89}$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n+\sin x} \text{ on } (0,2\,\pi); \tag{90}$$

$$\sum_{n=1}^{\infty} 2^n \sin\frac{1}{3^n\,x} \text{ on } (0,\infty); \tag{91}$$

$$\sum_{n=1}^{\infty} \frac{\sin x \sin(n\,x)}{\sqrt{n+x}} \text{ on } (0,\infty). \tag{92}$$

Exercise 61. (USTC3) Let $S_0(x)=1$ and define successively

$$S_n(x)=\sqrt{x\,S_{n-1}(x)}. \tag{93}$$

Prove that $S_n(x)$ converges uniformly on $[0,1]$.

Exercise 62. (FOLLAND) Let $f_n(x)=g(x)\,x^n$ where $g(x)$ is continuous on $[0,\,1]$ with $g(1)=0$. Prove that $f_n(x)\longrightarrow 0$ uniformly on $[0,1]$. (Hint:[31])

Exercise 63. (FOLLAND) Show that $\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}$ converges uniformly on any $[a,\,b]$ such that $[a,\,b]\cap(\mathbb{Z}-\{0\})=\varnothing$. Then prove that the sum is continuous on $\mathbb{R}-\{\mathbb{Z}-\{0\}\}$.

Exercise 64. (FOLLAND) Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{x^2+n}$ converges uniformly on $\mathbb{R}$, although the convergence is conditional at every x.

Exercise 65. (FOLLAND) Let $\{c_n\}$ be such that $\sum_{n=1}^{\infty} c_n$ converges. Consider the "Lambert series"

$$\sum_{n=1}^{\infty} c_n\,\frac{x^n}{1-x^n}, \qquad x\neq\pm 1. \tag{94}$$

a) Show that the series converges absolutely and uniformly on $[-a,a]$ for any $a<1$;

b) Show that the series converges uniformly on $(-\infty,b]$ and $[b,\infty)$ for any $b>1$, and that the convergence is absolute if and only if $\sum_{n=1}^{\infty}|c_n|<\infty$. (Hint:[32])

Exercise 66. (USTC3) Assume $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} a_n \exp[-n\,x]$ converges uniformly in $[0,\infty)$. Note that there is no condition on the sign of a_n.

Exercise 67. (USTC3) Let $\sum_{n=1}^{\infty} u_n(x)$ be convergent on $[a,b]$. Assume that

$$\exists M, \qquad \forall x\in[a,b], \forall n, \qquad \left|\sum_{k=1}^{n} u_k'(x)\right| \leqslant M. \tag{95}$$

31. For any $\varepsilon>0$, there is $\delta>0$ such that $|g(x)|<\varepsilon$ on $[1-\delta,\,1]$. Now take $N\in\mathbb{N}$ such that $(1-\delta)^N\,M<\varepsilon$ where $M=\max_{[0,1]}|g(x)|$.

32. When $|x|\geqslant b>1$, $\left|\frac{x^n}{1-x^n}\right|$ is uniformly bounded.

Prove that $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly. Then apply this result to $u_n(x) = \frac{\sin n\,x}{n}$ and $\frac{\cos n\,x}{n}$.

Exercise 68. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Show directly that $f(x)\,f(y) = f(x+y)$.

Exercise 69. (FOLLAND) Show that $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{1/2}}$ converges conditionally, and the Cauchy product of this series with itself diverges.

Exercise 70. (FOLLAND) Prove that

$$\sum_{m,n=1}^{\infty} \frac{1}{(m+n)^p} \tag{96}$$

converges if and only if $p > 2$. (Hint:[33])

Exercise 71. (FOLLAND) Let $a_{mn} = 1$ if $m = n$ and -1 if $m = n + 1$, and 0 otherwise. Prove that $\sum_{n=0}^{\infty} [\sum_{m=0}^{\infty} a_{mn}]$ and $\sum_{m=0}^{\infty} [\sum_{n=0}^{\infty} a_{mn}]$ both converge but the sums are unequal.

5.3. Problems

Problem 2. We say $f_n(x)$ converges to $f(x)$ weakly on $\mathbb{R}$ if and only if, for all functions $\phi \in C_0^{\infty}(\mathbb{R})$, that is ϕ is infinitely differentiable, and there is $R > 0$ such that $\phi(x) = 0$ for all $|x| > R$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,\phi(x)\,\mathrm{d}x. \tag{97}$$

Note that the integrals are not improper. Find $f_n(x)$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,\phi(x)\,\mathrm{d}x = \phi(0) \tag{98}$$

for all $\phi \in C_0^{\infty}(\mathbb{R})$. So in some sense $\lim_{n\to\infty} f_n(x) = \delta(x)$ the Dirac delta function.[34]

Problem 3. Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} a_n\,x^2 \cos(n\,x). \tag{99}$$

Problem 4. Generalize the continuity, differentiability, integrability theorems in the text to the case $f_n(x)$: $A \subseteq \mathbb{R}^N \mapsto \mathbb{R}$.

Problem 5. (USTC3) Let $a, b \in \mathbb{R}$. Assume that $\sum_{n=1}^{\infty} u_n(x) = S(x)$ on $[a, b]$ and each $u_n(x)$ is continuous and non-negative on $[a, b]$.

a) Prove that $S(x)$ attains minimum on $[a, b]$;

b) Must $S(x)$ attain its maximum on $[a, b]$? Justify.

c) Does the conclusion still hold if $[a, b]$ is not bounded?

d) Can the conclusion be generalized to replace $[a, b]$ by arbitrary compact set?

Problem 6. Find a non-negative convergent series $S(x) = \sum_{n=1}^{\infty} u_n(x)$ such that each $u_n(x)$ is continuous but $S(x)$ is not. Can you find a positive convergent series with the same property?

Problem 7. Order $\mathbb{Q}$ as $\{r_1, r_2, \ldots\}$. Define

$$f(x) := \sum_{n=1}^{\infty} \frac{|x - r_n|}{3^n}. \tag{100}$$

Prove that $f(x)$ is continuous, differentiable at every irrational point, but not differentiable at every rational point.

Problem 8. What is the property of

$$f(x) := \sum_{n=1}^{\infty} p^{-n}\,u_0(p^n\,x) \tag{101}$$

33. Since $a_{mn} > 0$, it doesn't matter how we sum it. Thus we can first sum up all terms with the same $m+n$ first.

34. According to ??, the Dirac delta function: $\delta(x) = 0$ for all $x \neq 0$ while $\int_{\mathbb{R}} \delta(x)\,\mathrm{d}x = 1$, was in fact proposed by Oliver Heaviside (1850 - 1925), who was kicked out of the Royal Society for this "academic sin". The function's acceptance by the community was due to its application by Paul Dirac (1902 - 1984), the real-life Sheldon Cooper. See ??.

where $p>1$ and $u_0(x)$ is as defined in (56).

Problem 9. (USTC3) Let $\sum_{n=1}^{\infty} a_n$ be convergent. Prove

$$\lim_{x\to 0+}\left(\sum_{n=1}^{\infty}\frac{a_n}{n^x}\right)=\sum_{n=1}^{\infty} a_n. \tag{102}$$