Math 317 Winter 2014 Homework 6 Solutions

Due Mar. 26 2p

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answers.

Question 1. *In the following $a > 0$ is a constant.*

a) *Calculate*
$$\int_L |x\, y|\, \mathrm{d}s \tag{1}$$
with $L: \begin{pmatrix} a \cos^3 t \\ a \sin^3 t \end{pmatrix}$, $t \in [0, 2\pi]$.

b) *Calculate*
$$\int_S (x + y + z)\, \mathrm{d}S \tag{2}$$
with S the first octant part of the unit sphere: $x^2 + y^2 + z^2 = a^2$, $x, y, z \geqslant 0$.

Question 2. *Let $a > 0$ be a constant. Calculate*
$$\int_L y\, \mathrm{d}x - z\, \mathrm{d}y + x\, \mathrm{d}z \tag{3}$$
where L is the intersection of $(x^2 + y^2)/2 + z^2 = a^2$ and $x = y$, oriented counter-clockwise when viewed from the positive x-axis.

a) *directly;*

b) *via Stokes's Theorem.*

Question 3. *Calculate*
$$\int_S \begin{pmatrix} x^2 \\ -y^2 \\ z^2 \end{pmatrix} \cdot \mathbf{d}\boldsymbol{S} \tag{4}$$
where $S = \partial V$ where $V = \{x^2 + y^2 + z^2 \leqslant 3\} \cap \{z \geqslant 0\} \cap \left\{z \geqslant \sqrt{x^2 + y^2 - 1}\right\}$, *oriented by the outer normal,*

a) *directly;*

b) *via Gauss's Theorem.*

Question 4. *Let $D \subset \mathbb{R}^2$. Let* $T(u, v) = \begin{pmatrix} X(u, v) \\ Y(u, v) \end{pmatrix}: \mathbb{R}^2 \mapsto \mathbb{R}^2$ *be a C^1 bijection such that* $\det \left(\frac{\partial(X, Y)}{\partial(u, v)} \right) \neq 0$ *everywhere. Assume that $D, T^{-1}(D)$ both satisfy the hypothesis of Green's Theorem.*

Prove
$$\mu(D) = \int_{T^{-1}(D)} \left| \det \left(\frac{\partial(X, Y)}{\partial(u, v)} \right) \right| \mathrm{d}(u, v) \tag{5}$$
using Green's Theorem.

Question 5. *Let S be a closed C^1 surface given by $\Phi = 0$ where $\Phi: \mathbb{R}^3 \mapsto \mathbb{R}$ is C^1 and satisfy* $\operatorname{grad} \Phi \neq 0$ *everywhere. Prove that the area of S is given by*
$$-\int_V \operatorname{div} \left(\frac{\operatorname{grad} \Phi}{\|\operatorname{grad} \Phi\|} \right) \mathrm{d}\boldsymbol{x} \tag{6}$$
where $V := \{\Phi > 0\}$ is the region enclosed by S.

Question 6. *Let $V \subset \mathbb{R}^3$ and let ∂V be C^1 oriented by outer normal $\boldsymbol{n}$. Let $u\colon \mathbb{R}^3 \mapsto \mathbb{R}$, $\boldsymbol{f}\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ be C^1. Prove that*

$$\int_{\partial V} u\, \boldsymbol{n}\, \mathrm{d}S = \int_V (\operatorname{grad} u)\, \mathrm{d}\boldsymbol{x}; \qquad \int_{\partial V} \boldsymbol{n} \times \boldsymbol{f}\, \mathrm{d}S = \int_V (\operatorname{curl} \boldsymbol{f})\, \mathrm{d}\boldsymbol{x}. \tag{7}$$

Here if $a \in \mathbb{R}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$, $a\, \boldsymbol{b}$ is defined as $\begin{pmatrix} a\, b_1 \\ a\, b_2 \\ a\, b_3 \end{pmatrix}$.