Math 317 Winter 2014 Homework 3

Due Feb. 5 2p

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answers.

Question 1. *Calculate the radius of convergence of the power series*

$$\sum_{n=1}^{\infty} \left(\frac{3}{n} + \frac{6}{n^2}\right) x^n. \tag{1}$$

Question 2. *Find all $x \in \mathbb{R}$ where the series*

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} e^{nx} \tag{2}$$

converges.

Question 3. *Consider the infinite series in Question 2.*

a) *Discuss the uniform convergence of the series.*

b) *Is the sum a continuous function (meaning: continuous at every x where it is defined)?*

Question 4.

a) *Prove*

$$\forall x \in (-1,1), \qquad \frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}. \tag{3}$$

b) *Then prove*

$$\forall x \in (-1,1), \qquad \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}. \tag{4}$$

Question 5. *Without using Abel's theorem, prove directly through the re-summation technique that*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \tag{5}$$

converges uniformly on $[0,1]$. Then prove

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots. \tag{6}$$

Question 6. *Let the radii of convergence for $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be R_1, R_2 respectively.*

a) *Prove that the radius of convergence R for the series $\sum_{n=0}^{\infty} (a_n b_n) x^n$ satisfies $R \geqslant R_1 R_2$.*

b) *Show through an example that strict inequality may hold: $R > R_1 R_2$.*

Note: For part a) you shouldn't assume the existence of any of $\lim_{n\to\infty} |a_n|^{1/n}, \lim_{n\to\infty} |b_n|^{1/n}$, or $\lim_{n\to\infty} |a_n b_n|^{1/n}$.