Math 317 Winter 2014 Homework 5 Solutions

Due Mar. 12 2p

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answers.

Question 1. *Let $x \in (0,1)$. Recall that it has decimal representation $x = 0.a_1 a_2 a_3 \ldots$, $a_n \in \{0,1,2,3,4,5,6,7,8,9\}$ means*

$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}. \tag{1}$$

Prove that

- *If $x \neq \frac{m}{10^n}$ for any $m, n \in \mathbb{N}$, then it has a unique decimal representation;*
- *If $x = \frac{m}{10^n}$ for some $m, n \in \mathbb{N}$, then it has exactly two decimal representations.*

Solution.

- Assume $x \neq \frac{m}{10^n}$ for any $m, n \in \mathbb{N}$.

  Then we determine $a_1, a_2, \ldots.$ one by one as follows.

  - Let $a_1 = \max \{i | i \in \{0, 1, \ldots., 9\}, i < 10\, x\}$.
  - Assume a_1, ..., a_n have been chosen. Set $e_n := x - 0.a_1 \ldots. a_n$. Set $a_{n+1} = \max \{i | i \in \{0, 1, \ldots., 9\}, i < 10^{n+1}\, e_n\}$.

  Now we prove that $x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$, which is equivalent to $\lim_{n \to \infty} e_n = 0$.

  First we notice that

$$e_1 = 10^{-1}\, [10\, x - a_1] \in (0, 10^{-1}). \tag{2}$$

  Note that by assumption $e_1 \neq 0$.

  Now assume $e_n \in (0, 10^{-n})$. Then we have

$$e_{n+1} = 10^{-(n+1)}\, [10^{n+1}\, e_n - a_{n+1}] \in \left(0, 10^{-(n+1)}\right). \tag{3}$$

  Therefore by induction we have

$$\forall n \in \mathbb{N}, \qquad 0 < e_n < 10^{-n} \tag{4}$$

  and the conclusion follows from Squeeze Theorem.

  Now we prove that this is the unique representation. Assume there is another one $0.b_1 b_2 \ldots = x$. Let $n_0 = \min \{k | a_k \neq b_k\}$. Wlog $a_k > b_k$. Then we have

$$0.a_1 a_2 \ldots. - 0.b_1 b_2 \ldots = \frac{a_k - b_k}{10^k} + \sum_{n=k+1}^{\infty} \frac{a_n - b_n}{10^n}. \tag{5}$$

  Thus we must have

$$\frac{1}{10^k} \leqslant \frac{a_k - b_k}{10^k} = \sum_{n=k+1}^{\infty} \frac{b_n - a_n}{10^n} \leqslant \sum_{n=k+1}^{\infty} \frac{9}{10^n} = \frac{1}{10^k}. \tag{6}$$

  This implies

$$\sum_{n=k+1}^{\infty} \frac{b_n - a_n}{10^n} = \sum_{n=k+1}^{\infty} \frac{9}{10^n} \Longrightarrow \sum_{n=k+1}^{\infty} \frac{9 - (b_n - a_n)}{10^n} = 0. \tag{7}$$

Since $9-(b_n-a_n)\geqslant 0$ for all n, the above series is non-negative. Therefore

$$0\leqslant \sum_{n=k+1}^{m} \frac{9-(b_n-a_n)}{10^n} \leqslant 0 \Longrightarrow 9=b_n-a_n \Longrightarrow b_n=9, a_n=0 \qquad \forall n\geqslant k+1. \tag{8}$$

But this implies

$$x=\frac{a_1....a_k}{10^k}, \tag{9}$$

contradiction.

- Assume $x=\frac{m}{10^n}$ for some $m,n\in\mathbb{N}$.

  The existence of two different representations is obvious:

$$x=0.a_1a_2....a_n000...\doteq 0.a_1'a_2'....a_n'999.... \tag{10}$$

with $m=a_1\,10^{n-1}+\cdots+a_0=\sum_{k=0}^{n-1} a_{n-k}\,10^k$ and $m-1=\sum_{k=0}^{n-1} a'_{n-k}\,10^k$. Now we prove that these are the only two.

Now we prove that any decimal $0.b_1b_2.... = x$ is either $0.a_1a_2....a_n000......$ or than $0.a_1'a_2'....a_n'999....$

For any such decimal there are only two cases:

i. $m=\sum_{k=0}^{n-1} a_{n-k}\,10^k \leqslant \sum_{k=0}^{n-1} b_{n-k}\,10^k$.

We have

$$\begin{aligned} 0=0.b_1b_2....-0.a_1a_2... &= \sum_{n=1}^{\infty} b_n\,10^{-n} - \sum_{k=1}^{n} a_k\,10^{-k} \\ &= \left[\sum_{k=1}^{n} b_k\,10^{-k} - \sum_{k=1}^{n} a_k\,10^{-k}\right] + \sum_{k=n+1}^{\infty} b_k\,10^{-k} \\ &= 10^{-n}\left[(b_1\,10^{n-1}+...+b_n)-(a_1\,10^{n-1}+...+a_n)\right] \\ &\quad + \sum_{k=n+1}^{\infty} b_k\,10^{-k}. \end{aligned} \tag{11}$$

Note that the re-arrangement of the sums is allowed because both series are absolutely convergent.

By assumption $(b_1\;10^{n-1}+...+b_n)-(a_1\;10^{n-1}+...+a_n)\geqslant 0$, by definition of decimals $b_k\geqslant 0$ for each $k=n+1,....$ and therefore $\sum_{k=n+1}^{\infty} b_k\;10^{-k}\geqslant 0$. Thus we conclude

$$(b_1\,10^{n-1}+...+b_n)-(a_1\,10^{n-1}+...+a_n)=0 \Longrightarrow a_k=b_k, \quad k=1,2,3,... \tag{12}$$

and

$$\sum_{k=n+1}^{\infty} b_k\,10^{-k}=0. \tag{13}$$

Notice that

$$0\leqslant b_{n+1}\,10^{-(n+1)} \leqslant \sum_{k=n+1}^{\infty} b_k\,10^{-k}=0 \Longrightarrow b_{n+1}=0. \tag{14}$$

Now by induction we can prove

$$b_k=0, \qquad k=n+1,n+2,... \tag{15}$$

ii. $m-1=\sum_{k=0}^{n-1} a'_{n-k}\,10^k \geqslant \sum_{k=0}^{n-1} b_{n-k}10^k$.

The proof for $b_k = a_k'$ for $k \leqslant n$ and $b_k = 9$ for all $k > n$ is similar to the previous case and is omitted.

Question 2. *Prove the following through explicit construction of bijections.*

a) $\mathbb{R} - \mathbb{Q} \sim \mathbb{R}$; *You can assume that* $\mathbb{Q}$ *has already been listed as* $\{r_1, r_2, \ldots.\}$.

b) $\mathbb{R} - A \sim \mathbb{R}$, *where* A *is the set of all algebraic numbers. You can assume that* A *has already been listed as* $\{a_1, a_2, a_3, \ldots.\}$.

Solution.

a) Since $\mathbb{Q}$ is countable, we list $\mathbb{Q} = \{r_1, r_2, \ldots.\}$. Now let $A := \mathbb{Q} + \sqrt{2} = \{r_1 + \sqrt{2}, r_2 + \sqrt{2}, \ldots.\}$. As $\sqrt{2}$ is irrational, we have $\mathbb{Q} \cap A = \varnothing$. Now define

$$f: \mathbb{R} \mapsto \mathbb{R} - \mathbb{Q} \qquad f(x) = \begin{cases} x & x \notin A \cup \mathbb{Q} \\ r_{2k} + \sqrt{2} & x = r_k + \sqrt{2} \in A \\ r_{2k-1} + \sqrt{2} & x = r_k \in \mathbb{Q} \end{cases}. \tag{16}$$

It is clear that this is a bijection.

b) The construction is almost identical as soon as we find $B \subset \mathbb{R}$ such that $B \cap A = \varnothing$ and $B \sim A$. To do this we consider the set

$$B := \alpha\, \mathbb{N} = \{\alpha, 2\,\alpha, 3\,\alpha, \ldots.\} \tag{17}$$

where $\alpha \in \mathbb{R} - A$. Note that since $\mathbb{R}$ is not countable, we know such α exists (for example $\alpha = \pi$ or e).

It is clear that $B \sim A$. All we need to show is $B \cap A = \varnothing$, that is $n\,\pi$ is not algebraic for any $n \in \mathbb{N}$. Assume the contrary. Then there are $n \in \mathbb{N}$ and $a_0, \ldots, a_m \in \mathbb{Z}$ with $a_0, \ldots., a_m$ not all 0, such that

$$a_m\,(n\,\alpha)^m + \cdots + a_1\,(n\,\alpha) + a_0 = 0. \tag{18}$$

This gives

$$(a_m\, n^m)\,\alpha^m + (a_{m-1}\, n^{m-1})\,\alpha^{m-1} + \cdots + (a_1\, n)\,\alpha + a_0 = 0. \tag{19}$$

Since α is not algebraic, we conclude

$$a_m\, n^m = a_{m-1}\, n^{m-1} = \cdots = a_1\, n = a_0 = 0 \Longrightarrow a_m = a_{m-1} = \cdots = a_1 = a_0 = 0. \tag{20}$$

Contradiction.

Question 3. *Recall that if* $(X, \leqslant)$ *is a partially ordered set, then* $x_0 \in Y \subseteq X$ *is said to be*

- *a* ***least element*** *of* Y *if and only if for every* $y \in Y$, $x_0 \leqslant y$.
- *a* ***minimal element*** *of* Y *if and only if there is no* $y \in Y$ *such that* $y < x_0$.

a) *Let* $(X, \leqslant)$ *be a partially ordered set. Assume that every non-empty subset of* X *has a least element. Prove that* $\leqslant$ *is in fact a well-ordering.*

b) *Does the conclusion in a) still hold if we instead assume that every non-empty subset has a minimal element?*

c) *Does the conclusion in a) still hold if we instead assume that every non-empty subset has a unique minimal element?*

d) *Find a partially ordered set* $(X, \leqslant)$ *such that it has a unique minimal element but no least element.*

Solution.

a) All we need to show is that $(X, \leqslant)$ is linearly ordered, that is any $x, y \in X$, at least one of $x \leqslant y, y \leqslant x, x = y$ holds.

Take any $x, y \in X$ such that $x \neq y$. Consider the non-empty subset $\{x, y\} \subseteq X$. We know that it has a least element, which must be either x or y. If x is the least element, then by definition of least element we have $x < y$. On the other hand if y is the least element, we have $y < x$.

b) No. For example consider $\mathbb{N} \cup \{a\}$ where a is not comparable to any other element.

c) Yes. First in this case the order must be linear, since otherwise we have $a, b \in X$ not comparable, then $\{a, b\} \subseteq X$ has two minimal elements. Now we show that in a linearly ordered set, any minimal element must also be a least element. Assume otherwise, then there is $y \in Y$ such that $x_0 \leqslant y$ does not hold. Since the order is linear, it must hold that $y < x_0$, contradicting the minimality of x_0.

d) Let $X = \{a, b\} \cup \mathbb{Z}$ be ordered by $a < b$ and the natural ordering on $\mathbb{Z}$. Then a is the only minimal element but there is no least element.

Question 4. *We define an ordered pair (a, b) as $\{\{a\}, \{a, b\}\}$.*

a) *Prove that $(a, b) = (c, d) \Longleftrightarrow a = c,\ b = d$. (Note: the case $a = b$ needs to be discussed separately).*

b) *Review lecture note for Weeks 5 & 6 and the 217 lecture note on "Numbers". Explain why any positive real number x can be identified as a*

set of sets of sets of sets of finite ordinals.

Solution.

a) $\Longleftarrow$ is trivial. We prove $\Longrightarrow$.

- First consider the case $a = b$. In this case
$$\{\{c\}, \{c, d\}\} = (c, d) = (a, b) = \{\{a\}, \{a, b\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}. \tag{21}$$
Therefore
$$\{c\} = \{c, d\} = \{a\} \Longrightarrow c = d = a. \tag{22}$$
- Now consider the case $a \neq b$. In this case we have
$$\{\{c\}, \{c, d\}\} = \{\{a\}, \{a, b\}\}. \tag{23}$$
Thus either
$$\{c\} = \{a\}, \qquad \{c, d\} = \{a, b\} \tag{24}$$
or
$$\{c\} = \{a, b\}, \qquad \{a\} = \{c, d\}. \tag{25}$$
The latter implies $c = a = b$, contradicting the assumption $a \neq b$. The former implies $a = c$, $b = d$, as desired.

b) We have

- According to a), each ordered pair of natural numbers (m, n) is a

set of sets of finite ordinals;

- positive rational numbers are defined as equivalent class of ordered pairs of natural numbers, therefore any $r \in \mathbb{Q}^+$ is a set of ordered pairs, and there for is a

  set of sets of sets of finite ordinals;

- positive real numbers are defined through Dedekind's "cuts", that is as subsets of $\mathbb{Q}^+$, therefore any $x \in \mathbb{R}^+$ is a

  set of sets of sets of sets of finite ordinals.

Question 5.

a) *Re-order* $\mathbb{N}$ *to obtain the following ordinal numbers:*

$$\omega + 7; \qquad \omega \cdot 2; \qquad \omega \cdot \omega + 1 \tag{26}$$

You don't need to justify your answers.

b) *Find ordinal numbers* α, β, γ*, calculate* $(\alpha + \beta) \cdot \gamma$ *and* $\alpha \cdot \gamma + \beta \cdot \gamma$ *to show* $(\alpha + \beta) \cdot \gamma \neq \alpha \cdot \gamma + \beta \cdot \gamma$*. You don't need to justify your calculations.*

Solution.

a) We have

- $\omega + 7$:

$$8 < 9 < 10 < \cdots < 1 < 2 < 3 < 4 < 5 < 6 < 7. \tag{27}$$

- $\omega \cdot 2$:

$$1 < 3 < 5 < \cdots < 2 < 4 < 6 < \cdots \tag{28}$$

- $\omega \cdot \omega + 1$:

$$\text{all prime numbers} < \text{numbers with two prime factors} < \cdots < 1. \tag{29}$$

b) Take $\alpha = \beta = 1$, $\gamma = \omega$. Then

$$(\alpha + \beta) \cdot \gamma = 2 \cdot \omega = \omega \neq \omega + \omega = 1 \cdot \omega + 1 \cdot \omega. \tag{30}$$

Question 6. *Let* $A_1 := \{E \subseteq \mathbb{R} | \, E \text{ is open}\}$*;* $A_2 := \{E \subseteq \mathbb{R} | \, E \text{ is closed}\}$*;* $A_3 := \{E \subseteq \mathbb{R} | \, E \text{ is Jordan measurable}\}$*;* $A_4 := \{E \subseteq \mathbb{R}^2 | \, E \text{ is Jordan measurable}\}$*;* $A_5 := \{f: \mathbb{R}^2 \mapsto \mathbb{R} | \, f \text{ is Riemann integrable}\}$*. Find the cardinalities of* $A_1 - A_5$*. (Hint:[1])*

Solution.

a) Since $\{(0, r) | \, r \in \mathbb{R}\} \subset A_1$, we have $A_1 \gtrsim \mathbb{R}$. On the other hand, any open set E is the union of all the rational open intervals contained in it:

$$E = \cup_{a, b \in \mathbb{Q}, (a, b) \subseteq E} (a, b). \tag{31}$$

Therefore we can construct a one-to-one mapping from A_1 to $\mathcal{P}(\mathbb{Q} \times \mathbb{Q})$ as follows[2]:

$$E \in A_1 \mapsto \{(a, b) \in \mathbb{Q} \times \mathbb{Q} | \, (a, b) \subseteq E\}. \tag{32}$$

Therefore we have $A_1 \lesssim \mathcal{P}(\mathbb{Q} \times \mathbb{Q}) \sim \mathcal{P}(\mathbb{N}) \sim \mathbb{R}$. Therefore by Schröder-Bernstein we have $A_1 \sim \mathbb{R}$. The cardinality is therefore $\mathfrak{c}$.

1. Cantor set has the same cardinality as $\mathbb{R}$.
2. Note that the first (a, b) is an ordered pair while the second (a, b) is an open interval.

b) Since $\{[r,r] | r \in \mathbb{R}\} \subseteq A_2$ we have $A_2 \gtrsim \mathbb{R}$. On the other hand, any closed set E can be written as

$$E = \cup_{k \in \mathbb{Z}} E_k \tag{33}$$

with each $E_k := E \cap [k, k+1]$. Therefore if we set $A_2' := \{E \subseteq [0, 1] | E \text{ is closed}\}$, then $A \lesssim (A_2')^{\mathbb{Z}}$.

Now notice that the following mapping from A_2' to A_1 is one-to-one:

$$E \mapsto \mathbb{R} - E. \tag{34}$$

Therefore we have $A_2' \lesssim \mathbb{R}$. This gives $A \lesssim \mathbb{R}^{\mathbb{Z}} \sim \mathbb{R}$.

Finally by Schröder-Bernstein we have $A_2 \sim \mathbb{R}$. The cardinality is therefore $\mathfrak{c}$.

Remark. We see that most sets in $\mathbb{R}$ are neither open nor closed.

c) It is clear that $A_3 \lesssim \mathcal{P}(\mathbb{R})$. On the other hand, since the Cantor set C has Jordan measure 0 and $C \sim \mathbb{R}$, any subset of C is Jordan measurable (with measure 0). Therefore $A_3 \gtrsim \mathcal{P}(C) \sim \mathcal{P}(\mathbb{R})$. By Schröder-Bernstein we have $A_3 \sim \mathcal{P}(\mathbb{R})$. The cardinality is therefore $2^{\mathfrak{c}}$.

d) It is clear that $A_4 \lesssim \mathcal{P}(\mathbb{R}^2) \sim \mathcal{P}(\mathbb{R})$. On the other hand, the segment $[0, 1] \times \{0\}$ is Jordan measurable with measure 0. Thus $A_4 \gtrsim \mathcal{P}([0,1] \times \{0\}) \sim \mathcal{P}(\mathbb{R})$. Consequently the cardinality is again $2^{\mathfrak{c}}$.

e) First consider all the characteristic functions for Jordan measurable sets

$$\chi_E(x, y) := \begin{cases} 1 & (x, y) \in E \\ 0 & (x, y) \notin E \end{cases}. \tag{35}$$

This means $A_5 \gtrsim A_4 \sim \mathcal{P}(\mathbb{R})$. On the other hand,

$$A_5 \lesssim \{\text{All functions } \mathbb{R}^2 \mapsto \mathbb{R}\} \lesssim \mathcal{P}(\mathbb{R}^2 \times \mathbb{R}) \sim \mathcal{P}(\mathbb{R}). \tag{36}$$

Consequently the cardinality is still $2^{\mathfrak{c}}$.

Question 7. (Extra 1 pt) *Study the wiki page http://en.wikipedia.org/wiki/JPEG about the JPEG format. Explain why discrete cosine (instead of discrete sine, or classical Fourier expansion/transform) are used in encoding every 8×8 block.*

Solution. Each 8×8 block can be viewed as

a) one period of a (doubly) periodic function; This corresponds to Fourier expansion which leads to Discrete Fourier transform;

b) one half (in each direction) period of a doubly periodic and odd function; This corresponds to Fourier Sine expansion which leads to Discrete Sine transform;

c) one half (in each direction) period of a double periodic and even function; This corresponds to Fourier cosine expansion which leads to Discrete Cosine transform.

From the convergence theory of Fourier series, we know that if this function is not continuous, Gibbs's phenomenon will appear. The only continuous function in the above three cases is c). This is why Discrete Cosine Transform is used.